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ON A MIXED PROBLEM IN BOUNDED DOMAIN FOR ONE EQUATION CORRECT BY PETROVSKII AND ESTIMATE OF ITS SOLUTION

Abstract

In this paper the existence and uniqueness of a mixed problem in bounded domain for one correct by Petrovskii equation is proved and the estimate of solution by data of the problem is obtained.

At studying the perturbance propagation in viscous gas there arises the following equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \omega \frac{\partial}{\partial t} \Delta_3 u(x,t) = a^2 \Delta_3 u(x,t), \quad x \in R_3, \ t \geq 0, \tag{1.0}$$

where $\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, a is the sound velocity in gas in the absence thereof viscosity , $\omega = \frac{4}{3}\nu$, ν is a kinematic coefficient of viscosity [1]. In the paper [2] the Cauchy problem has been studied for equation (1.0) in $L_2(R_3)$, and the uniform stabilization of solution of the Cauchy problem for equation (1.0) has been obtained.

1. Existence and uniqueness of solution of a mixed problem.

In this paper in $Q = \Omega \times (0, \infty)$ the following mixed problem is studied

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \omega \frac{\partial}{\partial t} \Delta_n u(x,t) = a^2 \Delta_n u(x,t) + f(x,t), \tag{1.1}$$

$$u(x,t)|_{t=0} = \varphi_0(x), \ \left. \frac{\partial u(x,t)}{\partial t} \right|_{t-0} = \varphi_1(x), \tag{1.2}$$

$$u(x,t)|_{\partial\Omega} = 0, \tag{1.3}$$

where Ω is a bounded domain with the sufficiently smooth boundary $\partial\Omega$ of n-dimensional Euclidean space R_n, $x = (x_1, x_2, ..., x_n) \in R_n$, $\varphi_0(x)$, $\varphi_1(x)$, $f(x,t)$ are the functions given in Q

$$\Delta_n = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}.$$

Denote by $C^{(0,0)}(Q)$ a space of the functions defined in $\overline{Q}$ and continuous with respect to (x,t).

Definition 1. *Denote by $B^{(2,2)}(Q)$ a space of functions defined in Q such that $\frac{\partial^{\beta+|\alpha|}}{\partial t^\beta \partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} u(x,t) = C^{(0,0)}(Q)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + |\alpha|_n$ and satisfying the estimate*

$$\|u(x,t)\|_{C^{(|\alpha|,\beta)}(\Omega)} \leq Ce^{-\gamma t}, \quad 0 \leq |\alpha|, \ \beta \leq 2, \tag{1.4}$$

where C and γ are some constants.

Denote by $H_D^{\mu_j}(\Omega)$, $(\mu_j \geq 1,\ j=1,2)$, $H_N^{\mu_3}(\Omega)$, $(\mu_j \geq 2)$ ([3], p.252) the subspaces of Sobolev-Slobodetskii space $H^{\mu_j}(\Omega)$, $j=1,2,3$ for whose elements respectively the following conditions are satisfied

$$F(x)|_{\partial\Omega} = 0, ..., \Delta^{\left[\frac{\mu_j-1}{2}\right]} F(x)|_{\partial\Omega} = 0,\ j=1,2;$$
$$\left.\frac{\partial F(x)}{\partial\tau}\right|_{\partial\Omega} = 0, ..., \frac{\partial}{\partial\tau}\Delta^{\left[\frac{\mu_3}{2}\right]-1} F(x)|_{\partial\Omega} = 0,$$

where τ is a normal to $\partial\Omega$, $[\sigma]$ denotes the entire part of σ.

We'll assume that $\varphi_0(x) \in H^{\mu_0}(\Omega)$, $\varphi_1(x) \in H^{\mu_1}(\Omega)$, $f(x,t) \in H^{\mu_2}(\Omega)$ at each $t \geq 0$, where μ_0, μ_1, μ_2 are some numbers, $H^{\mu_j}(\Omega)$, $j=0,1,2$; are Sobolev-Slobodetskii spaces.

