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ON APPROXIMATE SOLUTION OF ONE CLASS OF TWO-DIMENSIONAL SINGULAR INTEGRAL EQUATIONS

Abstract

In the paper the basis of collocation method is given for singular integral equation of the form

$$\rho\left(x\right)+\int\limits_{S}\frac{q\left(x,y\right)}{\left|x-y\right|^{2}}\rho\left(y\right)d\sigma_{y}=f\left(x\right),$$

where S is a closed smooth surface in R^3, $q\in C\left(S\times S\right)$ and $q\left(x,x\right)=0$ for

$x\in S$, $f\in C\left(S\right)$ and ρ is an unknown function from $C\left(S\right)$.

Solution of many theoretical and applied problems of mathematics, mechanics and physics leads to the investigation of the class of two-dimensional singular integral equations (SIE)

$$\rho\left(x\right)+\int\limits_{S}\frac{q\left(x,y\right)}{\left|x-y\right|^{2}}\rho\left(y\right)d\sigma_{y}=f\left(x\right)\tag{1}$$

where S is a closed smooth surface in R^3, $q\in C\left(S\times S\right)$ and $q\left(x,x\right)=0$ at $x\in S$, $f\in C\left(S\right)$ and ρ is an unknown function from $C\left(S\right)$, and the function from $C\left(S\right)$, and integral is understood as singular. It is known that equation of form (1) we can solve analitically only in very seldom cases. Therefore, both for theory and in particular, for its applications, creation of approximate methods of solutions of integral equations with corresponding theoretical foundation is very important. The given paper is devoted to grounds of collocation method for SIE of form (1).

Let $\omega_{\rho}\left(\delta\right)$ be "corrected" modulus of continuity of the function ρ and

$$\omega_{q}^{*}\left(\delta\right)=\delta\cdot\sup_{\tau\geq\delta}\frac{\overline{\omega}_{q}^{*}\left(\tau\right)}{\tau},\quad\delta>0,$$

where

$$\omega_{q}^{*}\left(\delta\right)=\sup_{\substack{\left|x-y\right|\leq\delta\\x,y\in S}}\left|q\left(x,y\right)\right|\ ;$$

$$\omega_{q}^{1,0}\left(\delta\right)=\sup_{\substack{\left|x_{1}-x_{2}\right|\leq\delta\\x_{1},\ x_{2}\in S}}\max_{y}\left|q\left(x_{1},y\right)-q\left(x_{2},y\right)\right|\ ;$$

$$\omega_{q}^{0,1}\left(\delta\right)=\sup_{\substack{\left|y_{1}-y_{2}\right|\leq\delta\\y_{1},\ y_{2}\in S}}\max_{x}\left|q\left(x,y_{1}\right)-q\left(x,y_{2}\right)\right|\ ;$$

$$\left\|q\left(x,y\right)\right\|_{\infty}=\max_{x,y\in S}\left|q\left(x,y\right)\right|\ .$$

It is obvious that

$$\omega_\rho,\ \omega_q \in \mathcal{E}_1 = \left\{ \varphi |\, \varphi \geq 0, \varphi \uparrow, \varphi(\delta)/\delta \downarrow, \lim_{\delta\to 0}\varphi(\delta) = 0, \varphi(\delta_1+\delta_2) \leq \varphi(\delta_1) + \varphi(\delta_2) \right\}$$

and

$$\omega_q^{1,0},\ \omega_q^{0,1} \in \mathcal{E}_2 = \left\{ \varphi |\, \varphi \geq 0, \varphi \uparrow, \lim_{\delta\to 0}\varphi(\delta) = 0 \right\}$$

The following holds

Lemma. *Let* $\rho \in C(S)$ *and*

$$\int_0^{diam S} \frac{\omega_q^*(\tau)}{\tau} d\tau < +\infty.$$

Then for any $x \in S$ *the integral*

$$\int_S \frac{q(x,y)}{|x-y|^2} \rho(y)\, d\sigma_y \tag{2}$$

converges as singular.

