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THE ASYMPTOTIC FORM OF THE SOLUTION OF BOUNDARY-VALUE PROBLEM FOR QUASILINEAR ELLIPTIC EQUATION IN THE RECTANGULAR DOMAIN

Abstract

In the domain with angular points the first boundary-value problem is considered for the second order quasilinear elliptic equation, containing small parameter at higher derivatives and degenerating to the non-linear hyperbolic equation. The asymptotic expansion of generalized solution of the considered problem is constructed with any exactness and the remained term is estimated.

By studying many real phenomena, where there are non-uniform passages from one physical characteristics to the another ones, we have to research the singular perturbed problems. Such boundary-value problems are well studied for linear differential equations. Non-linear singular perturbed boundary-value problems are not enough studied. Let's mark here the papers [1]-[9].

In the present paper in the rectangle $D = \{(x, y)\,|0 < x < 1, 0 < y < 1\}$ the following boundary-value problem is considered:

$$L_\varepsilon U \equiv -\varepsilon^p \left[\frac{\partial}{\partial x} \left(\frac{\partial U}{\partial x} \right)^p + \frac{\partial}{\partial y} \left(\frac{\partial U}{\partial y} \right)^p \right] -$$

$$-\varepsilon \Delta U + \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + F(x, U) = 0, \tag{1}$$

$$U\,|_\Gamma = 0, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, $p = 2k+1$, k is an arbitrary natural number, Γ is a boundary of the domain D, Δ is a Laplacian operator, $F(x, y, U)$ is a given smooth function, which can depend on U both linearly, i.e. $F(x, y, U) = a(x, y)\,U - f(x, y)$, $\left(a(x, y) > 0,\ (x, y) \in \overline{D}\right)$ and nonlinearly. It is supposed, that $F(x, y, U)$ satisfies conditions

$$F(x, y, 0) \neq 0 \quad \text{at} \quad (x, y) \in \overline{D}, \tag{3}$$

$$\frac{\partial F(x, y, U)}{\partial U} \geq \gamma^2 > 0 \quad \text{at} \quad (x, u, U) \in \overline{D} \times (-\infty, +\infty). \tag{4}$$

It is known, that at every fixed ε there exists a unique solution of problem (1), (2) in the class $\overset{\circ}{W}{}^1_{p+1}(D)$ (see [10]).

Our purpose in to construct the asymptotic expansion of generalized solution of problem (1), (2).

For constructing the asymptotic form we'll perform the iteration processes. In the first iteration process we'll search the approximate solution of equation (1) in the form

$$W = W_0 + \varepsilon W_1 + ... + \varepsilon^n W_n, \tag{5}$$

at that the functions $W_i(x, y)$; $i = 0, 1, ...n$ will be chosen so, that

$$L_\varepsilon W = 0\left(\varepsilon^{n+1}\right). \tag{6}$$

Substituting (5) to (6), expanding at the point (x, y, W_0) the function $F(x, y, W_0)$ in powers of ε and equating the members with the same powers of ε, for definition of W_i; $i = 0, 1, ..., n$ we'll obtain the following recurrently connected equations:

$$\frac{\partial W_0}{\partial x} + \frac{\partial W_0}{\partial y} + F(x, y, W_0) = 0, \tag{7}$$

$$\frac{\partial W_i}{\partial x} + \frac{\partial W_i}{\partial y} + \frac{\partial F(x, y, W_0)}{\partial W_0} W_i = f_i; i = 1, 2, ..., n, \tag{8}$$

where the functions f_i depend only on the first and the second derivatives $W_0, W_1, ..., W_{i-1}$; $i = 1, 2, ..., n$.

