MATHEMATICS

Bahram A. ALIYEV

ASYMPTOTIC BEHAVIOR OF EIGEN-VALUES OF A BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS FOR THE SECOND ORDER ELLIPTIC DIFFERENTIAL-OPERATOR EQUATION

Abstract

In this present paper we obtain the asymptotic formula for eigen values of the following boundary value problems

$$-u''(x)+Au(x)=\lambda u(x), \quad x\in(0,b),$$

$$u'(0)-\lambda u(0)=0, \quad u'(b)+\lambda u(b)=0,$$

where $A=A^*\geq\omega^2 I$ *in* H, A^{-1} *is completely continuous in* H, $\lambda>0$ *is a spectral parameter,* H *is a separable Hilbert space.*

Let H be a separable Hilbert space. By $\mathcal{L}_2((0,b);H)$ $(0<b<\infty)$ we denote a set of all vector-functions $x\to u(x):(0,b)\to H$, strongly measurable and such that $\int\limits_0^b \|u(x)\|_H^2\ dx<+\infty$. As is known $\mathcal{L}_2((0,b);H)$ is a Hilbert space with respect to the scalar product

$$(u,\upsilon)_{\mathcal{L}_2((0,b);H)}=\int\limits_0^b (u(x),\upsilon(x))_H\ dx.$$

Let A be a self-adjoint positive-definite operator in H $(A=A^*\geq\omega^2 I,\omega>0,\ I$ be a unit operator in $H)$ with domain of definition $D(A)$. Since A^{-1} is bounded in H, then

$$H(A)=\left\{u:u\in D(A),\|u\|_{H(A)}=\|Au\|_H\right\}$$

is a Hilbert space whose norm is equivalent to the norm of the graph of the operator A.

$$W_2^2((0,b);H(A),H)=\left\{u:Au,u''\in\mathcal{L}_2((0,b);H);\right.$$

$\left.\|u\|^2_{W_2^2((0,b);H(A).H)}=\|Au\|^2_{L_2((0,b);H)}+\|u''\|_{L_2((0,b);H)}\right\}$ is a Hilbert space [1, p.23]. In the space $\mathcal{L}_2((0,b);H)$ let's consider the boundary value problem

$$-u''(x)+Au(x)=\lambda u(x), \quad x\in(0,b), \tag{1}$$

$$u'(0)-\lambda u(0)=0,$$

$$u'(b) + \lambda u(b) = 0, \tag{2}$$

where $A = A^* \geq \omega^2 I$ in H, A^{-1} is completely continuous in H, $\lambda > 0$ is a spectral parameter.

The goal of the paper is to study the asymptotic behavior of eigen-values of problem (1)-(2) knowing the asymptotic distribution of eigen-values of the operator A.

Note that the asymptotics of eigen-values of boundary value problems for Sturm-Liouville differential operator equation on a finite segment with the same spectral parameter in the equation and in one of the boundary conditions was studied in papers of V.I. Gorbachuk and M.A.Rybak [2], M.A.Rybak [3]. More precisely, in particular in the papers [2]-[3] it was studied the asymptotic distribution of eigen-values of the following boundary-value problem

$$-u''(x) + Au(x) = \lambda u(x), \quad x \in (0, b), \tag{1}$$

$$u'(0) + \lambda u(0) = 0, \tag{3}$$

$$u(b) = 0$$

in $\mathcal{L}_2((0, b); H) \oplus H$. It is proved that if the spectrum of the operator A is discrete, then the operator generated by the boundary value problem (1), (3) have also discrete spectrum. Eigen-values of the problem (1), (3) form two infinite sequences $\lambda_k \sim \sqrt{\mu_k}$; $\lambda_{n,k} \sim \mu_k + \frac{\pi^2}{b^2} n^2$, where $\mu_k = \mu_k(A)$are eigen-values of the operator A.

