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ON DEPENDENCE OF EQUICONVERGENCE RATE ON THE MODULE OF CONTINUITY OF COEFFICIENT $P_2(x)$ OF A FOURTH ORDER DIFFERENTIAL OPERATOR

Abstract

In this paper differential operator of the fourth order

$$Lu = u^{(4)} + P_2(x)\, u^{(2)} + P_3(x)\, u^{(1)} + P_4(x)\, u$$

with summable complex-valued coefficients $P_l(x)$, $l = \overline{2,4}$ *on the interval* $G = (0,1)$ *is considered.*

Rate of uniform equiconvergence of biortogonal expansion of functions from $L_p(G)$, $p \geq 1$, *with their trigonometric series is investigated.*

Let's consider the formal differential operator

$$Lu = u^{(4)} + P_2(x)\, u^{(2)} + P_3(x)\, u^{(1)} + P_4(x)\, u \tag{1}$$

with summable coefficients $P_l(x)$, $l = \overline{2,4}$ on the interval $G = (0,1)$. Eigen and associated functions of the operator L are understood in the sense of the paper [1].

Let's denote by $D(G)$ a class of functions, absolutely continuous together with their derivatives to the third order inclusively on the closed interval $\overline{G}$. Let's consider arbitrary system $\{u_k(x)\}_{k=1}^{\infty}$, consisting of root functions of the operator L responding to the system of eigenvalues $\{\lambda_k\}$ and require that along with each root function of order $l \geq 1$ this system involve corresponding root functions of order less l and the rank of eigenfunctions be uniformly bounded. This means, that $u_k(x) \in D$ and satisfy the equation $Lu_k + \lambda_k u_k = \theta_k u_{k-1}$ almost everywhere in G, where θ_k equals either 0 (in this case $u_k(x)$ is an eigenfunction), or equals 1 (in this case we require $\lambda_k = \lambda_{k-1}$ and call $u_k(x)$ an associated function).

Let $\mu_k = (-\lambda_k)^{1/4}$, where $\left[re^{i\varphi}\right]^{1/4} = r^{1/4} e^{i\varphi/4}$, $-\pi/2 < \varphi < 3\pi/2$.

We'll require, that the system $\{u_k\}$ satisfy Il'in V.A. conditions and call them conditions A:

1) the system $\{u_k\}$ is closed and minimal in $L_p(G)$ for fixed $p \geq 1$;

2) Karleman and "unity sum" conditions are fulfilled:

$$|Jm\mu_k| \leq const,\ k = 1, 2, ... \tag{2}$$

$$\sum_{\tau \leq \rho_k \leq \tau+1} 1 \leq const,\ \forall\tau \geq 0,\ \rho_k = \operatorname{Re}\mu_k, \tag{3}$$

3) for any compact $K \subset G$ there exists a constant $C_0(K)$ such, that

$$\|u_k\|_{p,K} \|\upsilon_k\|_q \leq C_0(K), \tag{4}$$

$k = 1, 2, ...;\ q = p/(p-1),\ (p = 1, q = \infty);\ \{\upsilon_k\}$ is a biorthogonally adjoint system to $\{u_k\}$.

Let's construct partial sum of the spectral expansion

$$\sigma_\nu(x,f)=\sum_{\rho_k\le\nu}(f,\upsilon_k)\,u_k(x),\ \nu>0.$$

for the arbitrary function $f(x)\in L_p(G)$.

By $S_\nu(x,f)$ we'll denote the partial sum of trigonometric series of the function $f(x)$.

Let's introduce notation

$$\Delta_\nu(K,f)=\|\sigma_\nu(\cdot,f)-S_\nu(\cdot,f)\|_{C(K)};$$

$$f_k=(f,\upsilon_k),\ \hat{f}_k=f_k\,\|\upsilon_k\|_q^{-1};$$

$$\Omega(f,\nu/2,\alpha)=\nu^{-1}\sum_{1\le\rho_k\le\nu/2}\rho_k^{-\alpha}\left|\hat{f}_k\right|;$$

$$\Phi_p(f,\nu)=\nu^{-1}\|f\|_p+\max_{\rho_k\ge\nu/2}\left|\hat{f}_k\right|+\omega_1\left(f,\nu^{-1}\right);$$

$$D(\nu)=\inf_{\substack{\beta>1\\ m\ge2}}\left\{\omega_1\left(P_2,m^{-1}\right)\Omega(f,\nu/2,0)+\right.$$

$$\left.+\,m^{2\left(1-\beta^{-1}\right)}\|P_2\|_1\,\Omega\left(f,\nu/2,1-\beta^{-1}\right)\right\};$$

where $\alpha\ge0,\ \beta>1,\ \omega_1(\cdot,\delta)$ is an integral module of continuity.