Equations (7), (8) are solved under the following boundary-value conditions:

$$W_0 \left.\right|_{x=0} = 0, \quad W_0 \left.\right|_{y=0} = 0, \tag{9}$$

$$W_i \left.\right|_{x=0} = 0, \quad W_i \left.\right|_{y=0} = 0, \tag{10}$$

(7), (9) is called the degenerated problem, corresponding to problem (1), (2). It is true the following

Theorem 1. *Let $F(x, y, U) \in C^k\left(\overline{D} \times (-\infty, +\infty)\right)$ and satisfy the condition*

$$\left.\frac{\partial^i f(x, y)}{\partial x^{i_1} \partial y^{i_2}}\right|_{x=y} = 0;\ i = i_1 + i_2;\ i = 0, 1, ..., k,\ (0 \le x \le 1) \tag{11}$$

in case of linear dependence of F on U, condition

$$\left.\frac{\partial^i F(x, y, 0)}{\partial x^{i_1} \partial y^{i_2} \partial U^{i_3}}\right|_{x=y} = 0; \quad i = i_1 + i_2 + i_3; \quad i = 0, 1, ..., k, \quad (0 \le x \le 1) \tag{12}$$

in case of nonlinear dependence of F on U. Then problem (7), (9) has a unique solution, at that $W_0(x, y) \in C^k\left(\overline{D}\right)$ and

$$\left.\frac{\partial^i W_0(x, y)}{\partial x^{i_1} \partial y^{i_2}}\right|_{x=y} = 0;\ i = i_1 + i_2;\ i = 0, 1, ..., k,\ (0 \le x \le 1). \tag{13}$$

The proof of theorem 1 is done similarly to the proof of theorem 1 in the paper [9]. Let us mark here only some differences. When the function $F(t, x, W_0)$ linearly depends on W_0 the solution of problem (7), (9) is represented in the form

$$W_0=\begin{cases}\int\limits_0^x f(\xi,\xi+y-x)\exp\left[-\int\limits_\xi^x a(\tau,\tau+y-x)\,d\tau\right]d\xi,\\ 0\le x<y\le 1,\\ \int\limits_0^y f(x-y+\xi,\xi)\exp\left[-\int\limits_\xi^y a(x-y+\xi,\xi)\,d\tau\right]d\xi,\\ 0\le y<x\le 1.\end{cases}\qquad(14)$$

From (14) it follows, that the derivatives $W_0(x,y)$ can have ordinary discontinuities at $x=y$. If $f(x,y)$ satisfies condition (11), then $W_0(x,y)\in C^k\left(\overline{D}\right)$ and condition (13) is satisfied.

In case of nonlinear dependence of the function $F(x,y,W_0)$ on W_0 problem (7), (9) by the obvious transformations decomposes into two Cauchy problems for ordinary differential equations, whose proof on the existence of the local solutions is not difficult. Then with the help of a priori estimations the continuous extendability of these solutions to the triangles $D_1=\{(x,y)\in\overline{D}\,|x<y\}$ and $D_1=\{(x,y)\in\overline{D}\,|x<y\}$ is proved.

To study the differential properties of solution of problems (7), (9) in non-linear case let's represent it in the following form

$$W_0(x,y)=\begin{cases}-\int\limits_0^x F(\tau,\tau+y-x,W_0)\,d\tau,\ 0\le x<y\le 1,\\ -\int\limits_0^x F(x-y+\tau,\tau,W_0)\,d\tau,\ 0\le x<y\le 1.\end{cases}\qquad(15)$$

Using formula (15), it is possible to prove, that if condition (12) is satisfied, then $W_0(x,y)\in C^k\left(\bar{D}\right)$, and (13) is fulfilled.

Theorem 1 is proved.

Problems (8), (10), from which the functions W_i $i=1,2,...,n$ will be defined in turn are linear. The solution of these problems can be written in an evident form by formula (14).

If in theorem 1 we take $k=2n+2$, then from this theorem in case of linear dependence of $F(x,y,U)$ on U it follows, that $W_i\in C^{2(n-i)+2}\left(\bar{D}\right);\ i=0,1,...,n$. And from here and from (5) it implies, that $W\in C^2\left(\bar{D}\right)$. Hence, on the constructed function W we can act by the operator L.