Proof. For $x \in S,\ \varepsilon > 0$ we'll introduce the notation

$S_\varepsilon(x) = \{ y \in S : |x-y| \leq \varepsilon \}$,and we'll denote by d the radius of standard sphere for S.

Since,

$$\int_S \frac{q(x,y)}{|x-y|^2}\rho(y)\, d\sigma_y = \int_{S_d(x)} \frac{q(x,y)}{|x-y|^2}\rho(y)\, d\sigma_y + \int_{S\backslash S_d(x)} \frac{q(x,y)}{|x-y|^2}\rho(y)\, d\sigma_y,$$

then using reduction of superficial integral to double one we'll get

$$\left| \int_{S_d(x)} \frac{q(x,y)}{|x-y|^2}\rho(y)\, d\sigma_y \right| = \|\rho\| \cdot \int_{S_d(x)} \frac{\omega_q^*(|x-y|)}{|x-y|^2} d\sigma_y \leq const \cdot \|\rho\| \cdot \int_0^d \frac{\omega_q^*(\tau)}{\tau} d\tau$$

and

$$\left| \int_{S\backslash S_d(x)} \frac{q(x,y)}{|x-y|^2}\rho(y)\, d\sigma_y \right| \leq \|\rho\| \cdot \int_{S\backslash S_d(x)} \frac{\omega_q^*(|x-y|)}{|x-y|^2} d\sigma_y \leq$$

$$\leq \|\rho\| \cdot \frac{\omega_q^*(d)}{d} \cdot \int_{S\backslash S_d(x)} \frac{1}{|x-y|} d\sigma_y = const \cdot \|\rho\|$$

As a result we obtain that

$$\left| \int_S \frac{q(x,y)}{|x-y|^2}\rho(y)\, d\sigma_y \right| \leq const \cdot \|\rho\| \left(\int_0^d \frac{\omega_q^*(\tau)}{\tau} d\tau + 1 \right).$$

The lemma is proved.

Theorem 1. *Let*
1) $\omega_q^{1,0}(\delta) = 0\left(1n^{-1}\delta\right)$;
2) $\int\limits_0^{diamS} \frac{\omega_q^*(\tau)}{\tau} d\tau < +\infty$.
Then the integral operator

$$(K\rho)(x) = \int\limits_0 \frac{q(x,y)}{|x-y|^2} \rho(y)\, d\sigma_y$$

is compact on $C(S)$.

Proof. Let $x_1, x_2 \in S$, $|x_1 - x_2| = \delta$, $\delta \in \left(0, \frac{d}{2}\right]$ and $\rho \in C(S)$. It is obvious that

$$(K\rho)(x_1) - (K\rho)(x_2) = \int\limits_{S_{\delta/2}(x_1) \cup S_{\delta/2}(x_2)} \frac{q(x_1,y)}{|x_1-y|^2} \rho(y)\, d\sigma_y -$$

$$- \int\limits_{S_{\delta/2}(x_1) \cup S_{\delta/2}(x_2)} \frac{q(x_2,y)}{|x_2-y|^2} \rho(y)\, d\sigma_y +$$

$$+ \int\limits_{S \setminus \left(S_{\delta/2}(x_1) \cup S_{\delta/2}(x_2)\right)} \left[\frac{q(x_1,y)}{|x_1-y|^2} - \frac{q(x_2,y)}{|x_2-y|^2}\right] \rho(y)\, d\sigma_y.$$

The additives in the right hand-side of the last equality are denoted by A_1,A_2 and A_3, respectively.