Definition 2. *We'll call the function $u(x,t)$ a classical solution of problem (1.1)-(1.3) if $u(x,t) \in B^{(2,2)}(Q)$ and satisfies the equation, initial and boundary conditions in the ordinary sense.*

Theorem 1. *The classical solution of problem (1.1)-(1.3) is unique if it exists.*

Proof. We show that the classical solution of homogeneous problem $(1.1)_0$, $(1.2)_0$, (1.3) is a trivial solution, where a zero by the number of data means that they are equal to zero. For this multiplying $(1.1)_0$ by $u(x,t)$ and integrating by $\Omega \times [0,t]$, we obtain

$$\varepsilon(t) = \int_0^t\int_\Omega \left[\frac{\partial^2 u(x,t)}{\partial t^2} - \omega\frac{\partial\Delta u(x,t)}{\partial t} - a^2\Delta u(x,t)\right]\frac{\partial u(x,t)}{\partial t}dtdx \equiv$$
$$\equiv \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t) \equiv 0. \tag{1.5}$$

We transform each of addends in (1.5) using the initial and boundary conditions. Taking into account that for the homogeneous problem $\varphi_1(x) \equiv 0$, we obtain

$$\varepsilon_1(t) = \int_0^t\int_\Omega \frac{\partial^2 u(x,t)}{\partial t^2}\frac{\partial u(x,t)}{\partial t}dtdx =$$
$$\equiv \frac{1}{2}\int_\Omega\int_0^t \frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dtdx = \frac{1}{2}\int_\Omega \left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx. \tag{1.6}$$

Applying the first Green formula and allowing for boundary condition (1.3) we obtain

$$\varepsilon_2(t) = -\omega\int_0^t\int_\Omega \frac{\partial}{\partial t}\Delta u(x,t)\frac{\partial u(x,t)}{\partial t}dtdx = \omega\int_0^t\int_\Omega\sum_{j=1}^n\left(\frac{\partial^2 u(x,t)}{\partial x_j\partial t}\right)^2 dtdx. \tag{1.7}$$

Analogously

$$\varepsilon_3(t) = -a^2\int_0^t\int_\Omega \Delta u(x,t)\frac{\partial u(x,t)}{\partial t}dtdx = a^2\int_0^t\int_\Omega\sum_{j=1}^n\frac{\partial u(x,t)}{\partial x_j}\frac{\partial^2 u(x,t)}{\partial x_j\partial t}dtdx =$$

$$=\frac{a^2}{2}\int_0^t\int_\Omega\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dtdx=\frac{a^2}{2}\int_\Omega\sum_{j=1}^n\left(\frac{\partial u(x,t)}{\partial x_j}\right)^2 dx. \tag{1.8}$$

From (1.5)-(1.8) we obtain

$$\varepsilon(t)=\frac{1}{2}\int_\Omega\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx+\omega\int_0^t\int_\Omega\sum_{j=1}^n\left(\frac{\partial^2 u(x,t)}{\partial x_j\partial t}\right)^2 dtdx+$$

$$+\frac{a^2}{2}\int_\Omega\sum_{j=1}^n\left(\frac{\partial u(x,t)}{\partial x_j}\right)^2 dx\equiv 0.$$

Hence we obtain

$$\frac{\partial u(x,t)}{\partial t}=0,\ \frac{\partial^2 u(x,t)}{\partial x_j\partial t}=0,...,\frac{\partial u(x,t)}{\partial x_j}=0,\ (x,t)\in\overline{Q},\ j=1,2,...,n. \tag{1.9}$$

Taking into account that for homogeneous problem the initial data are identically equal to zero and boundary condition (1.3) we obtain that

$$u(x,t)\equiv 0.$$

Theorem 1 is proved.