So, we constructed the function W, which is an approximate solution of equation (1) in the sense of (6), and satisfies the boundary conditions:

$$W|_{x=0}=0,\ (0\le y\le 1),\quad W|_{y=0}=0,\quad (0\le x\le 1).\qquad(16)$$

The constructed approximate solution doesn't satisfy, generally speaking, homogeneous boundary conditions at $x=1$ and $y=1$. Therefore, we should construct the functions of type of boundary layers near the boundaries $x=1$ and $y=1$. Let's describe the scheme of construction of boundary layer functions.

First of all, making the iteration process, to the function W we'll add the function

$$V=V_0+\varepsilon V_1+...+\varepsilon^{n+1}V_{n+1}\qquad(17)$$

of type of boundary layer near the boundary $x=1$ such that the obtained sum $W+V$ would satisfy the boundary condition

$$(W+V)\left.\right|_{x=1}=0 \tag{18}$$

Besides, by choosing V the equality

$$L_{\varepsilon,1}(W+V)-L_{\varepsilon,1}W=0\left(\varepsilon^{n+1}\right), \tag{19}$$

should be fulfilled, where $L_{\varepsilon,1}$ is a new splitting of the operator L_{ε} near the boundary $x=1$, which we'll cite below.

Let us mark, that owing to the smoothing functions, V doesn't disturb the fulfilment of the first condition from (16), i.e. the sum $W+V$ will satisfy the condition

$$(W+V)\left.\right|_{x=0}=0. \tag{20}$$

But the function V can disturb the fulfilment of the second condition from (16). For the of condition

$$(W+V)\left.\right|_{y=0}=0, \tag{21}$$

to be fulfilled, we should provide the equality to zero of all functions V_j at $y=0$, i.e.

$$V_j|_{y=0}=0;\quad j=0,1,...,n+1; \tag{22}$$

Will be constructed then there the function

$$\eta=\eta_0+\varepsilon\eta_1+...+\varepsilon^{n+1}\eta_{n+1} \tag{23}$$

of type of boundary layer near the boundary $y=1$, which provides the fulfillment of the boundary condition

$$(W+V+\eta)\left.\right|_{y=1}=0. \tag{24}$$

At that, the equations, form which there will be defined the functions η_j; $j=0,1,...,n+1$, will be obtained from the equality

$$L_{\varepsilon,2}(W+V+\eta)-L_{\varepsilon,2}(W+V)=O\left(\varepsilon^{n+1}\right), \tag{25}$$

where $L_{\varepsilon,2}$ is an another splitting of the operator L_{ε} near the boundary $y=1$, which we should write later.

The function η will not disturb condition (21). Therefore the condition

$$(W+V+\eta)\left.\right|_{y=0}=0 \tag{26}$$

will be fulfilled.

But when we add the fucntion η to the sum $W+V$, for the obtained new sum conditions (18) and (20) may be disturbed. If for the fucntions η_j there will be satisfied the conditions

$$\eta_j|_{x=0}=\eta_j|_{x=1}=0;\quad j=0,1,...,n+1, \tag{27}$$

then, by virtue of (18), 20), (23), the sum $W+V+\eta$, besides the conditions (24), (26) will also satisfy the conditions

$$(W+V+\eta)\left|_{x=0}=(W+V+\eta)\right|_{x=1}=0. \tag{28}$$

Let's start to construct the functions of V_j of type of boundary layers near the boundary $x=1$, that are the members of expansions (17). For writing new splitting of the operator L_ε near the boundary $x=1$ we'll replace the variables $1-x=\varepsilon\tau$, $y=y$. Consider the auxiliary function

