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TRANSFORMATION OPERATOR FOR STURM-LIOUVILLE OPERATORS WITH SINGULARITY AND DISCONTINUITY CONDITIONS INSIDE AN INTERVAL

Abstract

In this article the existence of transformation operators is proved for a class of singular Sturm-Liouville differential operators with discontinuity conditions inside an interval. In addition, the classical relation between the potential of given operator and the kernel of transformation operator is given.

Introduction

In interval (a,b) when the given interval is finite which is generated by the differential expression $\ell(y) := -y''(x)+q(x)y(x)$ in theory of the Sturm-Liouville operator, the function $q(x)$ satisfies the condition $q(x) \in L_1(a,b)$ in general. As in singular case, i.e. when interval (a,b) is infinite or the function $q(x)$ has non-integrable singularity in extremity points of interval the condition of $q(x) \in L_{1,loc}(a,b)$ is given.

In study of [6] when $q(x)$ is a first order singular generalized function, Sturm-Liouville operator, i.e. singular Sturm-Liouville operator has been defined which has a potential as $q = u'$ by using concept of generalized derivative such that $u \in L_2(0,1)$.

Moreover, in this study, self-adjoint extensions of differential operators generated by differential expression $\ell(y)$ which has potential $q(x) = u'(x)$ such that $u \in L_2(0,1)$. When $\alpha \neq 2,4,6,\ldots$ generalized functions can be corresponded to the functions $|x|^{-\alpha}\, signx$ by using the method of canonic regularization [5]. When $\alpha < \dfrac{3}{2}$ generalized functions which are obtained by this way can be shown as generalized derivative of functions from the space L_2 and therefore Sturm-Liouville operator which is given by the differential expression $\ell(y)$ and which has a potential like $q(x) = |x|^{-\alpha}\, signx$ can be defined. In [1], when $q(x) = cx^{-\alpha}$ and $x < 3/2$, $C \in \Re$, a regularization of the constructing boundary value problems for Sturm-Liouville equation which has this type potential has been given.

As in this study of [4], when $q(x) = cx^{-\alpha}$ and $\alpha \in [1,2)$, all self-adjoint extensions of operators generated by the differential expression $\ell(y)$ which has this type potential according to boundary value conditions and therefore when $\alpha \in [1,2)$ regularization of constructing boundary value problems for Sturm-Liouville equation which has type potential has been given. Regularization in the studies of [4] and [6] coincide only when $\alpha < 3/2$.

Let us consider the differential expression

$$\ell(y) := -y''(x) + \frac{c}{x^{\alpha}}y(x) + q(x)y(x), \qquad 0 < x < \pi, \tag{1}$$

where c is a real number, $q(x)$ is a real valued bounded function.

We shall define an operator $L_0' : L_0'(y) = \ell y$, in the set of $D_0' = C_0^{\infty}(0,\pi)$. It is obvious that the operator L_0' is symmetric in the space of $L_2(0,\pi)$. We say that the operator L_0 which is the closure of L_0' is minimal operator generated by the differential expression (1). The conjugate L_0' of the operator L_0 is said to be maximal operator generated by the differential expression (1).

In [4], all maximal dissipative and accumulative and also self-adjoint extensions of the operator L_0 have been studied according to the domain and boundary conditions of minimal and maximal operators generated by differential expression (1).

We denote that $(\Gamma_\alpha y)(x) = y'(x) - u(x)y(x)$, where $u(x) = \dfrac{Cx^{1-\alpha}}{1-\alpha}$.

It has been shown, in [4], that if $y(x) \in D(L_0^*)$ then the function $(\Gamma_\alpha y)(x)$ has a limit as $x \to 0^+$, i.e.

$$\lim_{x\to 0^+} (\Gamma_\alpha y)(x) = (\Gamma_\alpha y)(0).$$

Hence the domain $D(L_0)$ of minimal operator L_0 generated by differential expression (1) contains only functions $y(x) \in D(L_0^*)$ such that function $y(x)$ satisfies the conditions $y(0) = y(\pi) = (\Gamma_\alpha y)(0) = y'(\pi) = 0$.

In the present paper the construction method of transform operators is given for one class of singular Sturm-Liouville operator in the case $q = u'$ with discontinuity conditions inside an interval. Here the differentiation is in the meaning of general differentiation. In the case when Sturm-Liouville operator has a singularity of Bessel type ($\ell(\ell-1)x^{-2}$, $\ell \in \mathbb{Z}_+$) on the finite interval the transform operator was construct in [5], [1] and in the case $[0,\infty)$ it was given in [4]. When Sturm-Liouville operator has a singularity of Coulomb type (Ax^{-1}, $A \in \Re - \{0\}$) on the finite interval, the transformation operator was constructed in [7]. Note that transformation operator for the Sturm-Liouville operator with the potential $x\,|q(x)| \in L_1(0,\pi)$ related with $x = 0$ was constructed by Amirov. In the paper [3] the classical relation between the potential of given operator and the kernel of transformation operator was not given obviously, but in this study the classical relation between them is given. In the case when Sturm-Liouville operator with $q(x) \in L_2(0,\pi)$ and discontinuity conditions inside an interval the transform operator was construct in [9].

2. Construction of the integral equations. Let us consider the boundary value problem $L = L(h, H, \beta)$ for the equation:

$$\ell(y) := -y''(x) + \frac{c}{x^\alpha} y(x) + q(x)y(x) = \lambda y(x), \quad \lambda = k^2, \tag{2}$$

on the interval $(0,\pi)$ with the boundary conditions

$$U(y) := (\Gamma_\alpha y)(0) - hy(0) = 0, \qquad V(y) := (\Gamma_\alpha y)(\pi) + Hy(\pi) = 0, \tag{3}$$

and with the jump conditions

$$y(a+0) = \beta y(a-0), \qquad y'(a+0) = \beta^{-1} y'(a-0), \tag{3'}$$

where λ is the spectral parameter, $\alpha \in (1, 3/2)$, β are real numbers and $a \in (0,\pi)$,$\beta > 0$, $\beta \neq 1$, $q(x) \in L_2(0,\pi)$ is real valued function.

Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering of the geosciences.

The first part of this work establishes estimates for solutions of (1). Once these estimates are established, many of the proofs from can be used with minor modifications. The main feature of the theory developed in [10], which is not present here, is the special role played by even potentials. Consequently, in the second part of this work we simply sketch the modified theory which applies to the problems (1). Although the "generalized Dirichlet" boundary conditions, $y(0) = 0$, $y'(\pi) + Hy(\pi) = 0$, are not explicitly considered, it should be noted that a development analogous to that in [10] can be found in [11].

To fix notation the Wronskian of the functions f and g is

$$[f, g] = fg' - f'g.$$

We begin with basic estimates valid for any solution of (1) with $\alpha \in [0, 2)$. By using variation of the parameters, every solution $Y(x,\lambda)$ of equation (1) can be written as a solution of the integral equation

$$Y(x,\lambda) = A\cos(k[\pi - x]) - B\frac{\sin(k[\pi - x])}{k} + \\ + \int_x^\pi \frac{\sin(k[\pi - t])}{k}\left[\frac{c}{t^\alpha} + q(t)\right] Y(t,\lambda)dt. \tag{4}$$

Estimate for $Y(x,\lambda)$ can be obtained using Picard iteration.

Lemma 1. *If $\alpha \in [0, 2)$ then $Y(x,\lambda)$ extends continuously to $[0, \pi]$ for each $\lambda \in C$ and $Y(x,\lambda)$ is an entire function of λ for each $x \in [0, \pi]$. $Y(x,\lambda)$ satisfies the estimates*

$$\left| Y(x,\lambda) - A\cos(k[\pi - x]) + B\frac{\sin(k[\pi - x])}{k} \right| \le \\ \le \begin{cases} K\exp(\sigma[\pi - x])\,|k|^{-1}\log(1 + |k|), & \alpha \le 1, \\ K\exp(\sigma[\pi - x])\,|k|^{\alpha-2}, & 1 < \alpha < 2. \end{cases}$$

Here the function $K(\|q\|_{L_2}, |A| + |B|)$ is bounded on bounded subset of $[0,\infty)\times$ $\times[0,\infty)$, $\sigma = |\mathrm{Im}(k)|$.

Lemma 2. *For $\alpha \in [0, 2)$ equation (1) has a unique solution $Y_2(x,\lambda)$ whose derivative extends continuously to $x = 0$ and satisfies $Y_2(0,\lambda) = 0$, $Y_2'(0,\lambda) = 1$. For each $x \in [0, \pi]$, $Y_2(x,\lambda)$ is an entire function of λ and $Y_2(x,\lambda)$ satisfies the estimate*

$$|Y_2(x,\lambda)| \le K\frac{x}{1 + |kx|}\exp(\sigma x) \tag{5}$$

Proof. This lemma we concentrate on the estimate, analysing the integral equation:

$$Y_2(x,\lambda) = \frac{\sin kx}{k} + \int_0^x \frac{\sin(k[\pi - t])}{k}\left[\frac{c}{t^\alpha} + q(t)\right] Y_2(t,\lambda)dt,$$

with Picard iteration.

Lemma 3. *For each $x \in [0, \pi]$ the function $Y_2(x,\lambda)$ satisfies the estimates*

$$\left| Y_2(x,\lambda) - \frac{\sin kx}{k} \right| \le \begin{cases} Kx\exp(\sigma x)\,|k|^{-1}\log(1 + |k|), & 0 \le \alpha \le 1, \\ Kx\exp(\sigma x)\,|k|^{\alpha-2+\varepsilon}, & 1 < \alpha < 2. \end{cases}$$

Proof lemma 3 analogously to the proof of lemma 1.

We will need a second solution, $Y_1(x,\lambda)$ of (1) which is linearly independent from Y_2 and so that the pair Y_1, Y_2 is analogous to the pair $\cos kx$, $\dfrac{\sin kx}{k}$ is the case $c=0=q(x)$. The solution $Y_1(x,\lambda)$ of (1) is defined by requiring

$$Y_1(1,\lambda)=\frac{Y_2'(\pi,\lambda)}{[Y_2(\pi,\lambda)]^2+[Y_2'(\pi,\lambda)]^2},\qquad Y_1'(1,\lambda)=\frac{-Y_2(\pi,\lambda)}{[Y_2(\pi,\lambda)]^2+[Y_2'(\pi,\lambda)]^2}$$

Such a choice makes Y_1 and Y_2 linearly independent, since the Wronskian is $[Y_1,Y_2]=1$. Notice that for λ real, depends analytically on λ.