Let's estimate A_1. It is clear that $A_1 = A_1' + A_1''$, where

$$A_1' = \int\limits_{S_{\delta/2}(x_1)} \frac{q(x_1,y)}{|x_1-y|^2} \rho(y)\, d\sigma_y,$$

$$A"_1 = \int\limits_{S_{\delta/2}(x_2)} \frac{q(x_1,y)}{|x_1-y|^2} \rho(y)\, d\sigma_y,$$

Using formula of reduction of superficial integral to double one, we'll obtain

$$\left|A_1'\right| \leq const \cdot \|\rho\| \cdot \int\limits_0^{\delta} \frac{\omega_q^*(\tau)}{\tau} d\tau.$$

Besides, taking into account

$$\frac{\delta}{2} \leq |y - x_1| \leq \frac{3\delta}{2},\ y \in S_{\frac{\delta}{2}}(x_2),$$

we'll obtain

$$A"_1 \leq \|\rho\| \cdot \int\limits_{S_{\delta/2}(x_1)} \frac{\omega_q^*(|x_1-y|)}{|x_1-y|^2} \cdot \frac{1}{|x_1-y|^2} d\sigma_y \leq$$

$$\leq \|\rho\| \cdot \frac{\omega_q^*\left(\frac{\delta}{2}\right)}{\frac{\delta}{2}} \cdot \frac{1}{\frac{\delta}{2}} \cdot \int\limits_{S_{\delta/2}(x_2)} d\sigma_y \leq const \cdot \|\rho\| \cdot \omega_q^*(\delta).$$

Finally, we have

$$|A_1| \leq const \cdot \|\rho\| \cdot \left(\omega_q^*(\delta) + \int\limits_0^\delta \frac{\omega_q^*(\tau)}{\tau} d\tau\right).$$

Analogously we can show that

$$|A_2| \leq const \cdot \|\rho\| \cdot \left(\omega_q^*(\delta) + \int\limits_0^\delta \frac{\omega_q^*(\tau)}{\tau} d\tau\right)$$

Let's estimate A_3. Let's denote

$$A_3' = \int\limits_{S\backslash\left(S_{\delta/2}(x_1)\cup S_{\delta/2}(x_2)\right)} \frac{q(x_1,y) - q(x_2,y)}{|x_1-y|^2} \rho(y)\, d\sigma_y,$$

$$A_3" = \int\limits_{S\backslash\left(S_{\delta/2}(x_1)\cup S_{\delta/2}(x_2)\right)} q(x_2,y) \cdot \left[\frac{1}{|x_1-y|^2} - \frac{1}{|x_2-y|^2}\right] \rho(y)\, d\sigma_y$$

For A_3' we have

$$\left|A_3'\right| \leq \|\rho\| \cdot \omega_q^{1,0}(\delta) \cdot \int\limits_{S\backslash\left(S_{\delta/2}(x_1)\cup S_{\delta/2}(x_2)\right)} \frac{1}{|x_1-y|^2} \rho(y)\, d\sigma_y \leq$$

$$\leq const \cdot \|\rho\| \cdot \omega_q^{1,0}(\delta) \cdot |1n\delta|.$$

Let's estimate A_3''. Allowing for the inequality $\frac{1}{3}|x_2-y| \leq |x_1-y| \leq 3\cdot|x_2-y|$ at $y \in S\backslash\left(S_{\delta/2}(x_1) \cup S_{\delta\backslash 2}(x_2)\right)$ we obtain

$$A_3" = \|\rho\| \cdot \int\limits_{S\backslash\left(S_{\delta/2}(x_1)\cup S_{\delta/2}(x_2)\right)} \frac{\omega_q^*(|x_1-y|)\cdot|x_1-x_2|\cdot(|x_1-y|+|x_2-y|)}{|x_1-y|^2\cdot|x_2-y|^2} d\sigma_y \leq$$

$$\leq \delta \cdot \|\rho\| \cdot const \int\limits_{S\backslash S_{\delta/2}(x_2)} \frac{\omega_q^*(|x_1-y|)}{|x_2-y|^3} d\sigma_y \leq const \cdot \|\rho\| \cdot \delta \int\limits_\delta^{diam S} \frac{\omega_q^*(\tau)}{\tau} d\tau.$$