In the present paper we prove the following theorem:

Theorem. *Let $P_2(x)\in L_r(G),\ r\ge1$ and the system $\{u_k\}$ satisfy the conditions A. Then the expansions of arbitrary function $f(x)\in L_p(G)$ in biortogonal series in the system $\{u_k\}$ and in trigonometric series uniformly equiconverge at any compact $K\subset G$, i.e.*

$$\Delta_\nu(K,f)\to0,\ \nu\to\infty \tag{5}$$

and $(\nu>2)$ are true

I. At $r>1$

$$\Delta_\nu(K,f)\le C(K)\left\{\Omega\left(f,\nu/2,1-r^{-1}\right)+\Phi_p(f,\nu)\right\}, \tag{6}$$

II. At $r=1$

$$\Delta_\nu(K,f)\le C(K)\{D(\nu)+\Phi_p(f,\nu)\}, \tag{7}$$

where $C(K)>0$ is independent of ν.

Proof. Note that estimations (5), (6) were established earlier in the paper [2]. We'll prove only estimation (7).

We fix the arbitrary segment $K=[a,b]\subset G$ and consider the function

$$W(t,\nu,R)=\begin{cases}\frac{\sin\nu t}{\pi t},\ t\le R\\ 0,\ t>R,\end{cases},$$

where $\nu>0,\ t=|x-y|,\ y\in G,\ R\in[R_0/2,\ R_0],\ R_0>0,\ dist(K,\partial G)>4C_0R_0$, where C_0 is a constant from the mean value formula of (50) of the paper [3].

Let's denote average value $\hat{W}(r,\nu,R_0) = S_{R_0}[W]$ by $\hat{W}(r,\nu,R_0)$, where $S_{R_0}[g] = 2/R_0 \int\limits_{R_0/2}^{R_0} g(R)\,dR$. Then Fourier coefficients of the function $\hat{W}$ by the system $\left\{\overline{u_k(y)}\right\}$ are calculated by the formula

$$\hat{W}_k = \hat{W}_k(x,\nu,R_0) = \frac{2}{\pi} S_{R_0}\left[\int\limits_0^R \frac{\sin\nu t}{t}\frac{u_k(x-t)+u_k(x+t)}{2}dt\right].$$

Let's consider difference $\sum\limits_{k=1}^{\infty}\hat{W}_k f_k - \sigma_\nu(x,f)$, $\quad f \in L_p(G)$.

In the paper [2] it is established, that

$$\sum_{k=1}^{\infty}\hat{W}_k f_k - \sigma_\nu(x,f) = \sum_{j=1}^{14}\gamma_j(x) \ ,$$

where for $\gamma_j(x)$ $j=\overline{1,7}$; $j=\overline{12,14}$ the estimation

$$\left\|\gamma_j\right\|_{C(K)} \le C(K)\left\{\nu^{-1}\left\|f\right\|_p + \max_{\rho_k\ge\nu/2}\left|\hat{f}_k\right|\right\} \tag{8}$$

is fulfilled, and for $\gamma_j(x)$ $j=\overline{8,11}$ the estimation

$$\left\|\gamma_j\right\|_{C(K)} \le C(K)\left\{\left\|P_2\right\|_r\Omega\left(f,\nu/2,1-r^{-1}\right) + \max_{\rho_k\ge\nu/2}\left|\hat{f}_k\right|\right\} \tag{9}$$

is fulfilled.