For construction of solution of mixed problem (1.1)-(1.3) we fulfil the Laplace transformation over problem (1.1)-(1.3), and take into account estimate (1.4). Then we obtain

$$-\varphi_1(x)-k\varphi_0(x)+k^2\hat{u}(x,k)-\omega\Delta_n\left[-\varphi_0(x)+k\hat{u}(x,k)\right]=a^2\Delta_n\hat{u}(x,k)+\hat{f}(x,k),$$

where $\operatorname{Re}k\geq-\gamma_0$, and the sign$\hat{}$over the function denotes the Laplace transformation of this function by t. Hence we obtain

$$\left(-k\omega-a^2\right)\Delta_n\hat{u}(x,k)+k^2\hat{u}(x,k)=\varphi_1(x)+k\varphi_0(x)-\omega\Delta_n\varphi_0(x)+\hat{f}(x,k).$$

Denote

$$\Phi(x,k)=\varphi_1(x)+k\varphi_0(x)-\omega\Delta_n\varphi_0(x)+\hat{f}(x,k).$$

Then we obtain the following boundary-value problem equivalent to problem (1.1)-(1.3)

$$\left(k\omega+a^2\right)\Delta_n\hat{u}(x,k)+k^2\Delta_n\hat{u}(x,k)=-\Phi(x,k),\ x\in\Omega, \tag{1.10}$$

$$\hat{u}(x,k)|_{\partial\Omega}=0. \tag{1.11}$$

For the solution of problem (1.10),(1.11) we introduce the following operator. Consider the differential expression $\tilde{A}=\Delta_n$ with domain of definition

$$D\left(\tilde{A}\right)=\left\{W(x):W(x)\in C^2(\Omega)\cap C\left(\bar{\Omega}\right),\,W(x)|_{\partial\Omega}=0,\ \Delta_nW(x)\in L_2(\Omega)\right\}.$$

The differential expression $\tilde{A}$ with the domain of definition $D\left(\tilde{A}\right)$ allows negative definite self-adjoint extension of A in $L_2(\Omega)$. The spectrum of the operator A is discrete, for its eigen values λ_i the inequality

$$0 > \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l \geq ..., \ \lim_{l\to\infty} \lambda_i = -\infty \tag{1.12}$$

holds.

The eigen functions $\psi_l(x)$ of the operator A corresponding to the eigen values λ_i form basis in the space $L_2(\Omega)$.

Using theorem 3.6 from [4] (p.177) for solution of problem (1.10),(1.11) we obtain

$$\hat{u}(x,k) = \sum_{l=1}^{\infty} \frac{c_l(k)\,\psi_l(x)}{(k\omega + a^2)\,\lambda_l - k^2}, \tag{1.13}$$

where λ_i are eigen values, $\psi_l(x)$ are eigen functions of the Dirichlet problem for the Laplace operator, and

$$c_l(k) = -\int_{\Omega} \Phi(x,k)\,\psi_l(x)\,dx. \tag{1.14}$$

Series in (1.13) converges in $L_2(\Omega)$. Later we'll show that the series in (1.13) as well as series obtained from it by termwise integration with respect to k converge uniformly with respect to x in $\overline{\Omega}$ under the definite condition on initial functions $\varphi_0(x)$, $\varphi_1(x)$ and on boundary $\partial\Omega$ of the domain Ω.

We transform the coefficients $c_l(k)$. From (1.14) we have

$$c_l(k) = -c_l^{(1)} - kc_l^{(0)} + \int_{\Omega} \Delta\varphi_0(x)\,\psi_l(x)\,dx - \int_{\Omega} \hat{f}(x,k)\,\psi_l(x)\,dx,$$

where

$$c_l^{(j)} = \int_{\Omega} \varphi_j(x)\,\psi_l(x)\,dx, \quad j = 0, 1.$$

Using the second Green formula and allowing for boundary condition (1.11), we obtain

$$\int_{\Omega} \Delta\varphi_0(x)\,\psi_l(x)\,dx = \int_{\Omega} \varphi_0(x)\,\Delta\psi_l(x)\,dx = \lambda_l c_l^{(0)},$$

