MECHANICS

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STRESS STATE OF NONLINEARLY VISCOELASTIC ANNULAR DISK OF VARIABLE THICKNESS AT ROTATION

Abstract

Stress, strain and permutation components in a rotating annular disk made of physically nonlinear viscous elastic material are determined by the sequential approximations method similar to the method of variable elasticity parameters.

The thickness of the disk is considered to be variable. Viscous elastic properties of the disk's material are describef by V.V.Moskvitin's nonlinear relations.

Let a circular disk of radius b or annular disk of internal radius a and external b rotate with angular speed $\omega(t)$, where t is time. We assume that the disk's material is incompressible and its mechanical properties are described by the V.V. Moskvitin's known viscoelastic relations [1] that are represented in the from of Volterra nonlinear integral equation of the second kind. In the case when the properties of disk's material are subjected to the stationary creeping law a similar problem was considered in [2]. Assume that the thickness h is variable: $h = h(r)$ where is r current radius of a disk. Contours of the disk are free from any efforts. Distribution of stresses along the thickness of the disk is assumed to be uniform, i.e. plane stress state condition is fulfilled.

Using a cylindrical system of coordinates $(r\varphi z)$, where $z = 0$ is a mean plane of the disk we adopt that $\sigma_{zz} = \sigma_z$ and σ_{rz} in planes parallel to the plane $z = 0$ are small in comparison with stress components $\sigma_{rr} = \sigma_r$, $\sigma_{\varphi\varphi} = \sigma_\varphi$ in planes normal to it. Moreover, because of axial symmetry $\sigma_{\varphi z} = \sigma_{\rho r} = 0$. From the components of deformation ε_{rz} is alov small in comparison with components $\varepsilon_{\varphi\varphi} = \varepsilon_\varphi$, $\varepsilon_{rr} = \varepsilon_r$. Moreover, $\varepsilon_{r\varphi} = 0$. Consequently, the stress components σ_φ, σ_r, deformation components ε_φ, ε_r, $\varepsilon_{zz} = \varepsilon_z$ differ from zero. As the disk's material is mechanically incompressible, the quantity ε_z is expressed by the strain ε_φ, ε_r.

Follwring [1], we represent the defining equations of the theory of physically nonlinear viscoelasticity in the form of

$$2G_0 e_{ij} = f(\sigma_+)\, s_{ij} + \int_0^t \Gamma(t-\tau)\, f(\sigma_+)\, s_{ij}\, d\tau; \qquad \theta = 0, \tag{1}$$

where $i = 1,2,3$. Besides, G_0 is an instant elasticity modulus; $e_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}$ is deviator of deformations ε_{ij}; $\varepsilon = \varepsilon_{ij}\delta_{ij}/3$ is a mean strain, δ_{ij} are Kronecker symbols; $\theta = 3\varepsilon$ is relative change of volume; $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is deviator of stresses σ_{ij}; $\sigma = \sigma_{ij}\delta_{ij}/3$ is mean stress; $\sigma_+ = \left(\frac{3}{2} s_{ij} s_{ij}\right)^{1/2}$ is stress intensity; $\Gamma(t)$ is a creeping function; $f(\sigma_+)$ is a function characterizing physical noulinearity of deformation.

In the considered case $\varepsilon = \theta/3 = (\varepsilon_r + \varepsilon_\varphi + \varepsilon_z)/3 = 0$; $\sigma = (\sigma_r + \sigma_\varphi)/3$; $e_r = \varepsilon_r$; $e_\varphi = \varepsilon_\varphi$, $e_z = \varepsilon_z$, $e_{rz} = 0$; $e_{\varphi z} = 0$; $e_{r\varphi} = 0$; $s_r = (2\sigma_r - \sigma_\varphi)/3$; $s_\varphi = (2\sigma_\varphi - \sigma_2)/3$; $s_z = -(\sigma_r + \sigma_\varphi)/3$; $s_{rz} = 0$; $s_{\varphi z} = 0$; $s_{r\varphi} = 0$. Allowing for these relations, only the two relations

$$2G_0(\varepsilon_\varphi - \varepsilon_r) = f(\sigma_+)(\sigma_\varphi - \sigma_r) + \int_0^t \Gamma(t-\tau) f(\sigma_+)(\sigma_\varphi - \sigma_r)\, d\tau, \tag{2}$$

$$6G_0(\varepsilon_\varphi + \varepsilon_r) = f(\sigma_+)(\sigma_\varphi + \sigma_r) + \int_0^t \Gamma(t-\tau) f(\sigma_+)(\sigma_\varphi + \sigma_r)\, d\tau. \tag{3}$$

remain from (1).

