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ON COMPLETENESS OF EXPONENT SYSTEM WITH COMPLEX COEFFICIENTS IN WEIGHT SPACES

Abstract

In the paper the exponents system with complex-valued coefficients is considered, the equivalence of completeness of this system in the weight space $L_{p,\rho}$ of trivial solvability of definite conjugation problem in Hardy weight class is proved.

Consider the following system of exponents

$$\left\{A(t)\,e^{int}, B(t)\,e^{-i(n+1)t}\right\}_{n\geq 0}, \tag{1}$$

with complex-valued coefficients $A(t) \equiv |A(t)|$ on the segment $[-\pi,\pi]$. Investigation of basis properties of such system of functions in different functional spaces is important not only from the view of applications in spectral theory of differential operators, but it is only of theoretical interest. Such a system in such general form is considered by B.T.Bilalov [1] and he proved the basicity of this system in $L_p \equiv L_p(-\pi,\pi)$, $1<p<+\infty$ at definite conditions on the coefficients $A(t)$ and $B(t)$. It should be noted than earlier the basic properties of system (1) at concrete $A(t)$ and $B(t)$ were investigated by many authors (see for example [2] and [3-5]).

Not much papers were devoted to basic properties of such systems in weight spaces. In the paper of K.S. Kazayran, P.I.Lizorkin [6] it is found the necessary and sufficient condition on the weight $\rho(t)$ for basicity of classical system of the exponents $\left\{e^{int}\right\}_{-\infty}^{+\infty}$ in the weight space $L_{p,\rho} : (1<p<+\infty)$

$$L_{p,\rho}(-\pi,\pi) \equiv \left\{f : \int_{-\pi}^{\pi} |f(t)|^p \rho(t)\,dt < +\infty\right\}, \tag{2}$$

with norm

$$\|f\|_{p,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^p \rho dt\right)^{1/p}.$$

E.I.Moiseev [7] obtained necessary and sufficient condition on the parameters $\alpha,\beta \in R$ (R is a set of real numbers) for basicity of system of the sines

$\{\sin[(n+\alpha)t+\beta]\}_{n\geq 1}$ in $L_{p,\rho}(0,\pi)$ when the weight ρ has concrete power form.

In the suggested paper the completeness of system (1) is investigated in weight space in $L_{p,\rho}$ when the weight ρ has definite power form.

So, in future under $L_{p,\rho}$ we'll assume the weight space (2) with the weight ρ:

$$\rho(t) \equiv \prod_{i=1}^{l} \left\{ \sin \left| \frac{t-\tau_i}{2} \right| \right\}^{\beta_i}.$$

where $\{\tau_i\} \subset (-\pi, \pi)$, $\{\beta_i\} \subset R$ is some set.

Assume the fulfilment of the following conditions:

1) $\left|A(t)^{\pm 1}\right|, |B(t)|^{\pm 1} \in L_\infty$;

2) $\alpha(t)$ and $\beta(t)$ are piece-wise Holder functions on $[-\pi, \pi]$;

$\{s_i : -\pi < s_i < ... < s_r < \pi\}$ is the set of break points of the function $\theta(t) \equiv \alpha(t) - \beta(t)$;

3) the sets $T \equiv \{\tau_i\}_1^l$ and $S \equiv \{s_i\}_1^r$ do not intersect: $T \cap S = \{\varnothing\}$

By obtaining the basic result the class $H_{q,\nu}^{+}$ introduced in S.G.Veliev paper [9] is used: let H_1^+ be an ordinary Hardy class analytical inside of unit circle of the functions and $\nu(t) \geq 0$ a.e. on $(-\pi, \pi)$ be some weight. Denote:

$$H_{q,\nu}^{+} \stackrel{def}{\equiv} \left\{ f \in H_1^+ : \int_{-\pi}^{\pi} \left| f^+ \left(e^{it}\right) \right|^q \nu(t)\, dt < +\infty \right\},$$

where $f^+\left(e^{it}\right)$ are non-tangent boundary values on a unit circle of the function $f(z)$ inside the unit circle.

The following is true.

Theorem. *Let the complex-valued functions $A(t)$, $B(t)$ satisfy conditions 1), 2) and the inequality $-1 < \beta_i < +\infty$, $i = \overline{1,l}$ be fulfilled. Then exponent system (1) is complete in the space $L_{p,\rho}$, $1 < p < +\infty$ only in the case if the homogeneous conjugate problem*

$$F_1^+(\tau) - \frac{A(t)}{\tau B(t)} \overline{F_2^+(\tau)} = 0 \quad \text{a.e. } t \in (-\pi, \pi), \quad t = \arg \tau, \tag{0}$$

has only trivial solution in the class $H_{q,\nu}^{+}$, where q is a conjugate to p number, $\nu = \rho^{-\frac{1}{p}}$.

Let's remind that under the solution of problem (0) in the class $H_{q,\nu}^{+}$ it is understood the pair of the functions $\{F_1(z); F_2(z)\}$ from this class whose boundary non-tangential values on a unit circle a.e. satisfy equality (0).