Then

$$c_l(k) = -c_l^{(1)} - kc + \omega\lambda_l c_l^{(0)} - \hat{f}_l(k),$$

where

$$\hat{f}_l(k) = \int_{\Omega} \hat{f}(x,k)\,\psi_l(x)\,dx$$

and for $u(x,t)$ we obtain the expression

$$u(x,t) = -\frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \sum_{l=1}^{\infty} \left[\frac{c_l^{(1)} + (k - \omega\lambda_l)\,c_l^{(0)}}{(k\omega - a^2)\,\lambda_l - k^2} e^{kt} dk \right] \psi_l(x)\,dx. \tag{1.15}$$

Later with the conditions on the data of problem (1.1)-(1.3) we show that we can termwise integrate the series in (1.15). Then

$$u(x,t)=\frac{1}{2\pi i}\sum_{l=1}^{\infty}\psi_l(x)\left\{c_l^{(1)}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{kt}dk}{k^2-(k\omega+a^2)\lambda_l}+c_l^{(0)}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{ke^{kt}dk}{k^2-(k\omega+a^2)\lambda_l}-\right.$$

$$\left.-\omega\lambda_l\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{kt}dk}{k^2-(k\omega+a^2)\lambda_l}+\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\hat{f}_l(k)\,e^{kt}dk}{k^2-(k\omega+a^2)\lambda_l}\right\}\equiv$$

$$\equiv u_{\varphi_1}(x,t)+u_{\varphi_0}(x,t)+u_f(x,t)\,. \tag{1.16}$$

where $\varepsilon>0$ is a sufficiently small number, $u_{\varphi_i}(x,t)$ is a solution of problem (1.1), (1.2), (1.3) with initial data $\varphi_i(x,t)$, and for data with index $j\neq i$ $\varphi_j(x,t)\equiv 0$, $u_f(x,t)$ is a solution of this problem at $\varphi_0(x)\equiv 0$, $\varphi_1(x)\equiv 0$.

We compute the integrals in (1.16). Denote

$$J_{1,l}(t)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{kt}dk}{k^2-(k\omega+a^2)\lambda_l},\quad J_{2,l}(t)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\hat{f}_l(k)\,e^{kt}dk}{k^2-(k\omega+a^2)\lambda_l}. \tag{1.17}$$

The poles of integrands in (1.17) are at the points

$$k_{1,2}=\frac{\omega\lambda_l}{2}\pm\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l},$$

that are situated in the half-plane $\mathrm{Re}\,k\leq 0$. Applying the residue method we obtain

$$J_{1,l}(t)=\frac{e^{\left(\frac{\omega\lambda_l}{2}+\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right)t}}{2\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}}-\frac{e^{\left(\frac{\omega\lambda_l}{2}-\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right)t}}{2\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}}$$

$$J_{2,l}(t)=\frac{e^{\left(\frac{3\omega\lambda_l}{2}+\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right)}}{2\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}}e^{\left(\frac{\omega\lambda_l}{2}+\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right)t}-$$

$$-\frac{e^{\left(\frac{3\omega\lambda_l}{2}-\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right)}}{2\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}}e^{\left(\frac{\omega\lambda_l}{2}-\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right)t}.$$

By G.Borel theorem ([5], p.475)

$$u_f(x,t)=\sum_{l=1}^{\infty}\psi_l(x)\int_{\Omega}\psi_l(x)\int_0^t\left[f(x,\tau)\,\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{(t-\tau)k}dk}{(k\omega+a^2)\lambda_l-k^2}\right]dx. \tag{1.18}$$

From (1.16)and (1.17) we obtain

$$u_{\varphi_0}(x,t)=\sum_{l=1}^{\infty}c_l^{(0)}\psi_l(x)\,J_{2,l}(t)\,, \tag{1.19}$$

$$u_{\varphi_1}(x,t)=\sum_{l=1}^{\infty}c_l^{(1)}\psi_l(x)\,J_{1,l}(t)\,, \tag{1.20}$$

$$u_f(x,t)=\sum_{l=1}^{\infty}\psi_l(x)\int_{\Omega}\psi_l(x)\int_0^t\left[f(x,\tau)\,J_{1,l}(t-\tau)\,d\tau\right]dx=$$

$$=\sum_{l=1}^{\infty}\psi_l(x)\int_0^t f_l(\tau)\,J_{1,l}(t-\tau)\,d\tau. \tag{1.21}$$

2. An estimate of a solution of mixed problem (1.1)-(1.3).

We first show some estimates that are necessary for estimating the solution of problem (1.1)-(1.3).