$$r=\sum_{j=0}^{n+1}\varepsilon^j r_j\left(\tau,y\right),$$

where $r_j\left(\tau,y\right)$ are some functions, defined near $x=1$. Taking into account this replacement, substituting the expressions r in $L_\varepsilon r$, expanding at the point $(1,y,r_0)$ the function $F\left(1-\varepsilon\tau,y,r\right)$ in powers of ε, after definite transformations we obtain the expansion of $L_\varepsilon\left(r\right)$ in the coordinates (τ,y) in the form

$$L_{\varepsilon,1}r\equiv\varepsilon^{-1}\left\{-\left[\frac{\partial}{\partial\tau}\left(\frac{\partial r_0}{\partial\tau}\right)^{2k+1}+\frac{\partial^2 r_0}{\partial\tau^2}+\frac{\partial r_0}{\partial\tau}\right]+\right.$$
$$+\sum_{j=1}^{n+1}\varepsilon^j\left[-\frac{\partial}{\partial\tau}\left(\left((2k+1)\left(\frac{\partial r_0}{\partial\tau}\right)^{2k}+1\right)\frac{\partial r_j}{\partial\tau}\right)-\right.$$
$$\left.\left.-\frac{\partial r_j}{\partial\tau}+H_j\left(r_0,r_1,...,r_{j-1}\right)\right]+0\left(\varepsilon^{n+2}\right)\right\}, \tag{29}$$

where H_j are the known functions, dependent on $\tau,y,r_0,r_1,...,r_{j-1}$ and their first and second derivatives. For example,

$$H_1\left(r_0\right)=\frac{\partial r_0}{\partial y}+F\left(1,y,r_0\right), \tag{30}$$

$$H_2\left(r_0\right)=\frac{(2k+1)\,(2k)}{2!}\frac{\partial}{\partial\tau}\left[\left(\frac{\partial r_0}{\partial y}\right)^{2k-1}\left(\frac{\partial r_1}{\partial\tau}\right)^2\right]+\frac{\partial r_1}{\partial y}-$$
$$-\frac{\partial F\left(1,y,r_0\right)}{\partial x}\tau-\frac{\partial\left(1,y,r_0\right)}{\partial r_0}r_1-\frac{\partial^2 r_0}{\partial y^2}\ . \tag{31}$$

The formula for the remaining H_j can be rewritten evidently.

Expanding each function $W_i\left(1-\varepsilon\tau,y\right)$ by the Tylor formula at the point $(1,y)$, constructed in the first iteration process, we obtain a new expansion in powers of ε of the function W in the form

$$W=\sum_{j=0}^{n+1}\varepsilon^j\omega_j\left(\tau,y\right)+o\left(\varepsilon^{n+2}\right), \tag{32}$$

where $\omega_0=W_0\left(1,y\right)$ doesn't depend on τ, and the remaining functions ω_k are defined by the formula:

$$\omega_k=\sum_{i+j=k}\left(-1\right)^i\frac{\partial^i W_j\left(1,y\right)}{\partial x^i}\tau^i;\ \ k=1,2,...,n+1.$$

We should take into account, that expansion (17) of the function V was written in the coordinates (τ, y), i.e.:

$$V = \sum_{j=0}^{n+1} \varepsilon^j V_j (\tau, y). \tag{33}$$

From (19), (29), (32), (33) it follows, that the functions V_j, included into the right hand side of (17) are solutions of the following equations respectively:

$$\frac{\partial}{\partial \tau} \left(\frac{\partial V_0}{\partial \tau} \right)^{2k+1} + \frac{\partial^2 V_0}{\partial \tau^2} + \frac{\partial V_0}{\partial \tau} = 0, \tag{34}$$

$$\frac{\partial}{\partial \tau} \left\{ \left[(2k+1) \left(\frac{\partial V_0}{\partial \tau} \right)^{2k} + 1 \right] \frac{V_j}{\partial \tau} \right\} + \frac{\partial V_j}{\partial \tau} = Q_j, \tag{35}$$