Using the representation (4) for Y_1, $A_1(\lambda)=Y_1(1,\lambda)$, $B_1(\lambda)=Y_1'(1,\lambda)$. Differentiating (4) gives

$$Y_1'(x,\lambda)=A_1k\sin(k[\pi-x])+B_1\cos(k[\pi-x])+$$
$$+\int_x^{\pi}\cos(k[\pi-t])\left[\frac{c}{t^{\alpha}}+q(t)\right]Y_1(t,\lambda)dt.$$

Lemma 2 shows that, for each real λ, $Y_1(x,\lambda)$ is bounded function and thus $Y_1'(x,\lambda)=O(x^{1-\alpha})$ (with the obvious modification if $\alpha=1$). This observation together with the estimates (5) means that $Y_2(x,\lambda)Y_1'(x,\lambda)$ extends continuously to $[0,\pi]$, with

$$(Y_2Y_1')(0,\lambda)=\lim_{x\to 0^+}(Y_2Y_1')(x,\lambda)=0.$$

We denote that $y_1(x)=y(x)$, $\quad y_2(x)=(\Gamma_\alpha y)(x)=y'(x)-u(x)y(x)$ and let us write the expression of left hand side of the equation (2) as follows

$$\ell(y):=-\left((\Gamma_\alpha y)(x)\right)'-u(x)(\Gamma_\alpha y)(x)-u^2(x)y(x) \tag{6}$$

Then the equation (2) reduces to the system

$$\begin{cases} y_1'-u(x)y_1=y_2, \\ y_2'+u(x)y_2+u^2(x)y_1=-k^2y_1 \end{cases} \tag{7}$$

or in matrice form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'=\begin{pmatrix} u(x) & 1 \\ -k^2-u^2(x) & -u(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{8}$$

The entries of the matrice

$$A(x)=\begin{pmatrix} u(x) & 1 \\ -k^2-u^2(x) & -u(x) \end{pmatrix}$$

are functions in $L_1[0,\pi]$.

For this reason [7], there exists only one solution of system (8) which satisfies the same initial conditions $y_1(\xi)=\alpha_1$, $y_2(\xi)=\alpha_2$ for each $\xi\in[0,\pi]$, $\alpha=(\alpha_1,\alpha_2)^t\in C$ especially the initial conditions $y_1(0)=1$, $y_2(0)=h$.

Definition 1. *First component of the solution which satisfies the initial conditions $y(\xi) = \alpha_1$, $(\Gamma_\alpha y)(\xi) = \alpha_2$ of the system (8) is called as the solution of the equation (2) which satisfies the same initial conditions.*

Let us denote a solution of system (7) by $\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}(x)$ in the case $q(x) \equiv 0$ satisfying the initial conditions $y_{01}(0) = 1$, $y_{02}(0) = ik$ and the jump conditions $(3')$.

It is obvious that function

$$\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}(x,\lambda)$$

is written as

$$\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}(x,\lambda) = \begin{cases} \begin{pmatrix} 1 \\ ik \end{pmatrix} e^{ikx}, & 0 \le x < a, \\ \beta^+ \begin{pmatrix} 1 \\ ik \end{pmatrix} e^{ikx} + \beta^- \begin{pmatrix} 1 \\ -ik \end{pmatrix} e^{ik(2a-x)}, & a < x \le \pi, \end{cases}$$

where $\beta^{\pm} = \frac{1}{2}(\beta \pm \frac{1}{\beta})$.

Let us prove that representation
for $0 \le x < a$

$$\begin{aligned} y_1(x,\lambda) &= e^{ikx} + \int_{-x}^{x} K_{11}(x,t)e^{ikt}dt, \\ y_2(x,\lambda) &= ike^{ikx} + b(x)e^{ikx} + \int_{-x}^{x} K_{21}(x,t)e^{ikt}dt + ik\int_{-x}^{x} K_{22}(x,t)e^{ikt}dt, \end{aligned} \tag{9}$$

for $a < x \le \pi$.

$$\begin{aligned} y_1(x,\lambda) &= \beta^+e^{ikx} + \beta^- e^{ik(2a-x)} + \int_{-x}^{x} K_{11}(x,t)e^{ikt}dt \\ y_2(x,\lambda) &= ik\left(\beta^+e^{ikx} - \beta^- e^{ik(2a-x)}\right) + b(x)\left(\beta^+e^{ikx} + \beta^- e^{ik(2a-x)}\right) + \\ &+ \int_{-x}^{x} K_{21}(x,t)e^{ikt}dt + ik\int_{-x}^{x} K_{22}(x,t)e^{ikt}dt. \end{aligned} \tag{9'}$$

Then it is clearly shown that integral equation for the solution $(y_1(x,\lambda), y_2(x,\lambda))^t$

is of the following type:

$$\begin{cases} y_1^{(1)}(x,\lambda)=e^{ikx}+\int\limits_0^x u(t)\cos k(x-t)y_1^{(1)}(t,\lambda)dt- \\ \qquad -\int\limits_0^x \frac{\sin k(x-t)}{k}\left\{u^2(t)y_1^{(1)}(t,\lambda)+u(t)y_2^{(1)}(t,\lambda)\right\}dt, \\ y_2^{(1)}(x,\lambda)=ike^{ikx}-k\int\limits_0^x u(t)\sin k(x-t)y_1^{(1)}(t,\lambda)dt- \\ \qquad -\int\limits_0^x \cos k(x-t)\left\{u^2(t)y_1^{(1)}(t,\lambda)+u(t)y_2^{(1)}(t,\lambda)\right\}dt, \end{cases}$$