Lemma 1. *For all $l=1,2,...$ the estimate*

$$\operatorname{Re}\left\{\frac{\omega\lambda_l}{2}+\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right\}\le\max\left\{-\frac{a^2}{\omega},\frac{\omega}{2}\lambda_1\right\}$$

holds.

Proof. Let $\dfrac{\omega}{2}\ge\dfrac{a}{|\lambda_1|^{\frac12}}$. Since λ_l satisfy the inequality (1.12), then

$$\frac{\omega\lambda_l}{2}+\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}=-\frac{a^2}{\dfrac{\omega}{2}+\sqrt{\dfrac{\omega^2}{4}-\dfrac{a^2}{|\lambda_l|}}}\le-\frac{a^2}{\omega}. \tag{2.1}$$

Let now $\dfrac{a}{|\lambda_{l_0+1}|^{\frac12}}<\dfrac{\omega}{2}\le\dfrac{a}{|\lambda_{l_0}|^{\frac12}}$. Then from

$$B_l=|\lambda_l|\left(-\frac{\omega}{2}+\sqrt{\frac{\omega^2}{4}-\frac{a}{|\lambda_l|}}\right)$$

and from inequality it follows that at $l=1,2,...,l_0$

$$\operatorname{Re}B_l=-\frac{\omega}{2}|\lambda_l|\le-\frac{\omega}{2}|\lambda_1|\,, \tag{2.2}$$

and at $l\ge l_0+1$ estimate (2.1) is valid for B_l. Then from (2.1) and (2.2) it follows that fort all l

$$\operatorname{Re}B_l\le\max\left\{-\frac{a^2}{\omega},\frac{\omega}{2}\lambda_1\right\}=\gamma.$$

Lemma 1 is proved.

We can also show that for all $l=1,2,3,...$ the inequality

$$\operatorname{Re}\left\{\frac{\omega\lambda_l}{2}-\sqrt{\frac{\omega^2\lambda_l^2}{4}+a^2\lambda_l}\right\}\le\frac{\omega}{2}\lambda_l\le\frac{\omega}{2}\lambda_1 \tag{2.3}$$

holds.

Besides, if $\omega \neq \frac{2a}{|\lambda_l|^{\frac{1}{2}}}$, $l = 1, 2, 3, ...$ then

$$\left|\sqrt{\frac{\omega^2\lambda_l^2}{4} + a^2\lambda_l}\right| \geq \varepsilon\omega^{\frac{1}{2}}\left|\lambda_l\right|, \tag{2.4}$$

where $\varepsilon > 0$ is a sufficiently small number.

We now pass to estimates $u_{\varphi_0}(x,t)$, $u_{\varphi_1}(x,t)$ and $u_f(x,t)$ and their derivatives.

Theorem 2. *Let* $\omega \neq \frac{2a}{|\lambda_l|^{\frac{1}{2}}}$, $l = 1, 2, 3, ...$, $\partial\Omega \in C^{\left(2\left[\frac{n}{2}\right]+8\right)}$, $\varphi_0(x) \in H_D^{\left(2\left[\frac{n}{2}\right]+8\right)}(\Omega)$. *Then for* $u_{\varphi_0}(x,t)$ *the estimate*

$$\left\|u_{\varphi_0}(x,t)\right\|_{C^{(|\alpha|,\beta)}(\Omega)} \leq C(\omega)\, e^{\gamma t}\left\|\varphi_0(x)\right\|_{H^{\theta_0}(\Omega)} \tag{2.5}$$

holds, where $\theta_0 = 2\left(\left[\frac{n}{2}\right]+1\right) + |\alpha| + 2\beta$, $0 \leq \alpha,\ \beta \leq 2$, *and* γ *has been defined in lemma 1,* $C(\omega)$ *is some constant depending on* ω.