where Q_j are the known functions, dependent on τ, y, V_0, V_1, ..., V_{j-1}, ω_0, ω_1,... ,ω_{j-1}, and their first and second derivatives. For example,

$$Q_1 = -\frac{\partial}{\partial \tau} \left[(2k+1) \left(\frac{\partial V_0}{\partial \tau} \right)^{2k} \frac{\partial \omega_1}{\partial \tau} \right] +$$

$$+ \frac{\partial V_0}{\partial y} + F(1, y, \omega_0 + V_0) - F(1, y, \omega_0), \tag{36}$$

$$Q_2 = -\frac{\partial}{\partial \tau} \left[(2k+1) \left(\frac{\partial V_0}{\partial \tau} \right)^{2k} \frac{\partial \omega_2}{\partial \tau} \right] + \frac{(2k+1)(2k)}{2!} \left(\frac{\partial V_0}{\partial y} \right)^{2k-1} \times$$

$$\times \left(\frac{\partial V_1}{\partial \tau} + \frac{\partial \omega_1}{\partial \tau} \right)^2 + \frac{\partial V_1}{\partial y} - \frac{\partial^2 V_0}{\partial y^2} + \left[\frac{\partial F(1, y, V_0 + \omega_0)}{\partial x} - \frac{\partial F(1, y, \omega_0)}{\partial x} \right] \tau +$$

$$+ \left[\frac{\partial F(1, y, V_0 + \omega_0)}{\partial U} - \frac{\partial F(1, y, \omega_0)}{\partial U} \right] \omega_1 + \frac{\partial F(1, y, V_0 + \omega_0)}{\partial U} V_1. \tag{37}$$

Formula for the rest Q_j also can be written in the evident form, however they are sufficiently cumbrous.

The boundary conditions for equations (34), (35) are found by the substitution of expressions for W and V from (5) and (17) to (18) respectively and equating the members with the same powers of ε.

The have the form

$$V_j|_{\tau=0} = -W_j|_{x=1}; \quad j = 0, 1, ..., n; \quad V_{n+1}|_{\tau=0} = 0. \tag{38}$$

At that we search V_j as the function of type of boundary layer ($\lim\limits_{\tau \to \infty} V_j = 0$, $j = 0, 1, ..., n+1$), that replaces the second boundary condition.

It is true the following

Theorem 2. *At any fixed $y \in [0, 1]$ problem (34), (38) at $j = 0$ has a unique solution of type of boundary layer, which is infinitely differentiable with respect to τ and has continuous derivatives by y to the $(2n+2)$th order inclusively, at that the fucntion $V_0(\tau, y)$ and its derivatives exponentially tend to zero as $\tau \to +\infty$.*

Proof. Uniqueness. If $V_0^{(1)}(\tau,y)$, $V_0^{(2)}(\tau,y)$ are two solutions of (34), (38) at $j=0$ of type of boundary layers, then integrating by parts we have

$$\int\limits_0^{+\infty}\left[\left(\frac{\partial V_0^{(1)}}{\partial\tau}\right)^{2k+1}-\left(\frac{\partial V_0^{(2)}}{\partial\tau}\right)^{2k+1}\right]\left(\frac{\partial V_0^{(1)}}{\partial\tau}-\frac{\partial V_0^{(2)}}{\partial\tau}\right)d\tau+$$

$$+\int\limits_0^{+\infty}\left(\frac{\partial V_0^{(1)}}{\partial\tau}-\frac{\partial V_0^{(2)}}{\partial\tau}\right)^2 d\tau=0,$$

whence it follows $V_0^{(1)}(\tau,y)\equiv V_0^{(2)}(\tau,y)$.