In the paper substantially using the ideas and method of the papers [2], [3] it is proved that the problem (1)-(2) has only one set of eigen-values: $\lambda_{n,k} \sim \mu_k + \frac{\pi^2}{b^2} n^2$, where μ_k are the eigen-values of the operator.

Theorem 1. *Let $A = A^* \geq \omega^2 I$ in H and A^{-1} be completely continuous in H. Then for the eigen-values of the problem (1)-(2) it is valid the following asymptotic formula*

$$\lambda_{n,k} = \mu_k + \xi_n \quad (k = 1, 2, ...; \ n = N, N + 1, ...),$$

where $\mu_k = \mu_k(A)$ are the eigen-values of the operator $A, \xi_n \sim \frac{\pi^2}{b^2} n^2$, N is a natural number.

Proof. The eigen-elements of the operator A that correspond to eigen-values $\mu_k(A)$ we denote by φ_k. It is known that $\{\varphi_k\}$ forms an orthonormed basis in H. Then allowing for spectral expansion, for coefficients $u_k = (u, \varphi_k)$ we get the following problem

$$-u_k''(x) + (\mu_k - \lambda) u_k(x) = 0, \tag{4}$$

$$u_k'(0) - \lambda u_k(0) = 0,$$

$$u_k'(b) + \lambda u_k(b) = 0. \tag{5}$$

The general solution of ordinary differential equations (4) is of the form:

$$u_k(x) = c_1 e^{-x\sqrt{\mu_k-\lambda}} + c_2 e^{-(b-x)\sqrt{\mu_k-\lambda}}, \tag{6}$$

where $c_i\ (i = 1, 2)$ are arbitrary constants.

Having put (6) into (5) we get a system with respect to $c_i\ (i = 1, 2)$ whose determinant is of the form

$$K(\lambda) = -\left(\sqrt{\mu_k - \lambda} + \lambda\right)^2 + \left(\sqrt{\mu_k - \lambda} - \lambda\right)^2 e^{-2b\sqrt{\mu_k-\lambda}}.$$

So, the eigen-values of the problem (1)-(2) these zero are for the equation

$$e^{2b\sqrt{\mu_k-\lambda}}\left(\sqrt{\mu_k - \lambda} + \lambda\right)^2 - \left(\sqrt{\mu_k - \lambda} - \lambda\right)^2 = 0. \tag{7}$$

Write equation (7) in the form of a system of equations

$$e^{b\sqrt{\mu_k-\lambda}}\left(\sqrt{\mu_k - \lambda} + \lambda\right) - \left(\sqrt{\mu_k - \lambda} - \lambda\right) = 0, \tag{8}$$

$$e^{b\sqrt{\mu_k-\lambda}}\left(\sqrt{\mu_k - \lambda} + \lambda\right) + \left(\sqrt{\mu_k - \lambda} - \lambda\right) = 0. \tag{9}$$

Thus, the eigen-values of the problem (1)-(2) consist of that real $\lambda \neq \mu_k$ that even at k satisfy the equations (8)-(9).

Look for the eigen-values of the problem (1)-(2) lesser than μ_k. Put $\sqrt{\mu_k - \lambda} = y$. Equations (8) and (9) in this case are equivalent to the equations

$$e^{by}\left(y^2 - y - \mu_k\right) + \left(y^2 + y - \mu_k\right) = 0,\ 0 < y < \sqrt{\mu_k}, \tag{10}$$

$$e^{by}\left(y^2 - y - \mu_k\right) - \left(y^2 + y - \mu_k\right) = 0,\ 0 < y < \sqrt{\mu_k}, \tag{11}$$

respectively.