Proof. Using lemma 1, estimates (2.3),(2.4) from (1.19) we obtain

$$\left\|u_{\varphi_0}(x,t)\right\|_{C(\overline{\Omega})} \leq C(\omega)\, e^{\gamma t}\sum_{l=1}^{\infty}\left|c_l^{(0)}\right|\left\|\psi_l(x)\right\|_{C(\overline{\Omega})}. \tag{2.6}$$

It's known that [6]

$$\left\|\psi_l(x)\right\|_{C^{(\nu)}(\overline{\Omega})} \leq C\left|\lambda_l\right|^{\frac{1}{2}\left(\left[\frac{n}{2}\right]+\gamma+1\right)} \tag{2.7}$$

and [3] (p.253)

$$c_0 l^{\frac{2}{n}} \leq |\lambda_l| \leq c_1 l^{\frac{2}{n}}, \tag{2.8}$$

where C, c_0, c_1 are constants independent of l. From (2.6) and (2.7) we have

$$\left\|u_{\varphi_0}(x,t)\right\|_{C(\overline{\Omega})} \leq C(\omega)\, e^{\gamma t}\sum_{l=1}^{\infty}\left|c_l^{(0)}\right|\left|\lambda_l\right|^{\frac{1}{2}\left(\left[\frac{n}{2}\right]+1\right)}. \tag{2.9}$$

Applying the Cauchy-Bunyakowskii inequality in (2.8) we obtain

$$\left\|u_{\varphi_0}(x,t)\right\|_{C(\overline{\Omega})} \leq C(\omega)\, e^{\gamma t}\left\{\sum_{l=1}^{\infty}\left|c_l^{(0)}\right|^2\left|\lambda_l\right|^{2\left(\left[\frac{n}{2}\right]+1\right)}\right\}^{\frac{1}{2}} \times$$

$$= \left\{\quad \left|\lambda_l\right|^{-\left(\left[\frac{n}{2}\right]+1\right)}\right\}^{\frac{1}{2}}. \tag{2.10}$$

Since at any natural n

$$\frac{2}{n}\left(\left[\frac{n}{2}\right]+1\right) \geq 1 + \frac{1}{n},$$

then the second series in (2.10) by virtue of estimate (2.8) converges. Further, by virtue of theorem 8 from [3] (p.253)

$$\left\{\sum_{l=1}^{\infty}\left|c_l^{(0)}\right|^2|\lambda_l|^{2\left(\left[\frac{n}{2}\right]+1\right)}\right\}^{\frac{1}{2}}\leq C\left\|\varphi_0(x)\right\|_{H^{2\left(\left[\frac{n}{2}\right]+1\right)}(\Omega)}, \tag{2.11}$$

where C is a constant independent of $\varphi_0(x)$.

From (2.10) and (2.11) we obtain

$$\left\|u_{\varphi_0}(x,t)\right\|_{C(\overline{\Omega})}\leq C(\omega)e^{\gamma t}\left\|\varphi_0(x)\right\|_{H^{2\left(\left[\frac{n}{2}\right]+1\right)}(\Omega)}. \tag{2.12}$$

We estimate now the derivatives $u_{\varphi_0}(x,t)$. Under the conditions of the theorem we can termwise differentiate the series in (1.19) with respect to x and t up to the second order inclusively. Estimating the series obtained by differentiation of (1.19) just as in (2.9)-(2.12) we obtain

$$\left\|u_{\varphi_0}(x,t)\right\|_{C^{(|\alpha|,\beta)}(\overline{\Omega})}\leq C(\omega)e^{\gamma t}\left\|\varphi_0(x)\right\|_{H^{\theta_0}(\Omega)},$$

where $\theta_0=2\left(\left[\frac{n}{2}\right]+1\right)+|\alpha|+|2\beta|$, $C(\omega)$ is a constant depending on ω.

Theorem 2 is proved.