Finally we've

$$|A_3| \leq const \cdot \|\rho\| \cdot \left(\omega_q^{1,0}(\delta)\,|1n\delta| + \delta \cdot \int\limits_\delta^{diam S} \frac{\omega_q^*(\tau)}{\tau} d\tau\right).$$

As a result we obtain that

$$|(K\rho)(x_1) - (K\rho)(x_2)| \leq const \cdot \|\rho\| \cdot$$

$$\cdot\left(\omega_q^*(\delta)+\omega_q^{1,0}(\delta)\left|1n\delta\right|+\int\limits_0^\delta\frac{\omega_q^*(\tau)}{\tau}d\tau+\delta\cdot\int\limits_\delta^{diamS}\frac{\omega_q^*(\tau)}{\tau}d\tau\right)=\|\rho\|\cdot Z(\delta).$$

For proving compactness of the operator K on $C(S)$ we introduce the continuous functions $Q_n : S\times S\to R$ $(n\in N)$ of the form

$$Q_n(x,y)=\begin{cases}0, \text{ if } |x-y|\le\frac{1}{2n}\\ (2n|x-y|-1)\cdot q(x,y)/|x-y|^2, \text{ if } \dfrac{1}{2n}\le|x-y|\le\dfrac{1}{n},\\ \frac{q(x,y)}{|x-y|^2}, \text{ if } |x-y|>\frac{1}{n}.\end{cases}$$

Let

$$(G_n\rho)(x)=\int\limits_S Q_n(x,y)\rho(y)\,d\sigma_y.$$

Then

$$|(K\rho)(x)-(G_n\rho)(x)|\le\int\limits_{S_{1/n}(x)}\left|\frac{q(x,y)}{|x-y|^2}\rho(y)\right|d\sigma_y\le$$

$$\le const\cdot\|\rho\|\cdot\int\limits_0^{1/n}\frac{\omega_q^*(\tau)}{\tau}d\tau \tag{3}$$

From estimation (3) we obtain

$$\|K-G_n\|\to 0 \text{ as } n\to 0.$$

Thus K is a uniform limit of sequence of the operators G_n compact on $C(S)$. The theorem is proved.

Theorem 2. *Let*

1) $\omega_q^{1,0}(\delta)=0\left(1n^{-1}\delta\right)$;

2) $\displaystyle\int\limits_0^{diamS}\frac{\omega_q^*(\tau)}{\tau}d\tau<+\infty$ *and*

3) $Ker(I+K)=\{0\}$

Then SIE (1) has a unique solution from $C(S)$ at any $f\in C(S)$.

Now we'll give the ground of collocation method for SIE of form (1). Let $\{h\}\subset R_+$ be a set of values of discretization parameter tending to zero. Let's decompose S into "regular" elementary domains: $S=\bigcup\limits_{l=1}^{N(h)}S_l^h$ (see [2]) and let $x_l\in S_l^h,\ l=\overline{1,N(h)}$ be support.

We'll accept the expression

$$\left(K^{N(h)}\rho\right)(x_l)=\sum_{\substack{j=1\\ j\neq l}}^{N(h)}\frac{q(x_l,x_j)}{|x_l-x_j|^2}\rho(x_j)\,mesS_j^h \tag{4}$$

as cubic formula (c.f.) for integral (2) at the points $x_l,\ l=\overline{1,N(h)}$. For estimation of error of c.f. (4) the following is true.