for $0\le x<a$

$$\begin{cases} y_1^{(2)}(x,\lambda)=\beta^+e^{ikx}+\beta^-e^{ik(2a-x)}+ \\ +\int\limits_0^a u(t)\left[\beta^+\cos k(x-t)+\beta^-\cos k(x+t-2a)\right]y_1^{(1)}(t,\lambda)dt- \\ -\frac{1}{k}\int\limits_0^a \left[\beta^+\sin k(x-t)-\beta^-\sin k(x+t-2a)\right]\left\{u^2(t)y_1^{(1)}(t,\lambda)+u(t)y_2^{(1)}(t,\lambda)\right\}dt+ \\ +\int\limits_a^x u(t)\cos k(x-t)y_1^{(2)}(t,\lambda)dt-\int\limits_a^x \frac{\sin k(x-t)}{k}\left\{u^2(t)y_1^{(2)}(t,\lambda)+u(t)y_2^{(2)}(t,\lambda)\right\}dt, \\ y_2^{(2)}(x,\lambda)=ik\left(\beta^+e^{ikx}-\beta^-e^{ik(2a-x)}\right)- \\ -k\int\limits_0^a u(t)\left[\beta^+\sin k(x-t)-\beta^-\sin k(x+t-2a)\right]y_1^{(1)}(t,\lambda)dt- \\ -\int\limits_0^a \left[\beta^+\cos k(x-t)+\beta^-\cos k(x+t-2a)\right]\left\{u^2(t)y_1^{(1)}(t,\lambda)+u(t)y_2^{(1)}(t,\lambda)\right\}dt- \\ -k\int\limits_a^x u(t)\sin k(x-t)y_1^{(2)}(t,\lambda)dt-\int\limits_a^x \cos k(x-t)\left\{u^2(t)y_1^{(2)}(t,\lambda)+u(t)y_2^{(2)}(t,\lambda)\right\}dt. \end{cases} \tag{10}$$

In order to be solution of system equations (10) of the functions which has representations (9) and (9'), the equality

$$\int\limits_{-x}^x K_{11}(x,t)e^{ikt}dt=\int\limits_0^a u(t)\left[\beta^+\cos k(x-t)+\beta^-\cos k(x+t-2a)\right]\left[e^{ikt}+\right.$$

$$\left.+\int\limits_{-t}^t K_{11}(t,s)e^{iks}ds\right]dt-\frac{1}{k}\int\limits_0^a \left[\beta^+\sin k(x-t)-\beta^-\sin k(x+t-2a)\right]\times$$

$$\times\left\{u^2(t)e^{ikt}+u^2(t)\int\limits_{-t}^{t}K_{11}(t,s)e^{iks}ds+u(t)\left[ike^{ikt}+b(t)e^{ikt}+\int\limits_{-t}^{t}K_{21}(t,s)e^{iks}ds+\right.\right.$$

$$\left.\left.+ik\int\limits_{-t}^{t}K_{22}(t,s)e^{iks}ds\right]\right\}+\int\limits_{0}^{x}u(t)\cos k(x-t)\left[\beta^+e^{ikt}+\beta^-e^{ik(2a-t)}+\right.$$

$$\left.+\int\limits_{-t}^{t}K_{11}(t,s)e^{iks}ds\right]dt-\int\limits_{a}^{x}\frac{\sin k(x-t)}{k}\left\{u^2(t)\left[\beta^+e^{ikt}+\beta^-e^{ik(2a-t)}+\right.\right.$$

$$\left.+\int\limits_{-t}^{t}K_{11}(t,s)e^{iks}ds\right]+u(t)\left[ik\left(\beta^+e^{ikt}-\beta^-e^{ik(2a-t)}\right)+b(t)\left(\beta^+e^{ikt}-\beta^-e^{ik(2a-t)}\right)+\right.$$

$$\left.\left.+\int\limits_{-t}^{t}K_{21}(t,s)e^{iks}ds+ik\int\limits_{-t}^{t}K_{22}(t,s)e^{iks}ds\right]\right\}dt,$$

$$b(x)\left(\beta^+e^{ikx}+\beta^-e^{ik(2a-x)}\right)+\int\limits_{-x}^{x}K_{21}(x,t)e^{ikt}dt+ik\int\limits_{-x}^{x}K_{22}(x,t)e^{ikt}dt=$$

$$=-k\int\limits_{0}^{a}u(t)\left[\beta^+\sin k(x-t)-\beta^-\sin k(x+t-2a)\right]\left[e^{ikt}+\int\limits_{-t}^{t}K_{11}(t,s)e^{iks}ds\right]dt-$$

$$-\int\limits_{0}^{a}\left[\beta^+\cos k(x-t)+\beta^-\cos k(x+t-2a)\right]\left\{u^2(t)\left[e^{ikt}+\int\limits_{-t}^{t}K_{11}(t,s)e^{iks}ds\right]+\right.$$

$$\left.+u(t)\left[ike^{ikt}+b(t)e^{ikt}+\int\limits_{-t}^{t}K_{21}(t,s)e^{iks}ds+ik\int\limits_{-t}^{t}K_{22}(t,s)e^{iks}ds\right]\right\}dt-$$

$$-k\int\limits_{a}^{x}u(t)\sin k(x-t)\left[\beta^+e^{ikt}+\beta^-e^{ik(2a-t)}+\int\limits_{-t}^{t}K_{11}(t,s)e^{iks}ds\right]dt-$$

$$-\int\limits_{a}^{x}\cos k(x-t)\left\{u^2(t)\left[\beta^+e^{ikt}+\beta^-e^{ik(2a-t)}+\int\limits_{-t}^{t}K_{11}(t,s)e^{iks}ds\right]+\right.$$

$$+u(t)ik\left(\beta^+e^{ikt}-\beta^-e^{ik(2a-t)}\right)+b(t)\left(\beta^+e^{ikt}+\beta^-e^{ik(2a-t)}\right)+$$

$$\left.\left.+\int\limits_{-t}^{t}K_{21}(t,s)e^{iks}ds+ik\int\limits_{-t}^{t}K_{22}(t,s)e^{iks}ds\right]\right\}dt$$

must be satisfied.