Proof. The conjugate to $L_{p,\rho}$ space is $L_{q,\rho}$, where $q : \frac{1}{p} + \frac{1}{q} = 1$ is a conjugate to p number. Any functional L on $L_{p,\rho}$ is representable in the following form:

$$L(g)=\int_{-\pi}^{\pi}\overline{f}g\rho dt,\quad g\in L_{p,\rho},$$

where $(\bar{\cdot})$ means the complex conjugation for some $f\in L_{q,\rho}$. Therefore completeness of system (1) in $L_{p,\rho}$ is equivalent to the existence of only zero function f from $L_{q,\rho}$ for which the following relations are fulfilled:

$$\int_{-\pi}^{\pi}\overline{f}(t)A(t)e^{int}\rho(t)dt=0,\quad \int_{-\pi}^{\pi}f(t)B(t)e^{-i(n+1)t}\rho(t)dt=0,\quad \forall n\geq 0. \tag{3}$$

In turn from these equalities we have:

$$\int_{-\pi}^{\pi}\overline{f}(t)A(t)\rho(t)e^{int}dt=\frac{1}{i}\int\limits_{|\tau|=1}\overline{f}(\arg\tau)A(\arg\tau)\rho(\arg\tau)\overline{\tau}\tau^{n}d\tau=0,\quad \forall\tau\geq 0.$$

Denote

$$\Phi_1(\tau)=\bar{f}(\arg\tau)A(\arg\tau)\rho(\arg\tau)\overline{\tau},\ |\tau|=1.$$

Consequently,

$$\int\limits_{|\tau|=1}\Phi_1(\tau)\tau^{n}dt=0,\ \forall n\geq 0. \tag{4}$$

Let $\Gamma=\{z\in C:|z|=1\}$ be a unit circle on a complex surface. It is absolutely obvious that $\Phi_i\in L_1(\Gamma)$. Then by the results of monograph of I.I.Privalov [8], equalities (4) are equivalent to the existence of the function $F_1(z)$ from the Hardy class H_1^{+}, for which non-tangential boundary values it holds $F_1^{+}(\tau)=\Phi_1(\tau)$ a.e. on Γ. Thus:

$$F_1^{+}(\tau)=\overline{f}(t)A(t)\rho(t)\overline{\tau},\ \tau\in\Gamma,\quad t=\arg\tau.$$

Analogously from the second relation (3) we obtain:

$$\int\limits_{|\tau|=1}\Phi_2(\tau)\tau^{n}dt=0,\ \forall n\geq 0,$$

where

$$\Phi_2(\tau)=f(\tau)\overline{B(t)}\rho(t),\quad t=\arg\tau.$$

From those considerations we obtain the existence of the functions $F_2(z)$ from the Hardy class H_1^+ for some $F_2^+(\tau) = \Phi_2(\tau)$ a.e. on Γ. Consequently,

$$F_2^+(\tau) = f(t)\,\overline{B}(t)\,\rho(t), \quad t = \arg\tau.$$

By comparing the obtained expressions for the function $f(t)$ we have:

$$\frac{F_1^+(\tau)}{A(t)\,\rho(t)\,\bar{\tau}} = \frac{\overline{F_2^+(\tau)}}{B(t)\,\rho(t)}, \quad t = \arg\tau, \text{ a.e. on } \Gamma.$$

As a result we obtain the conjugate problem

$$F_1^+(\tau) + G(\tau)\,\overline{F_2^+(\tau)} = 0 \qquad \text{a.e. on } \Gamma, \tag{5}$$

where $G(\tau) = -\frac{A(t)e^{-it}}{B(t)}$, $t = \arg\tau$.

On the other hand we can represent $\Phi_1(\tau)$ in the form: $(t = \arg\tau)$

$$|\Phi_1(\tau)| = |f(t)|\,|A|\,\rho = |f(t)|\,\rho^{\frac{1}{q}}(t)\,|A(t)|\,\rho^{\frac{1}{p}}(t).$$

Hence we easily conclude that $|\Phi_1(\tau)|\,\rho^{-\frac{1}{p}}(t) \in L_q$. Thus $F_1^+\left(e^{it}\right)\rho^{-\frac{1}{p}} \in L_q$ and as a result the function $F_1(z)$ belongs to the class $H_{q,\nu}^+$, where $\nu = \rho^{-\frac{1}{p}}$. It is easy to note that the function $F_2(z)$ also belongs to the class $H_{q,\nu}^+$, thus if system (1) is not complete in $L_{p,\rho}$ then conjugation problem (5) is nontrivially solvable in the class $H_{q,\nu}^+$.

And now we assume that the homogeneous conjugation problem (5) is nontrivially solvable in the class $H_{q,\nu}^+$. Denote by $f(t)$ the following function:

$$f(t) = \frac{\overline{F_1^+(e^{it})}}{e^{it}\overline{A(t)}\rho(t)}, \quad t \in [-\pi, \pi].$$

We have:

$$\int_{-\pi}^{\pi} |f(t)|^q \rho(t)\,dt \le c\int_{-\pi}^{\pi} \left|F_1^+\left(e^{it}\right)\right|^q \rho^{1-q}(t)\,dt =$$

$$= C\int_{-\pi}^{\pi} \left|F_1^+\left(e^{it}\right)\rho^{-\frac{1}{p}}\right|^q dt < +\infty.$$

The last inequality follows from $F_1 \in H_{q,\nu}^+$. By the definition of the class $H_{q,\nu}^+$ the function $\mathrm{F}_1(z)$ also belongs to the space H_1^+. Then by the V.I.Smirnov known theorem we obtain

$$\int\limits_{|\tau|=1} F_1^+(\tau)\,\tau^n d\tau = 0,$$

and as a result

$$\int_{-\pi}^{\pi} A(t)\,\rho(t)\,e^{int}\overline{f(t)}dt = 0, \quad \forall n \geq 0.$$

Analogously we have;

$$\int_{-\pi}^{\pi} B(t)\,\rho(t)\,e^{i(n+1)t}\overline{f(t)}dt = 0, \quad \forall n \geq 0.$$

As a result from the previous considerations it follows that system (1) is not complete in $L_{p,\rho}$.

The theorem is proved.

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