Theorem 3 ([3]). *Let the conditions*

$$a)\ \ \omega_q^{0,1}(\delta) = 0\left(1n^{-1}\delta\right),$$

$$b)\ \ \omega_q^{1,0}(\delta) = 0\left(1n^{-1}\delta\right)$$

and

$$c)\ \ \int\limits_0^{diamS} \frac{\omega_q^*(\tau)}{\tau} d\tau < +\infty.$$

Then it holds

$$\max_{l=\overline{1,N(h)}} \left| (K\rho)(x_l) - \left(K^{N(h)}\rho\right)(x_l) \right| \le$$

$$\le M \left((\omega_\rho(R(h)) + \|\rho\| \left(\omega_\rho^{0,1}(R(h)) \left|1nR(h)\right| + \int\limits_0^{R(h)} \frac{\omega_q^*(\tau)}{\tau} d\tau + \right.\right.$$

$$\left.\left. + R(h) \int\limits_{r(h)}^{diamS} \frac{\omega_q^*(\tau)}{\tau} d\tau \right)\right),$$

where M is a positive constant depending only on S and

$$R(h) = \max_{l=\overline{1,N(h)}} \sup_{x\in\partial S_l^h} |x - x_l|, \quad r(h) = \min_{l=\overline{1,N(h)}} \inf_{x\in\partial S_l^h} |x - x_l|.$$

Denote by $\mathbb{C}^{N(h)}$ the space of $N(h)$ dimensional vectors $w^{N(h)} = \left(w_1, w_2, ..., w_{N(h)}\right)$, $w_l \in \mathbb{C}^{N(h)}$, $l = \overline{1, N(h)}$ with the norm $\left\|w^{N(h)}\right\| = \max\limits_{l=\overline{1,N(h)}} |w_l|$. For $w^{N(h)} \in \mathbb{C}^{N(h)}$ assume

$$K_l^{N(h)} w^{N(h)} = \sum_{\substack{j=1\\ j\neq l}}^{N(h)} \frac{q(x_l, x_j)}{|x_l - x_j|^2} mesS_j^h \cdot w_j, \ \ l = \overline{1, N(h)}$$

$$K^{N(h)} w^{N(h)} = \left(K_1^{N(h)} w^{N(h)}, ..., K_{N(h)}^{N(h)} w^{N(h)}\right)$$

According to collocation method, SIE (1) with using c.f. (4) is substituted by the system of algebraic equations relative to w_l approximate values $\rho(x_l)$, $l = \overline{1, N(h)}$ which we'll write in the following form:

$$w^{N(h)} + K^{N(h)} w^{N(h)} = f^{N(h)},$$

where

$$f^{N(h)} = p^{N(h)} f = \left(f_1, ..., f_{N(h)}\right);\ f_l = f(x_l),\ l = \overline{1, N(h)};\ p^{N(h)} \in \mathcal{L}\left(C(S), \mathbb{C}^{N(h)}\right)$$

is operator of simple drift.

Theorem 4. *Under conditions a), b), c) and* $Ker\,(I+K)=\{0\}$ *equaitons (1) and (5) have a unique solution* $\rho_* \in C\,(S)$ *and* $w_*^{N(h)} \in \mathbb{C}^{N(h)}\,(N\,(h) \geq n_0)$, *moreover* $\left\| w_*^{N(h)} - p^{N(h)}\rho \right\| \to 0$ *as* $h \to 0 \to$*with the estimation*

$$C_1 \delta_{N(h)} \leq \left\| w_*^{N(h)} - p^{N(h)}\rho_* \right\| \leq C_2 \delta_{N(h)},$$

where

$$C_1 = \left(\sup_{N(h)\geq n_0} \left\| I^{N(h)} + K^{N(h)} \right\| \right)^{-1} > 0,$$

$$C_2 = \sup_{N(h)\geq n_0} \left\| \left(I^{N(h)} + K^{N(h)} \right)^{-1} \right\| < \infty$$

$$\delta_{N(h)} = \max_{l=\overline{1,N(h)}} \left| K_l^{N(h)} \left(p^{N(h)}\rho_* \right) - (K\rho_*)\,(x_l) \right|$$

and $I^{N(h)}$ *is a unique operator in* $\mathbb{C}^{N(h)}$.