The existence of the solution of this problem is proved in the paper [9] (see theorem 2). Here we'll mark, that the solution of problem (34), (38) at $j=0$ in the parametric form has the form

$$\tau=\ln\left|\frac{P_0}{P}\right|+\frac{2k+1}{2k}\left(P_0^{2k}-P^{2k}\right),\ V_0=-P^{2k+1}-P, \tag{39}$$

where P is a parameter, $P_0(y)$ is a real root of the algebraic equation

$$P^{2k+1}+P+\varphi(y)=0 \tag{40}$$

and $\varphi(y)$ is the known function, such that $\varphi(y)=-W_0(1,y)$.

Let us mark, that if $\varphi(y_0)=0$ for some $y_0>0$, then the corresponding real root $P_0=P_0(\varphi(y_0))$ of algebraic equation (40) also vanishes and the expression for τ in (39) looses the sence. On the other hand, at $\varphi(y_0)$, i.e. at $W_0(1,y_0)=0$ as the solution $V_0(\tau,y_0)$ of problem (34), (38) at $j=0$ it is possible to take $V_0(\tau,y_0)\equiv 0$. So, the desired solution of problem (34), (38) at $j=0$ is given in a parametric form (39), if $\varphi(y)\neq 0$ and is defined by the identical zero if $\varphi(y)=0$.

Using the evident form of the solution V_0 it is possible to prove, that the function V_0 is identically differentiable with respect to y, and the estimation

$$\left|\frac{\partial^k V_0}{\partial\tau^k}\right|\le ce^{-\tau},\quad (c>0) \tag{41}$$

is true for all $y\in[0,1]$.

Let us research the behavior of the solution $V_0(\tau,y)$ by y. First of all mark, that as equation (40) has a unique real root $P_0=P_0(\varphi(y))$ at all values of the parameter $y\in[0,1]$, and the function $\varphi(y)=-W_0(1,y)\in C^{2n+2}[0,1]$, then the function $P_0=P_0(\varphi(y))$ will also have continuous derivatives to the $(2n+2)$th order inclusively. From here and from (39) it implies the smoothness by y of the function $V_0(\tau,y)$ for such y for which $\varphi(y)\neq 0$.

Now estimate the derivatives by y of the function $V_0(\tau,y)$ as $\tau\to+\infty$. The function $\dfrac{\partial V_0}{\partial y}\equiv z$ satisfies the equation in variations, which is obtained from equation (34) by differentiation with respect to y:

$$\frac{\partial}{\partial\tau}\left\{\left[(2k+1)\left(\frac{\partial V_0}{\partial\tau}\right)^{2k}+1\right]\frac{\partial z}{\partial\tau}\right\}+\frac{\partial z}{\partial\tau}=0, \tag{42}$$

where V_0 is taken as the known function.

From (38) at $j=0$ we obtain, that the function z must satisfy the condition

$$z|_{\tau=0} = -\frac{\partial W_0\left(1,y\right)}{\partial y} = \varphi'\left(y\right). \tag{43}$$

Let us show that problem (42), (43) has also solution of type of boundary layer. Really having solved the equation

$$\left[(2k+1)\left(\frac{\partial V_0}{\partial \tau}\right)^{2k}+1\right]\frac{\partial z}{\partial \tau}+z=0$$

under conditions (43), we have

$$z=\varphi'\left(y\right)\exp\left\{-\int\limits_0^{\tau}\frac{d\xi}{(2k+1)\left[\dfrac{\partial V_0\left(\xi,y\right)}{\partial \xi}\right]^{2k}+1}\right\}. \tag{44}$$

Using estimation (41), from (44) we have

$$|x|\le|\varphi'\left(y\right)|\exp\left[-\int\limits_0^{\tau}\tfrac{e^{2k\xi}d\xi}{(2k+1)+e^{2k\xi}}\right]=c|\varphi'\left(y\right)|\exp\left[-\tfrac{1}{2}\ln\left|e^{2k\tau}+2k+1\right|\right]=$$

$$=c|\varphi'\left(y\right)|\ \left|e^{2k\tau}+2k+1\right|^{-\frac{1}{2k}}\le c\left|\varphi'\left(y\right)\right|e^{-\tau}. \tag{45}$$

So, the function $z=\dfrac{\partial V}{\partial y}$, given by formula (44), exponentially decreases by y.