Stress intensity σ_+ in the case of plane stress state is of the form:

$$\sigma_+ = \left(\sigma_r^2 - \sigma_r\sigma_\varphi + \sigma_\varphi^2\right)^{1/2}. \tag{4}$$

Equilibrium equation of a rotating disk of variable thickness under conditions of plane stress state (by fulfilling the condition $|dh/dr| << 1$) is represented in the form [3]

$$\frac{\partial}{\partial r}(h\sigma_r) + h\frac{\sigma_r - \sigma_\varphi}{r} + h\rho\omega^2(t)\, r = 0. \tag{5}$$

here ρ - is the density of disk's viscoelastic material.

To equations (2), (3) and (5) we add deformation compatability equation

$$\frac{\partial \varepsilon_\varphi}{\partial r} + \frac{\varepsilon_\varphi - \varepsilon_r}{r} = 0. \tag{6}$$

Boundary conditions are expressed by the relations

$$\sigma_r|_{r=a} = 0; \qquad \sigma_r|_{r=b} = 0. \tag{7}$$

The problem on deformation of σ_r, σ_φ, ε_r, ε_φ is the problem (2), (3), (5)-(7). Moreover, notice that deformation ε_z and permutation u are connected with deformation components ε_r and ε_φ by formulae:

$$\varepsilon_z = -(\varepsilon_r + \varepsilon_\varphi); \qquad \varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\varphi = \frac{u}{r}. \tag{8}$$

Let's solve problem (2), (3), (5)-(7). Preliminarily, following [4] represent its solution in the form of

$$\sigma_\varphi = \sigma'_\varphi;\ \sigma_r = \sigma'_r;\ \varepsilon_\varphi = \varepsilon'_\varphi + \int_0^t \Gamma(t-\tau)\,\varepsilon'_\varphi d\tau;\ \ \varepsilon_r = \varepsilon'_r + \int_0^t \Gamma(t-\tau)\,\varepsilon'_r d\tau. \tag{9}$$

By immediate substitution we can see that formulae (9) reduce problem (2), (3), (5), (6), (7) to the problem

$$2G_0\left(\varepsilon'_\varphi - \varepsilon'_r\right) = f\left(\sigma'_+\right)\left(\sigma'_\varphi - \sigma'_r\right), \tag{10}$$

$$6G_0\left(\varepsilon'_\varphi+\varepsilon'_r\right)=f\left(\sigma'_+\right)\left(\sigma'_\varphi+\sigma'_r\right), \tag{11}$$

$$\frac{\partial}{\partial r}\left(h\sigma'_r\right)+h\frac{\sigma'_r-\sigma'_\varphi}{r}+h\rho\omega^2\left(t\right)r=0 \tag{12}$$

$$\frac{\partial\varepsilon'_\varphi}{\partial r}+\frac{\varepsilon'_\varphi-\varepsilon'_r}{r}=0 \tag{13}$$

$$\left.\sigma'_r\right|_{r=a}=0; \qquad \left.\sigma'_r\right|_{r=b}=0 \tag{14}$$

The intensity σ'_+ contained in relations (10) and (11) is expressed as follows

$$\sigma'_+=\left(\sigma'^2_r-\sigma'_r\sigma'_\varphi+\sigma'^2_\varphi\right)^{1/2}. \tag{15}$$

according to formula (4)

As we see the problem on determination of $\sigma'_\varphi,\ \sigma'_r,\ \varepsilon'_\varphi, \varepsilon'_r$ is a problem of nonlinear elasticity or the theory of small elastic plastic deformations of A.A.Ilyushin at monotone loading.

Accept that the nonlinearity function is $f\left(\sigma_+\right)$ represented by a power function: $f\left(\sigma_+\right)=A\sigma_+^\alpha$, where A and α are material constants that for each viscoelastic material are determined from corresponding experiments [1]. Here $A>0,\ \alpha\geq 0$. When $A=1, \alpha=0$ the material is a physically linear viscoelastic material.

By η denote a ratio of σ_r per $\sigma_\varphi:\eta=\sigma'_r/\sigma'_\varphi$. With (15) we have: $f\left(\sigma_+\right)=A\sigma_+^\alpha=A\left(\sigma'_\varphi\right)^\alpha\left(\eta^2-\eta+1\right)^{\alpha/2}$. Use this expression of the function $f\left(\sigma_+\right)$ and from the relations (10) and (11) we determine the quantities $\varepsilon'_\varphi-\varepsilon'_r$ and ε'_φ:

$$\varepsilon'_\varphi-\varepsilon'_r=\frac{A}{2G_0}\left(\sigma'_\varphi\right)\left(1-\eta\right)\left(\eta^2-\eta+1\right)^{\alpha/2}, \tag{16}$$

$$\varepsilon'_\varphi=\frac{A}{6G_0}\left(\sigma'_\varphi\right)\left(2-\eta\right)\left(\eta^2-\eta+1\right)^{\alpha/2}. \tag{17}$$