Proof. Let's apply G.M.Vaynikko theorem on convergence for linear operator equations (see [4]).

Let $\mathcal{P} = \left\{ p^{N(h)} \right\}$ be a system of operators of simple drift. It is obvious that the system $\mathcal{P}$ is connecting for $C\,(S)$ and $\mathbb{C}^{N(h)}$, $N\,(h) = 1, 2, ...$ Besides $f^{N(h)} \xrightarrow{\mathcal{P}} f$ as $h \to 0 \to$ and the operators $I^{N(h)} + K^{N(h)}$ are Fredholm with zero indeces.

It remains to show that $K^{N(h)} \xrightarrow{\mathcal{PP}} K$ is compact. Since $K^{N(h)} \xrightarrow{\mathcal{PP}} K$ then by the known suggestion (see [4], p.8), it suffices to prove that there exists a relatively compact sequence $\left\{ K_{N(h)} w^{N(h)} \right\} \subset C\,(S)$, where $w^{N(h)} \in \mathbb{C}^{N(h)}$ and $\left\| w^{N(h)} \right\| \leq C = const$ such that

$$\left\| K^{N(h)} w^{N(h)} - p^{N(h)} \left(K_{N(h)} w^{N(h)} \right) \right\| \to 0 \text{ as } h \to 0.$$

As $\left\{ K_{N(h)} w^{N(h)} \right\}$we'll choose the sequence

$$\left(K_{N(h)} w^{N(h)} \right)(x) = \sum_{j=1}^{N(h)} \left[\int_{S_j^h} \frac{q\,(x,y)}{|x-y|^2} d\sigma_y \right] \cdot w_j, \quad x \in S.$$

It is easy to see that $\left\{ K_{N(h)} w^{N(h)} \right\} \subset C\,(S)$. Since

$$\left| K_l^{N(h)} w^{N(h)} - \left(K_{N(h)} w^{N(h)} \right)(x_l) \right| =$$

$$= \left| \sum_{\substack{j=1\\ j\neq l}}^{N(h)} \omega_j \left[\int_{S_j^h} \frac{q\,(x_l,x_j) - q\,(x_l,y)}{|x_l-y|^2} d\sigma_y + \int_{S_j^h} q\,(x_l,x_j) \times \right.\right.$$

$$\left.\left. \times \left(\frac{1}{|x_l-x_j|^2} - \frac{1}{|x_l-y|^2} \right) d\sigma_y \right] + w_l \int_{S_l^h} \frac{q\,(x_l,x_j)}{|x_l-y|^2} d\sigma_y \right| \leq const \times$$

$$\times\left[\int\limits_{S\backslash S_j^h}\frac{\omega_q^{0,1}\left(R\left(h\right)\right)}{\left|x_l-y\right|^2}d\sigma_y+R\left(h\right)\cdot\int\limits_{S\backslash S_j^h}\frac{\omega_q^*\left(\left|x_l-y\right|\right)}{\left|x_l-y\right|^3}d\sigma_y+\int\limits_{S_l^h}\frac{\omega_q^*\left(\left|x_l-y\right|\right)}{\left|x_l-y\right|^2}d\sigma_y\right]\leq$$

$$\leq const\cdot\left[\int\limits_0^{R(h)}\frac{\omega_q^*\left(\tau\right)}{\tau}d\tau+\omega_q^{0,1}\left(R\left(h\right)\right)\cdot\left|1nR\left(h\right)\right|+R\left(h\right)\cdot\int\limits_{R(h)}^{diamS}\frac{\omega_q^*\left(\tau\right)}{\tau}d\tau\right],$$

then

$$\left\|K^{N(h)}w^{N(h)}-p^{N(h)}\left(K_{N(h)}w^{N(h)}\right)\right\|\to 0\ \text{ as } h\to 0.$$

And relative compactness $\left\{K_{N(h)}w^{N(h)}\right\}$ follows from Arzeli theorem (see [5]). The theorem is proved.

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