By the similar way it is possible to establish estimations for the following derivatives of V_0 by y.

Theorem 2 is proved.

The construction of the functions $V_1, V_2, ..., V_{n+1}$ as the solutions of type of boundary layers of problems (35), (38) at $j=1,2,...,n+1$ relies on the theorem, whose proof is done in the paper [9] (see theorem 3).

Indicate the formulae for the functions V_j, which are solutions of type of boundary layers of problems (35), (38) at $j=1,2,...,n+1$, not coming to a stop on obtaining of this formula

$$V_j=\left[\varphi_j\left(y\right)-\int\limits_0^{\tau}g^{-1}\left(z,y\right)\left(\int\limits_z^{+\infty}Q_j\left(\xi,y\right)d\xi\right)\exp\left(\int\limits_0^{\xi}g^{-1}\left(\xi,y\right)d\xi\right)dz\right]\times$$

$$\times\exp\left(-\int\limits_0^{\tau}g^{-1}\left(\xi,y\right)d\xi\right), \tag{46}$$

where $g\left(\tau,y\right)=(2k+1)\left(\dfrac{\partial V_0}{\partial \tau}\right)^{2k}+1$, $\varphi_j\left(y\right)=-W_j\left(1,y\right);\ j=1,2,...,n;\ \varphi_{n+1}\equiv 0$, such that the estimations of the form

$$\left|\frac{\partial^k V_j}{\partial \tau^k}\right| < ce^{-\tau}\left(a_0 + a_1\tau + a_2\tau^2 + ... + a_j\tau^j\right) \tag{47}$$

are true.

Multiply all functions $V_j; j = 0, 1, ..., n+1$ by the smoothing factor and for the obtained new functions we'll preserve the previous denotation.

So, we have constructed the sum $W + V$, which satisfies conditions (18)-(20). Now find the conditions on $F(x, y, U)$, that will provide the truth of (22), whence (21) will follow.

The condition $V_0|_{y=0} = 0$ is fulfilled. Really, as it was marked above, the real root of algebraic equation (40) at $\varphi(0) = -W_0(1, 0)$ vanishes: $P_0|_{y=0} = 0$. Then, from formula (39) for τ we have

$$|P| = |P_0| \exp(-\tau) \exp\left(-\frac{2k+1}{2k}P^{2k}\right) \exp\left(\frac{2k+1}{2k}P_0^{2k}\right),$$

whence it follows, that $P|_{y=0} = 0$. Hence, from second formula of (39) it implies, that $V_0|_{y=0} = 0$.

From (47) it follows, that for fulfilment of (22) at $j = 1, 2, ..., n+1$ it suffices, that $\varphi_j(0) = 0$, $Q_j(\tau, y)|_{y=0} = 0$.

As $\varphi_j(y) = -W_j(1, y)$; $j = 1, 2, ..., n$; $\varphi_{n+1} \equiv 0$, from (10) it follows, that $\varphi_j(0) = 0$; $j = 1, 2, ..., n+1$.

From (36) it is clear, that if $\left.\dfrac{\partial V_0}{\partial y}\right|_{y=0} = 0$, then $Q_1|_{y=0} = 0$. Formula (44) shows, that if $\varphi'(0) = -\frac{\partial W_0(1,0)}{\partial y} = 0$, then $\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = 0$.

Further researches shows that the fulfilment of all conditions of (22) reduces to the fact that the function $W_0(x, y)$ with all derivatives to the $(2n+2)$th order inclusively must vanish at the angular point $x = 1$, $y = 0$.