We get the following integral equations from the last equality:

1) for $a<x<2a,\ -x<t<2a-x,$

$$K_{11}(x,t)=\frac{\beta^{+}}{2}u\left(\frac{x+t}{2}\right)+\frac{\beta^{-}}{2}u\left(a-\frac{x-t}{2}\right)-\frac{\beta^{+}}{2}\int\limits_{0}^{(x+t)/2}\left\{u^{2}(s)+u(s)b(s)\right\}ds+$$

$$+\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}\left\{u^{2}(s)+u(s)b(s)\right\}ds+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{11}(s,t+s-x)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{11}(s,t-x+s)ds+\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t-x+s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t+s-x)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t-x+s)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{(x-t)/2}^{x}u(s)K_{22}(s,t-x+s)ds\right\}+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{22}(s,t+x-s)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{22}(s,t+x-s)ds\right\}+\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-s-x+2a)ds+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u^{2}(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds+\int\limits_{a}^{x}u^{2}(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds+\int\limits_{a}^{x}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds\right\}+$$

$$+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u^{2}(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{11}(t,\xi)d\xi ds+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{21}(t,\xi)d\xi ds+$$

$$+\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-x-s+2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{22}(s,t+x+s-2a)ds+$$

$$+\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{11}(s,t+s+x-2a)ds,$$

$$K_{21}(x,t)=-\frac{\beta^+}{4}\left[u^2\left(\frac{x+t}{2}\right)+u\left(\frac{x+t}{2}\right)b\left(\frac{x+t}{2}\right)\right]-$$

$$-\frac{\beta^-}{4}\left[u^2\left(a-\frac{x-t}{2}\right)+u\left(a-\frac{x-t}{2}\right)b\left(a-\frac{x-t}{2}\right)\right]-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u^2(s)K_{11}(s,t+s-x)ds+\int\limits_{(x-t)/2}^{x}u^2(s)K_{11}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u^2(s)K_{11}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u^2(s)K_{11}(s,t-s+x)ds+\int\limits_{a}^{x}u^2(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{21}(s,t+s-x)ds+\int\limits_{(x-t)/2}^{x}u(s)K_{21}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u(s)K_{21}(s,t-s+x)ds+\int\limits_{a}^{x}u(s)K_{21}(s,t+x-s)ds\right\}-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u^2(s)K_{11}(s,t+s+x-2a)ds-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u^2(s)K_{11}(s,t-s-x+2a)ds-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{21}(s,t+s+x-2a)ds-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{21}(s,t-s-x+2a)ds,$$

$$K_{22}(x,t)=-\frac{\beta^+}{2}u\left(\frac{x+t}{2}\right)-\frac{\beta^-}{2}u\left(a-\frac{x-t}{2}\right)+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{11}(s,t-x+s)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t-x+s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t+x-s)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t+s-x)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{(x-t)/2}^{x}u(s)K_{22}(s,t-x+s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{22}(s,t-s+x)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t+x-s)ds\right\}-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-s-x+2a)ds+\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{11}(s,t+s+x-2a)ds-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-s-x+2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{22}(s,t+s+x-2a)ds;$$

2) for $x>2a,\ \ -x<t<2a-x,$

$$K_{11}(x,t)=\frac{\beta^{+}}{2}u\left(\frac{x+t}{2}\right)-\frac{\beta^{+}}{2}\int\limits_{0}^{(x+t)/2}\left\{u^{2}(s)+u(s)b(s)\right\}ds+$$

$$+\frac{1}{2}\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t-x+s)ds-\frac{1}{2}\int\limits_{(x-t)/2}^{x}u(s)K_{22}(s,t-x+s)ds+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t-s+x)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t+x-s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{22}(s,t+x-s)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t+x-s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u^{2}(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds+\int\limits_{a}^{x}u^{2}(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds+\int\limits_{a}^{x}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds\right\}+$$

$$+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u^{2}(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{11}(t,\xi)d\xi ds+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{21}(t,\xi)d\xi ds-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a} u(s)K_{22}(s,t+x+s-2a)ds-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a} u(s)K_{11}(s,t+x+s-2a)ds,$$

$$K_{21}(x,t)=-\frac{\beta^+}{4}\left[u^2\left(\frac{x+t}{2}\right)+u\left(\frac{x+t}{2}\right)b\left(\frac{x+t}{2}\right)\right]-$$

$$-\frac{1}{2}\int\limits_{(x-t)/2}^{x} u^2(s)K_{11}(s,t-x+s)ds-\frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{21}(s,t-x+s)ds-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a} u^2(s)K_{11}(s,t-s+x)ds+\int\limits_{a}^{x} u^2(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\left\{\beta^+\int\limits_{(x+t)/2}^{a} u(s)K_{21}(s,t-s+x)ds+\int\limits_{a}^{x} u(s)K_{21}(s,t+x-s)ds\right\}-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a} u^2(s)K_{11}(s,t+s+x-2a)ds-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a} u(s)K_{21}(s,t+s+x-2a)ds,$$

$$K_{22}(x,t)=-\frac{\beta^+}{2}u\left(\frac{x+t}{2}\right)+\frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{11}(s,t-x+s)ds+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a} u(s)K_{11}(s,t+x-s)ds+\int\limits_{a}^{x} u(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{22}(s,t-x+s)ds-\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a} u(s)K_{22}(s,t-s+x)ds+\right.$$

$$\left.+\int\limits_{a}^{x} u(s)K_{22}(s,t+x-s)ds\right\}+\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a} u(s)K_{11}(s,t+s+x-2a)ds-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a} u(s)K_{22}(s,t+s+x-2a)ds,$$