Allow for (16) and (17) in equation (13) and perform integration and determine the quantity σ'_φ

$$\sigma'_\varphi=c_1\left(t\right)\varphi\left(\eta\right) \tag{18}$$

Here $c_1\left(t\right)$ is an integration function, the function $\varphi\left(\eta\right)$ is represented by the formula

$$\varphi\left(\eta\right)=\left|2-\eta\right|^{-\frac{1}{1+\alpha}}\left(\eta^2-\eta+1\right)^{-\frac{\alpha}{2(1+\alpha)}}\exp\left(-\frac{3}{1+\alpha}\int\limits_a^r\frac{1-\eta}{2-\eta}\frac{dr}{r}\right). \tag{19}$$

Now let's integrate differential equation (12)

$$\sigma'_r=\frac{1}{rh\left(r\right)}\left[\int\limits_a^r\sigma_\varphi\left(r\right)h\left(r\right)dr-\rho\omega^2\left(t\right)\int\limits_a^r h\left(r\right)r^2dr\right]+\frac{c_2\left(t\right)}{rh}. \tag{20}$$

Here $c_2\left(t\right)$ is an integration function.

Satisfy the first boundary condition of (14). We get $c_2(t) = 0$. Notice that if the disk is entire we also have $c_2(t)$, since at the center $(r = 0)$ of the disk the stress $\sigma_r = \sigma_r'$ must be finite quantity. And we write (20) in the form:

$$\sigma_r' = \frac{1}{rh(r)}\left[\int_a^r \sigma_\varphi(r)\,h(r)\,dr - \rho\omega^2(t)\int_a^r h(r)\,r^2dr\right]. \qquad (21)$$

Now satisfy the second boundary condition of (14):

$$\int_a^b \sigma_\varphi(r)\,h(r)\,dr = \rho\omega^2(t)\int_a^b h(r)\,r^2dr.$$

In this relation allow for (18) and determine the integration function $c_1(t)$:

$$c_1(t) = \frac{\rho\omega^2(t)\int_a^r h(r)\,r^2dr}{\int_a^b \varphi(\eta(r))\,h(r)\,dr}.$$

By using the last relation in (18), the expression for σ_φ' is of the form

$$\sigma_\varphi' = \frac{\varphi(\eta)\,\rho\omega^2(t)\int_a^r h(r)\,r^2dr}{\int_a^b \varphi(\eta(r))\,h(r)\,dr}. \qquad (22)$$

Using denotation $\sigma_r'/\sigma_\varphi' = \eta$, we write expression for σ_r' as well:

$$\sigma_r' = \frac{\eta\varphi(\eta)\,\rho\omega^2(t)\int_a^r h(r)\,r^2dr}{\int_a^b \varphi(\eta(r))\,h(r)\,dr}. \qquad (23)$$

Relations (22) and (23) together with formula (19) are the equations to determine σ_φ' and σ_r'. Notice that if the disk is entire $(a = 0)$, then at the center of the disk $(r = 0)$ there should be $\sigma_\varphi'\big|_{r=0} = \sigma_r'\big|_{r=0}$. This means that for $r = 0$ we have $\eta = 1$. In this case the integrand function in formula (19) turns into indeterminacy. We can show that $\lim\limits_{\substack{r\to 0\\ \eta\to 1}}\left(\frac{1-\eta}{2-\eta}\frac{1}{r}\right) = 0$. Allowing for this, in the center of the entire disk

$\varphi(\eta) = 1$. And it follows from relations (22) and (23) that stress σ_φ and σ_r at the center of the entire rotating disk is expressed by the formula:

$$\sigma_\varphi = \sigma_r = \sigma'_\varphi = \sigma'_r = \frac{\rho\omega^2(t)\int\limits_a^r h(r)\,r^2 dr}{\int\limits_0^b \varphi(\eta(r))\,h(r)\,dr}$$

Equations (22) and (23) may be solved by the following sequential approximations method that is the analogy of variable elastic parameters method [5]. For zero approximation we accept corresponding solution of the elastic problem at constant thickness of a disk [3]. By using these solutions we determine $\eta^{(0)} = \sigma_r'^{(0)}/\sigma_\varphi'^{(0)}$. For the known quantity $\eta^{(0)}$ we calculate $\varphi\left(\eta^{(0)}\right)$ by formula (19). And for the given function $h(r)$, from relation (22) we find $\sigma_\varphi'^{(1)}$, from (23) find - $\sigma_r'^{(1)}$. Then we conduct similar calculation procedure. We determine $\eta^{(1)} = \sigma_r'^{(1)}/\sigma_\varphi'^{(1)}, \varphi\left(\eta^{(1)}\right)$ that allow to find $\sigma'{}_\varphi^{(2)}$ from (22) and- $\sigma_r'^{(2)}$ from (23). Notice that the applied method is similar to the variable elasticity parameters method [5], whose proof of convergence is in [6]. It is noted that already at the second approximation it gives high accuracy degree.

As soon as σ'_φ and σ'_r, are determined, we find ε'_φ and ε'_r by relations (10) and (11). Finally by transient formulae (9) we find the desired stress components σ_φ and σ_r, deformations ε_φ and ε_r. Deformation ε_z found by the first formula of (8), permutation u -by the third formula of (8).

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