Let $r=1$. In this case estimate the sum $\gamma_j(x)$, $j=\overline{8,11}$ in another way. Let's introduce $\gamma_8(x)$ in the form

$$\gamma_8(x) = \sum_{\rho_k>\rho_0} f_k \sum_{j=0}^{m_k}\mu_k^{-3(j+1)} S_{R_0}\left[\int\limits_x^{x+R} P_2(\xi)\,u_{k-j}^{(2)}(\xi)\,J_{1j}(\xi-x,R,\mu_k,\nu)\,d\xi\right] +$$

$$+\sum_{\rho_k>\rho_0} f_k \sum_{j=0}^{m_k}\mu_k^{-3(j+1)} S_{R_0}\left[\int\limits_x^{x+R}\sum_{l=3}^{4} P_l(\xi)\,u_{k-j}^{(4-l)}(\xi)\,J_{1j}(\xi-x,R,\mu_k,\nu)\,d\xi\right] =$$

$$= \gamma_8^1(x) + \gamma_8^2(x).$$

For $\gamma_8^2(x)$ the estimate [2]

$$\left\|\gamma_8^2\right\|_{C(K)} \le C(K)\left\{\Omega(f,\nu/2,1) + \max_{\rho_k\ge\nu/2}\left|\hat{f}_k\right|\right\}$$

is true.

Let's introduce $\gamma_8^1(x)$ as $\gamma_8^1(x) = \gamma_8^1(x,P_2-Q_m) + \gamma_8^1(x,Q_m)$, where $Q_m(x)$ is an algebraic polynomial of optimal approximation of the function $P_2(x)$ in metrics $L_1(G)$ of m power.

Apply the estimation (9) at $r=1$ for sum $\gamma_8^1(x,P_2-Q_m)$. As a result we have

$$\max_{x\in K}\left|\gamma_8^1(x,P_2-Q_m)\right|\le$$
$$\le C(K)\left\{\|P_2-Q_m\|_1\,\Omega(f,\nu/2,0)+\max_{\rho_k\ge\nu/2}\left|\hat f_k\right|\right\}, \tag{10}$$

Since $Q_m(x)$ is a polynomial, it belongs to $L_\beta(G)$, $\beta>1$. Therefore, for $\gamma_8^1(x,Q_m)$ we can use the estimate (9). As a result we find, that

$$\max_{x\in K}\left|\gamma_8^1(x,Q_m)\right|\le C(K)\left\{\|Q_m\|_\beta\,\Omega\left(f,\nu/2,1-\beta^{-1}\right)+\max_{\rho_k\ge\nu/2}\left|\hat f_k\right|\right\}, \tag{11}$$

Having applied the known inequalities

$$\|P_2-Q_m\|_1\le const\,\omega_1\left(P_2,m^{-1}\right)\ \text{(see[4])},$$

$$\|Q_m\|_\beta\le C(\beta)\,m^{2\left(1-\beta^{-1}\right)}\|Q_m\|_1\ \text{(see[5])},$$

in the estimates (10) and (11) we'll get

$$\left\|\gamma_8^1\right\|_{C(K)}\le C(K)\left\{D(\nu)+\max_{\rho_k\ge\nu/2}\left|\hat f_k\right|\right\}.$$

Consequently,

$$\|\gamma_8\|_{C(K)}\le C(K)\left\{D(\nu)+\max_{\rho_k\ge\nu/2}\left|\hat f_k\right|\right\}. \tag{12}$$

The sums $\gamma_j(x)$, $j=\overline{9,11}$ are estimated in the same way and for them the estimate (12) is fulfilled. Consequently subject to (8) we get

$$\left\|\sum_{k=1}^{\infty}\hat W_k f_k-\sigma_\nu(\cdot,f)\right\|_{C(K)}\le C(K)\left\{D(\nu)+\Phi_p(f,\nu)\right\}.$$

Since, in the metrics $C(K)$ the equality [2]

$$\sum_{k=1}^{\infty}\hat W_k f_k=\int_G f(y)\,\hat W(|x-y|,\nu,R_0)\,dy,$$

is true, then

$$\left\|\int_G f(y)\,\hat W(|\cdot-y|,\nu,R_0)\,dy-\sigma_\nu(\cdot,f)\right\|_{C(K)}\le$$
$$\le C(K)\left\{D(\nu)+\Phi_p(f,\nu)\right\}. \tag{13}$$

Since trigonometric system is a system of eigenfunctions of the operator $Lu=u^{(4)}$, then for it the inequality $(P_l(x)\equiv 0,\ l=\overline{2,4}$ (see [2]))

$$\left\|\int_G f(y)\,\hat W(|\cdot-y|,\nu,R_0)\,dy-S_\nu(\cdot,f)\right\|_{C(K)}\le$$

$$\leq C(K)\left\{\omega_1\left(f,\nu^{-1}\right)+\nu^{-1}\|f\|_p\right\}. \tag{14}$$

is fulfilled.