3) for $a<x<2a,\ \ x-2a<t<2a-x,$

$$K_{11}(x,t)=\frac{\beta^+}{2}u\left(\frac{x+t}{2}\right)+\frac{\beta^-}{2}u\left(a-\frac{x-t}{2}\right)-$$

$$-\frac{\beta^+}{2}\int\limits_{0}^{(x+t)/2}\left\{u^2(s)+u(s)b(s)\right\}ds-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}\left\{u^2(s)+u(s)b(s)\right\}ds+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{11}(s,t+s-x)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t-x+s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t-s+x)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t-x+s)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t-x+s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u(s)K_{22}(s,t+x-s)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t+x-s)ds\right\}+$$

$$+\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-x-s+2a)ds+\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{11}(s,t+x+s-2a)ds+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{0}^{a}u^2(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds+\int\limits_{a}^{x}u^2(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{0}^{a}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds+\int\limits_{a}^{x}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds\right\}+$$

$$+\frac{\beta^-}{2}\int\limits_{0}^{a}u^2(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{11}(t,\xi)d\xi ds+\frac{\beta^-}{2}\int\limits_{0}^{a}u(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{21}(t,\xi)d\xi ds-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-x-s+2a)ds-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{22}(s,t+x+s-2a)ds,$$

$$K_{21}(x,t)=-\frac{\beta^+}{4}\left[u^2(\frac{x+t}{2})+u(\frac{x+t}{2})b(\frac{x+t}{2})\right]-$$

$$-\frac{\beta^-}{4}\left[u^2(a-\frac{x-t}{2})+u(a-\frac{x-t}{2})b(a-\frac{x-t}{2})\right]-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u^2(s)K_{11}(s,t+s-x)ds+\int\limits_{a}^{x}u^2(s)K_{11}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u^2(s)K_{11}(s,t-s+x)ds+\int\limits_{a}^{x}u^2(s)K_{11}(s,t+s-x)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{21}(s,t+s-x)ds+\int\limits_{a}^{x}u(s)K_{21}(s,t+x-s)ds\right\}-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u^{2}(s)K_{11}(s,t+x+s-2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u^{2}(s)K_{11}(s,t-x-s+2a)ds-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{21}(s,t+x+s-2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{21}(s,t-x-s+2a)ds,$$

$$K_{22}(x,t)=-\frac{\beta^{+}}{2}u(\frac{x+t}{2})-\frac{\beta^{-}}{2}u(a-\frac{x-t}{2})+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{11}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{11}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t+x-s)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t+s-x)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{22}(s,t-s+x)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t+x-s)ds\right\}-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-x-s+2a)ds+\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{11}(s,t+x+s-2a)ds-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-x-s+2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{22}(s,t+x+s-2a)ds$$

4) for $x>2a$, $2a-x<t<x-2a$,

$$K_{11}(x,t)=\frac{\beta^{+}}{2}u(\frac{x+t}{2})+\frac{\beta^{-}}{2}u(a+\frac{x-t}{2})-\frac{\beta^{+}}{2}\int\limits_{0}^{(x+t)/2}\left\{u^{2}(s)+u(s)b(s)\right\}ds-$$

$$-\frac{\beta^-}{2}\int\limits_a^{a+(x-t)/2}\left\{u^2(s)+u(s)b(s)\right\}ds+\frac{1}{2}\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t-x+s)ds+$$

$$+\frac{1}{2}\int\limits_{(x+t)/2}^{x}u(s)K_{11}(s,t+x-s)ds-\frac{1}{2}\int\limits_{(x-t)/2}^{x}u(s)K_{22}(s,t-x+s)ds+$$

$$+\frac{1}{2}\int\limits_{(x+t)/2}^{x}u(s)K_{22}(s,t+x-s)ds+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_0^a u^2(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds+\int\limits_a^x u^2(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_0^a u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds+\int\limits_a^x u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds\right\}+$$

$$+\frac{\beta^-}{2}\int\limits_0^a u^2(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{11}(t,\xi)d\xi ds+\frac{\beta^-}{2}\int\limits_0^a u(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{21}(t,\xi)d\xi ds+$$

$$+\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-x-s+2a)ds-\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{22}(s,t+x+s-2a)ds,$$

$$K_{21}(x,t)=-\frac{\beta^+}{4}\left[u^2(\frac{x+t}{2})+u(\frac{x+t}{2})b(\frac{x+t}{2})\right]-$$

$$-\frac{\beta^-}{4}\left[u^2(a+\frac{x+t}{2})+u(a+\frac{x+t}{2})b(a+\frac{x+t}{2})\right]-$$

$$-\frac{1}{2}\int\limits_{(x-t)/2}^{x}u^2(s)K_{11}(s,t-x+s)ds-\frac{1}{2}\int\limits_{(x+t)/2}^{a}u^2(s)K_{11}(s,t+x-s)ds-$$

$$-\frac{1}{2}\int\limits_{(x-t)/2}^{x}u(s)K_{21}(s,t-x+s)ds-\frac{1}{2}\int\limits_{(x+t)/2}^{a}u(s)K_{21}(s,t+x-s)ds,$$

$$K_{22}(x,t)=-\frac{\beta^+}{2}u(\frac{x+t}{2})+\frac{1}{2}\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t-x+s)ds-$$

$$-\frac{1}{2}\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t+x-s)ds-\frac{1}{2}\int\limits_{(x-t)/2}^{x}u(s)K_{22}(s,t-x+s)ds-$$

$$-\frac{1}{2}\int\limits_{(x+t)/2}^{x} u(s)K_{22}(s,t+x-s)ds;$$

5) for $x>2a$, $2a-x<t<x$

$$K_{11}(x,t)=\frac{\beta^{+}}{2}u(\frac{x+t}{2})+\frac{\beta^{-}}{2}u(a-\frac{x-t}{2})+\frac{\beta^{-}}{2}u(a+\frac{x-t}{2})-$$