We now estimate $u_{\varphi_1}(x,t)$. The following theorem holds

Theorem 3. *Let* $\omega\neq\frac{2a}{|\lambda_l|^{\frac{1}{2}}}$, $l=1,2,3,...$, $\partial\Omega\in C^{\left(2\left[\frac{n}{2}\right]+6\right)}$, $\varphi_1(x)\in H_D^{\left(2\left[\frac{n}{2}\right]+6\right)}(\Omega)$. *Then for solution of problems (1.1)-(1.3)* $u_{\varphi_1}(x,t)$ *the following estimate holds*

$$\left\|u_{\varphi_1}(x,t)\right\|_{C^{(|\alpha|,\beta)}(\overline{\Omega})}\leq C(\omega)e^{\gamma t}\left\|\varphi_1(x)\right\|_{H^{\theta_1}(\Omega)}, \tag{2.13}$$

where $\theta_1=2\left(\left[\frac{n}{2}\right]\right)+|\alpha|+2\beta$, $0\leq|\alpha|,\ \beta\leq 2$.

Proof. Using (1.20) and estimate (2.7) we obtain

$$\left\|u_{\varphi_1}(x,t)\right\|_{C(\overline{\Omega})}\leq\sum_{l=1}^{\infty}\left|c_l^{(1)}\right|\left\|\psi_l(x)\right\|_{C(\overline{\Omega})}\left|J_{1,l}(t)\right|\leq\omega^{-\frac{1}{2}}\varepsilon^{-1}\sum_{l=1}^{\infty}\left|c_l^{(1)}\right||\lambda_l|^{\frac{1}{2}\left(\left[\frac{n}{2}\right]-1\right)}.$$

Further acting as in the estimate $u_{\varphi_0}(x,t)$ we obtain

$$\left\|u_{\varphi_1}(x,t)\right\|_{C(\overline{\Omega})}\leq C(\omega)e^{\gamma t}\left\|\varphi_1(x)\right\|_{H^{2\left(\left[\frac{n}{2}\right]\right)}(\Omega)}.$$

For the derivatives $\frac{\partial^{\beta+|\alpha|}}{\partial t^\beta\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}u(x,t)$ analogously we obtain

$$\left\|u_{\varphi_1}(x,t)\right\|_{C^{(|\alpha|,\beta)}(\overline{\Omega})}\leq C(\omega)e^{\gamma t}\left\|\varphi_1(x)\right\|_{H^{\theta_1}(\Omega)},$$

$C(\omega,\varepsilon)$ is a constant depending on ω and ε, $\theta_1=2\left(\left[\frac{n}{2}\right]\right)+|\alpha|+2\beta$.

Theorem 3 is proved.

Theorem 4. *Let* $\omega \neq \dfrac{2a}{|\lambda_l|^{\frac{1}{2}}}$, $l = 1, 2, 3, ...,$ $\partial\Omega \in C^{\left(2\left[\frac{n}{2}\right]+6\right)}$ *and*

$$\int_0^t \|f(x,\tau)\|^2_{H^{2\left[\frac{n}{2}\right]+6}(\Omega)} d\tau < +\infty.$$

Then for the solution $u_f(x,t)$ *of problem (1.1)-(1.3) the following estimate holds*

$$\|u_f(x,t)\|_{C^{(|a|,\beta)}\left(\overline{\Omega}\right)} \leq C(\omega)\left\{\int_0^t \|f(x,\tau)\|^2_{H^{\theta_1}(\Omega)} d\tau\right\}^{\frac{1}{2}}, \qquad (2.14)$$

where $\theta_1 = 2\left(\left[\dfrac{n}{2}\right]\right) + |\alpha| + 2\beta$, $C(\omega)$ *is a constant depending on* ω.