From (13) and (14) according to inequality of triangle we get

$$\Delta_\nu(K,f)\leq C(K)\left\{D(\nu)+\Phi_p(f,\nu)\right\}.$$

Theorem 1 is proved.

Teorem 2. *Let all conditions of theorem 1 at $p=1,\ r=1$ be fulfilled and for the Fourier coefficients of the functions $f(x)\in L_1(G)$ the estimate*

$$\left|\hat{f}_k\right|\leq const\left\{\omega_1\left(f,\rho_k^{-1}\right)+\rho_k^{-1}\|f\|_1\right\},\ \rho_k\geq 1. \tag{15}$$

is true.

Then the estimate

$$\Delta_\nu(K,f)\leq C(K)\left\{\nu^{-1}E(\nu)+\varphi_1(f,\nu)\right\}, \tag{16}$$

is true, where $\varphi_1(f,\nu)=\omega_1\left(f,\nu^{-1}\right)+\nu^{-1}\|f\|_1$.

$$E(\nu)=\inf_{\substack{\beta>1\\ m\geq 2}}\left\{\omega_1\left(P_2,m^{-1}\right)\left(\Phi\left(f,[\nu/2],0\right)+\ln\nu\|f_1\|\right)+\right.$$

$$\left.+m^{2\left(1-\beta^{-1}\right)}\|P_2\|_1\left(\Phi\left(f,[\nu/2],1-\beta^{-1}\right)+\frac{1}{1-\beta^{-1}}\|f_1\|\right)\right\},$$

$$\Phi(f,l,\varepsilon)=\begin{cases}\sum\limits_{i=1}^{l}i^{-\varepsilon}\omega_1\left(f,i^{-1}\right),\ \varepsilon\neq 1\\ \sum\limits_{i=1}^{l}i^{-1}\ln(1+i)\,\omega_1\left(f,i^{-1}\right),\ \varepsilon=1\end{cases}$$

Proof. By theorem 1 the estimate (7) will be fulfilled. Hence, by estimate (15) and monotonicity of $\omega_1(f,t)$ we get

$$\Phi_1(f,\nu)\leq const\left\{\nu^{-1}\|f\|_1+\omega_1\left(f,\nu^{-1}\right)\right\}\leq const\varphi_1(\nu);$$

$$D(\nu)\leq const\inf_{\substack{\beta>1\\ m\geq 2}}\left\{\omega_1\left(P_2,m^{-1}\right)\left[\nu^{-1}\Phi\left(f,[\nu/2],0\right)+\nu^{-1}\|f\|_1\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{-1}\right]+\right.$$

$$\left.+m^{2\left(1-\beta^{-1}\right)}\|P_2\|_1\left[\nu^{-1}\Phi\left(f,[\nu/2],1-\beta^{-1}\right)+\nu^{-1}\|f\|_1\sum_{1\leq\rho_k\leq\nu/2}\rho_k^{-2+\beta^{-1}}\right]\right\}\leq$$

$$\leq\nu^{-1}E(\nu).$$

Corollary. *If in theorem 2 the function $f(x)$ belongs to $W_1^1(G)$, then the estimate*

$$\Delta_\nu(K,f)\leq C(K)\,\nu^{-1}\left(1+T(\nu)\right)\|f\|_{W_1^1(G)},$$

where

$$T(\nu)=\inf_{m\geq 2}\left\{\omega_1\left(P_2,m^{-1}\right)\ln\nu+\|P_2\|_1\ln m\right\}$$

is true .

Let's note, that the similar results for a second order operator were earlier obtained in the papers [6] and [7].

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