$$-\frac{\beta^{+}}{2}\int\limits_{a}^{(x+t)/2}\left\{u^{2}(s)+u(s)b(s)\right\}ds+\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}\left\{u^{2}(s)+u(s)b(s)\right\}ds-$$

$$-\frac{\beta^{+}}{2}\int\limits_{a}^{a+(x-t)/2}\left\{u^{2}(s)+u(s)b(s)\right\}ds+\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{11}(s,t+s-x)ds+\right.$$

$$+\int\limits_{a}^{x}u(s)K_{11}(s,t-x+s)ds+\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t-x+s)ds+$$

$$+\frac{1}{2}\int\limits_{(x+t)/2}^{x}u(s)K_{11}(s,t+x-s)ds-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{22}(s,t-x+s)ds+\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t-x+s)ds\right\}+$$

$$+\frac{1}{2}\int\limits_{(x+t)/2}^{x}u(s)K_{22}(s,t+x-s)ds+\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-s-x+2a)ds+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u^{2}(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds+\int\limits_{a}^{x}u^{2}(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds+\int\limits_{a}^{x}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds\right\}+$$

$$+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u^{2}(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{11}(t,\xi)d\xi ds+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{21}(t,\xi)d\xi ds+$$

$$+\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-x-s+2a)ds$$

$$K_{21}(x,t)=-\frac{\beta^{+}}{4}\left[u^{2}(\frac{x+t}{2})+u(\frac{x+t}{2})b(\frac{x+t}{2})\right]-$$

$$-\frac{\beta^-}{4}\left[u^2(a-\frac{x-t}{2})+u(a-\frac{x-t}{2})b(a-\frac{x-t}{2})\right]-$$

$$-\frac{\beta^-}{4}\left[u^2(a+\frac{x-t}{2})+u(a+\frac{x-t}{2})b(a+\frac{x-t}{2})\right]-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{x}u^2(s)K_{11}(s,t+s-x)ds\int\limits_{(x-t)/2}^{x}u^2(s)K_{11}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u^2(s)K_{11}(s,t-x+s)ds\right\}-\frac{1}{2}\int\limits_{(x+t)/2}^{x}u^2(s)K_{11}(s,t+x-s)ds-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{21}(s,t+s-x)ds+\int\limits_{(x-t)/2}^{x}u(s)K_{21}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{21}(s,t-x+s)ds\right\}-\frac{1}{2}\int\limits_{(x+t)/2}^{x}u(s)K_{21}(s,t+x-s)ds-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}u^2(s)K_{11}(s,t-x-s+2a)ds-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{21}(s,t-x-s+2a)ds,$$

$$K_{22}(x,t)=-\frac{\beta^+}{2}u(\frac{x+t}{2})-\frac{\beta^-}{2}u(a-\frac{x-t}{2})+\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{11}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{11}(s,t-x+s)ds+\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t+x-s)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t+x-s)ds\right\}-$$

$$-\frac{1}{2}\int\limits_{(x+t)/2}^{x}u(s)K_{11}(s,t+x-s)ds-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t+s-x)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{22}(s,t-x+s)ds+\int\limits_{(x-t)/2}^{x}u(s)K_{22}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\int\limits_{(x+t)/2}^{x}u(s)K_{22}(s,t+x-s)ds-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-x-s+2a)ds-$$

$$-\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a} u(s)K_{22}(s,t-x-s+2a)ds$$

6) for $a<x<2a$, $x-2a<t<x$,

$$K_{11}(x,t)=\frac{\beta^+}{2}u(\frac{x+t}{2})+\frac{\beta^-}{2}u(a-\frac{x-t}{2})+\frac{\beta^-}{2}u(a+\frac{x-t}{2})+$$

$$+\int\limits_a^x u(s)K_{22}(s,t+x-s)ds-\frac{\beta^+}{2}\int\limits_a^{(x+t)/2}\left\{u^2(s)+u(s)b(s)\right\}ds-$$

$$-\frac{\beta^+}{2}\int\limits_a^{(x+t)/2}\left\{u^2(s)+u(s)b(s)\right\}ds+\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}\left\{u^2(s)+u(s)b(s)\right\}ds-$$

$$-\frac{\beta^-}{2}\int\limits_a^{a+(x-t)/2}\left\{u^2(s)+u(s)b(s)\right\}ds+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{11}(s,t+s-x)ds+\int\limits_a^x u(s)K_{11}(s,t-x+s)ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u(s)K_{11}(s,t-s+x)ds+\int\limits_a^x u(s)K_{11}(s,t+s-x)ds+\right.$$

$$\left.+\int\limits_{(x+t)/2}^{x}u(s)K_{11}(s,t+x-s)ds\right\}-\frac{1}{2}\left\{\beta^+\int\limits_{(x-t)/2}^{a}u(s)K_{22}(s,t-x+s)ds+\right.$$

$$\left.+\int\limits_a^x u(s)K_{22}(s,t-x+s)ds\right\}+\frac{1}{2}\left\{\beta^+\int\limits_{(x+t)/2}^{a}u(s)K_{22}(s,t+x-s)ds+\right.$$

$$\left.+\int\limits_a^x u(s)K_{22}(s,t+x-s)ds+\int\limits_{(x+t)/2}^{x}u(s)K_{11}(s,t+x-s)ds\right\}+$$

$$+\frac{\beta^-}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-s-x+2a)ds+\frac{\beta^-}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{11}(s,t+s+x-2a)ds+$$

$$+\frac{1}{2}\left\{\beta^+\int\limits_0^a u^2(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds+\int\limits_a^x u^2(s)\int\limits_{t-x+s}^{t+x-s}K_{11}(t,\xi)d\xi ds\right\}+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds+\int\limits_{a}^{x}u(s)\int\limits_{t-x+s}^{t+x-s}K_{21}(t,\xi)d\xi ds\right\}+$$