Let's prove the absence of solutions of equations (10) and (11) on the interval $\left(0, \sqrt{\mu_k}\right)$. Rewrite equation (10) in the form

$$y\ sh\frac{by}{2} + \left(\mu_k - y^2\right) ch\frac{by}{2} = 0,\quad 0 < y < \sqrt{\mu_k}. \tag{12}$$

Let's consider the function $\Phi_k(y) = y\ sh\frac{by}{2} + \left(\mu_k - y^2\right) ch\frac{by}{2},\quad 0 < y < \sqrt{\mu_k}$. Obviously, at each k and for all $y \in \left(0, \sqrt{\mu_k}\right)$ $\Phi_k(y) > 0$. Therefore, equation (12), consequently equation (10) has no solutions on the interval k for any $\left(0, \sqrt{\mu_k}\right)$.

In a similar way it is shown that equation (11) has no solutions on the interval $\left(0, \sqrt{\mu_k}\right)$ for any k. Thus, equation (7) has no zeros in the case when $\lambda < \mu_k$. Now let's study that solutions of equation (7) that are greater than $\lambda > \mu_k$. In this case equations (8) and (9) are equivalent to the equations

$$e^{ibz}\left(z^2 + iz + \mu_k\right) + \left(-z^2 + iz - \mu_k\right) = 0,\ z \in (0, +\infty), \tag{13}$$

$$e^{ibz}\left(z^2+iz+\mu_k\right)-\left(-z^2+iz-\mu_k\right)=0,\ z\in(0,+\infty), \tag{14}$$

respectively, where $z=\sqrt{\lambda-\mu_k}$.

The left hand side of equation (13) is a quasipolynomial. Applying Langer's theory on finding the asymptotics of zeros of a quasipolynomial [4, p.427] we find the asymptotics of (13). Equation (13) is equivalent to the condition

$$e^{ibz}=1+O\left(\frac{1}{z}\right). \tag{15}$$

The zeros of equation (15) at sufficiently great z are close (i.e. approximately equal) to the zeros of the equation

$$e^{ibz}=1.$$

The zeros of the last equation are of the form:

$$z_n=\frac{2\pi n}{b},\ n=1,2,\dots\ .$$

Hence for the zeros of equation (13) and hereby for the zeros of equation (8) we get the asymptotic formulae

$$\lambda_{n,k}^{(1)}=\mu_k+\xi_n, k=1,2,...;\quad n=N,\ N+1,... \tag{16}$$

where $\xi_n\sim\frac{\pi^2}{b^2}(2n)^2$, N is a natural number.

Equation (14) is equivalent to the equation

$$e^{ibz}=-1+O\left(\frac{1}{z}\right). \tag{17}$$

The zeros of equation (17) are found in a similar way. They are of the form:

$$\lambda_{n,k}^{(2)}\sim\mu_k+\frac{\pi^2}{b^2}(2n+1)^2,\ k=1,2,...;\ n=N,N+1...\ . \tag{18}$$

From (16) and (18) we get the asymptotic formula for the eigen-values of problem (1)-(2):

$$\lambda_{n,k}\sim\mu_k+\frac{\pi^2}{b^2}n^2,\quad k=1,2,...;\ n=N,N+1....$$

Theorem 1 is proved.

Corollary 1. Let the conditions of theorem 1 be fulfilled.

Let the eigen-values of the operator A arranged increase order satisfy the condition

$$\mu_k(A)\sim ak^\alpha\left(\lim_{k\to\infty}\frac{\mu_k(A)}{k^\alpha}=a, 0<a,\alpha<\infty\right)$$

then the eigen-values of problem (1)-(2) have the following asymptotics

$$\lambda_m \sim dm^{\frac{2\alpha}{2+\alpha}},$$

where $0 < d < \infty$ are constants.

The proof of Corollary 1 follows from the following statement that is in the paper [5] (see also [6]).