Proof. Estimating $u_f(x,t)$ by modulo from (1.21), using at that estimate (2.7) and applying the Cauchy-Bunyakovskii inequality we obtain

$$\|u_f(x,t)\|_{C\left(\overline{\Omega}\right)} \leq C(\omega)\sum_{l=1}^{\infty}|\lambda_l|^{\frac{1}{2}\left(\left[\frac{n}{2}\right]-1\right)}\int_0^t |f_l(\tau)|\, e^{\gamma(t-\tau)}d\tau \leq$$

$$\leq C(\omega)\left[\sum_{l=1}^{\infty}|\lambda_l|^{-\left(\left[\frac{n}{2}\right]+1\right)}\right]^{\frac{1}{2}}\left[\sum_{l=1}^{\infty}|\lambda_l|^{2\left[\frac{n}{2}\right]}\left(\int_0^t |f_l(\tau)|\, e^{\gamma(t-\tau)}d\tau\right)^2\right]^{\frac{1}{2}}. \qquad (2.15)$$

Applying (2.7) and (2.8) in (2.15) we obtain

$$\|u_f(x,t)\|_{C\left(\overline{\Omega}\right)} \leq C(\omega)\left[\sum_{l=1}^{\infty}|\lambda_l|^{2\left[\frac{n}{2}\right]}\int_0^t |f_l(\tau)|^2\, d\tau\int_0^t e^{2\gamma(t-\tau)}d\tau\right]^{\frac{1}{2}} \leq$$

$$\leq C(\omega)\left[1-e^{2\lambda t}\right]\left\{\int_0^t\left[\sum_{l=1}^{\infty}|\lambda_l|^{2\left[\frac{n}{2}\right]}|f_l(\tau)|^2\right]d\tau\right\}^{\frac{1}{2}}. \qquad (2.16)$$

By virtue of theorem 8 from [3] (p.253) and (2.16) we obtain

$$\|u_f(x,t)\|_{C\left(\overline{\Omega}\right)} \leq C(\omega)\left\{\int_0^t \|f(x,\tau)\|^2_{H^{2\left[\frac{n}{2}\right]}(\Omega)} d\tau\right\}^{\frac{1}{2}},. \qquad (2.17)$$

Differentiating $u_f(x,t)$ with respect to (x,t) and acting just as at estimate $u_f(x,t)$ we obtain

$$\|u_f(x,t)\|_{C^{(|a|,\beta)}\left(\overline{\Omega}\right)} \leq C(\omega)\left[\int_0^t\sum_{l=1}^{\infty}|\lambda_l|^{2\left[\frac{n}{2}\right]+|\alpha|+2\beta}|f_l(\tau)|^2\, d\tau\right]^{\frac{1}{2}}.$$

Hence we obtain

$$\|u_f(x,t)\|_{C^{(|a|,\beta)}(\overline{\Omega})} \leq C(\omega)\left\{\int_0^t \|f(x,\tau)\|^2_{H^{\theta_1}(\Omega)}\,d\tau\right\}^{\frac{1}{2}}.$$

Theorem 4 is proved.

Corollary of theorems 2-4. Let the conditions of theorems 2-4 be fulfilled. Then for the solution of problems (1.1)-(1.3) the following estimate holds

$$\|u(x,t)\|_{C^{(|a|,\beta)}(\overline{\Omega})} \leq C(\omega)\left\{e^{\gamma t}\left[\|\varphi_0(x)\|_{H^{(\theta_0)}(\Omega)} + \|\varphi_1(x)\|_{H^{(\theta_1)}(\Omega)}\right] + \right.$$

$$\left. + \left[\int_0^t \|f(x,\tau)\|^2_{H^{\theta_1}(\Omega)}\,d\tau\right]^{\frac{1}{2}}\right\}, \qquad (2.18)$$

where θ_0, θ_1 are determined in theorems 2 and 3.

Note that at $\omega = \dfrac{2a}{|\lambda_l|^{\frac{1}{2}}}$ the statements of theorems 2-4 remain valid only with the difference that in estimates (2.5),(2.13),(2.14),(2.18) it appears the multiplier t that is a contribution to the functions $J_{1,l}(t)$ and $J_{2,l}(t)$ at the points $\omega = \dfrac{2a}{|\lambda_l|^{\frac{1}{2}}}$.

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