$$+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u^{2}(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{11}(t,\xi)d\xi ds+\frac{\beta^{-}}{2}\int\limits_{0}^{a}u(s)\int\limits_{t-s-x+2a}^{t+s+x-2a}K_{21}(t,\xi)d\xi ds+$$

$$+\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-x-s+2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{22}(s,t+x+s-2a)ds,$$

$$K_{21}(x,t)=-\frac{\beta^{+}}{4}\left[u^{2}(\frac{x+t}{2})+u(\frac{x+t}{2})b(\frac{x+t}{2})\right]-$$

$$-\frac{\beta^{-}}{4}\left[u^{2}(a-\frac{x-t}{2})+u(a-\frac{x-t}{2})b(a-\frac{x-t}{2})\right]-$$

$$-\frac{\beta^{-}}{4}\left[u^{2}(a+\frac{x-t}{2})+u(a+\frac{x-t}{2})b(a+\frac{x-t}{2})\right]-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u^{2}(s)K_{11}(s,t+s-x)ds+\int\limits_{a}^{x}u^{2}(s)K_{11}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u^{2}(s)K_{11}(s,t-s+x)ds+\int\limits_{a}^{x}u^{2}(s)K_{11}(s,t+x-s)ds+\right.$$

$$\left.+\int\limits_{(x+t)/2}^{x}u^{2}(s)K_{11}(s,t+x-s)ds\right\}-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x-t)/2}^{a}u(s)K_{21}(s,t+s-x)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{21}(s,t-x+s)ds\right\}-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{21}(s,t-s+x)ds+\right.$$

$$\left.+\int\limits_{a}^{x}u(s)K_{21}(s,t+x-s)ds+\int\limits_{(x+t)/2}^{x}u(s)K_{21}(s,t+x-s)ds\right\}-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u^{2}(s)K_{11}(s,t+x+s-2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u^{2}(s)K_{11}(s,t-x-s+2a)ds-$$

$$-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{21}(s,t+x+s-2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{21}(s,t-x-s+2a)ds,$$

$$K_{22}(x,t)=-\frac{\beta^{+}}{2}u(\frac{x+t}{2})-\frac{\beta-}{2}u(a-\frac{x-t}{2})+$$

$$+\frac{1}{2}\left\{\beta^{+}\int\limits_{(+-t)/2}^{a}u(s)K_{11}(s,t-x+s)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(++t)/2}^{a}u(s)K_{11}(s,t+x-s)ds+\int\limits_{a}^{x}u(s)K_{11}(s,t+x-s)ds+\right.$$

$$\left.+\int\limits_{(x-t)/2}^{x}u(s)K_{11}(s,t+x-s)ds\right\}-\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{22}(s,t+x+s-2a)ds-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(+-t)/2}^{a}u(s)K_{22}(s,t+s-x)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t-x+s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{(x+t)/2}^{a}u(s)K_{22}(s,t-s+x)ds+\int\limits_{a}^{x}u(s)K_{22}(s,t+x-s)ds+\right.$$

$$\left.+\int\limits_{(x+t)/2}^{x}u(s)K_{22}(s,t+x-s)ds\right\}-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{11}(s,t-x-s+2a)ds+$$

$$+\frac{\beta^{-}}{2}\int\limits_{a-(x+t)/2}^{a}u(s)K_{21}(s,t+x+s-2a)ds-\frac{\beta^{-}}{2}\int\limits_{a-(x-t)/2}^{a}u(s)K_{22}(s,t-x-s+2a)ds.$$

It is shown by the successive approximations method that (see [12]) the following theorem is true.

Theorem 1. *For every solutionsof theproblem $L=L(h,H,\beta)$ which satisfying the initial conditions $y_1(0,\lambda)=1$, $y_2(0,\lambda)=ik$ and the jump conditions (3') has the form (9), (9') and also $\int\limits_{-x}^{x}|K_{ij}(x,t)|\,dt\le e^{c\sigma(x)}-1$, where*

$$\sigma(x)=\frac{1}{2}\int\limits_{0}^{x}\left(|u(t)|+\left|u^{2}(t)\right|\right)dt,\ c=\beta^{+}+\left|\beta^{-}\right|+1,$$

$$b(x)=-\frac{1}{2}\int\limits_{0}^{x}u^{2}(s)e^{-1/2\int\limits_{s}^{x}u(\xi)d\xi}ds,\quad K(x,t)=\begin{pmatrix}K_{11}(x,t) & 0\\ K_{21}(x,t) & K_{22}(x,t)\end{pmatrix},$$

$$K_{11}(x,x)=-\frac{1}{2}\beta^{+}u(x)-\beta^{+}b(x)-\beta^{+}b(a)(\beta^{+}-1),$$

$$K_{22}(x,x)=-\frac{1}{2}\beta^{+}u(x)-2\beta^{+}b(x)+\beta^{+}b(a)(\beta^{+}-1),$$

$$K_{21}(x,x)=-\frac{1}{2}\beta^{+}b'(x)-\frac{1}{2}\beta^{-}b'(a)-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u^{2}(s)K_{11}(s,s)ds+\int\limits_{0}^{x}u^{2}(s)K_{11}(s,s)ds+\int\limits_{a}^{x}u^{2}(s)K_{11}(s,s)ds\right\}-$$

$$-\frac{1}{2}\left\{\beta^{+}\int\limits_{0}^{a}u(s)K_{21}(s,s)ds+\int\limits_{0}^{x}u(s)K_{21}(s,s)ds\right\}.$$

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