Lemma. *Let be given the two sequences* $\{\mu_k\}$ *and* $\{\nu_n\}$ *such that* $\mu_k \sim ak^{\alpha}, \nu_n = cn^{\beta}$, $0 < a, c, \alpha, \beta < \infty; k = 1, 2, ...; n = 1, 2, ...$ *Compose the sum* $\mu_k + \nu_n$ *with all possible* k *and* n. *We numerate the obtained numbers by increase and denote by* λ_m. *Then the sequences* $\{\lambda_m\}$ *have the asymptotics* $\lambda_m \sim dm^{\delta}$, *where*

$$\delta = \frac{\alpha\beta}{\alpha+\beta}; \; d = \left(\frac{\alpha}{2\gamma}\right)^{\delta} a^{\frac{\beta}{\alpha+\beta}} c^{\frac{\alpha}{\alpha+\beta}}, \gamma = \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{\alpha}-1} t \cos^{1+\frac{2}{\beta}} t dt.$$

Example. In the rectangle $[0, b] \times [0, 1]$, $(0 < b < \infty)$ consider the eigen-value problem

$$-\frac{\partial^2 \upsilon(x,y)}{\partial x^2} + \frac{\partial^4 \upsilon(x,y)}{\partial y^4} + \omega\upsilon(x,y) = \lambda\upsilon(x,y), \tag{19}$$

$$\frac{\partial \upsilon(0,y)}{\partial x} - \lambda\upsilon(0,y) = 0, \tag{20}$$

$$\frac{\partial \upsilon(b,y)}{\partial x} + \lambda\upsilon(b,y) = 0, \tag{21}$$

$$\upsilon_y^{(j)}(x,0) = \upsilon_y^{(j)}(x,1), \quad j = \overline{0,3},$$

where $\omega > 0$ is a number.

The problem (19)-(20) is led to the form (1)-(2) where $u(x,\cdot)$ is a vector-function with values in a Hilbert space $H = \mathcal{L}_2(0,1)$. The operaotr A is determined as follows:

$$D(A) = W_2^4\left((0,1); u^{(j)}(0) = u^{(j)}(1), j = \overline{0,3}\right), Au = \frac{d^4 u}{dy^4} + \omega u, \tag{22}$$

Obviously, the operator A determined by formula (27) is self-adjoint and for sufficiently great $\omega > 0$ is positive-definite, a A^{-1} is completely continuous in $\mathcal{L}_2(0,1)$.

Simple calculations show that the eigen-values of the operator A are of the form: $\mu_k(A) = 16k^4 + \omega$. Then on the basis of corollary 1 the eigen-values of the problem (19)-(21) have the asymptotics: $\lambda_m \sim const\; m^{\frac{4}{3}}$

The author is thankful to professors S.Ya. Yakubov and B.A. Iskenderov for their useful discussions and advices.

References

[1]. Lions J.L., E. Majenes. *Homogeneous boundary-value problems and their applications.* M.: "Mir", 1971. (Russian)

[2]. Gorbachuk V.I., Rybak M.A. *On boundary-value problems for the Shturm-Liouville operator equation with spectral parameters in the equations and in the boundary-value conditions. Direct and inverse problems of the scattering theory.* Kiev, 1981, pp. 3-16. (Russian)

[3]. Rybak M.A. *On asymptotic eigen-value distributions of some boundary-value problems for the Shutrm-Liouville operator equation.* Ukr.Math.Journ. 1980, v.32., No2, pp.248-252. (Russian)

[4]. R.Belman, K.L.Kuk. *Differential-difference equations.* M.: “Mir”, 1967. (Russian)

[5]. Gorbachuk V.I., Corbachuk M.L. *On some boundary-value problems for the Sturm-Liouville equation with potential operator.* Ukr.Math.Journ., 1972, v.24, No3, pp.291-305. (Russian)

[6]. Mamedov K.S. *Asymptotics of the function of distribution of eigen-numbers of abstract differential operator.* Math.Zametky. 1982, v.31, No1, pp.41-51. (Russian)

Bahram A. Aliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 468 64 62 (apt.); 439 47 20 (off.)

Received June 01, 2005; Revised September 06, 2005.
Translated by Nazirova S.H.