When $F(x, y, U)$ linearly depends on U, following formula (14), it is possible to prove, that if $f(x, y)$ satisfies the condition

$$\frac{\partial^i f(1, 0)}{\partial x^{i_1}\partial y^{i_2}} = 0; \quad i = i_1 + i_2; \ i = 0, 1, ..., 2n+1, \tag{48}$$

then

$$\frac{\partial^i W_0(1, 0)}{\partial x^{i_1}\partial y^{i_2}} = 0; \quad i = i_1 + i_2; \quad i = 0, 1, ..., 2n+2. \tag{49}$$

In case of nonlinear dependence of $F(x, y, U)$ on U, following formula (15), it is possible to prove, that if $F(x, y, U)$ satisfies the condition

$$\frac{\partial^i F(1, 0, 0)}{\partial x^{i_1}\partial y^{i_2}} = 0; \quad i = i_1 + i_2; \quad i = 0, 1, ..., 2n+1, \tag{50}$$

(49) will be true.

So, the constructed sum $W + V$, besides (18), (20), satisfies condition (21) also.

The construction of the functions η_j, included into the right-hand side of expansion (23), is closely approximated to procedure of finding the function V_j. Therefore we'll not stop on constructions of η_j. Let us mark just some moments.

Here the replacement of variables near the boundary $y=1$ is done by formulae: $x=x$, $1-y=\varepsilon\xi$. The new splitting $L_{\varepsilon,2}$ of the operator L_ε near the boundary $y=1$ for the function

$$R=\sum_{j=0}^{n+1}\varepsilon^j R_j\left(x,\xi\right)$$

has the form exactly as (29), only τ must be replaced by ξ.

Expanding each function $W_i\left(x,1-\varepsilon\xi\right)$ and $V_j\left(\tau,1-\varepsilon\xi\right)$ by the Teylor formula at the points $(x,1)$ and $(\tau,1)$ respectively from (25) the equations for definition of η_j; $j=0,1,...,n+1$ are obtained. These equations have exactly the same forms as (34), (35), whose right-hand sides for η_j will depend on $\xi,x,\eta_0,\eta_1...,\eta_{j-1}$ and on the members of new expansions in ε of the functions $W_0,W_1,...,W_n$; $V_0,V_1,...,V_{n+1}$, written in the coordinates (x,ξ). The boundary conditions for these equations are found from (24), (5), (17), (23).

In connection with the above-marked, the question on the existence and uniqueness of the solution (of type of boundary layer near the boundary $y=1$) of the problems for η_0 and $\eta_1,\eta_2,...,\eta_{j-1}$ are solved with the help of theorem 2 and theorem 3 in [9]. Multiply the constructed functions $\eta_0,\eta_1,...,\eta_{n+1}$ by the smoothing functions.

The similar discussions, which were done for V_j, show that for fulfilment of (27) it suffices to fulfil the conditions

$$\frac{\partial^i W_0\left(0,1\right)}{\partial x^{i_1}\partial y^{i_2}}=0,\quad \frac{\partial^i W_0\left(1,1\right)}{\partial x^{i_1}\partial y^{i_2}}=0;\quad i=i_1+i_2;\quad i=0,1,...,2n+2. \tag{51}$$

From (11), (12) and (13) at $k=2n+2$ it follows, that the second condition from (51) is fulfilled. And for fulfilment of the first condition from (51) it suffices, that the function $F\left(x,y,U\right)$ satisfies the condition

$$\frac{\partial^i f\left(0,1\right)}{\partial x^{i_1}\partial y^{i_2}}=0;\quad i=i_1+i_2;\quad i=0,1,...,2n+1, \tag{52}$$

in case of linear dependence of F on U, the condition

$$\frac{\partial^i F\left(0,1,0\right)}{\partial x^{i_1}\partial y^{i_2}}=0;\quad i=i_1+i_2;\quad i=0,1,...,2n+1, \tag{53}$$

in case of nonlinear dependence F on U.

So, we have constructed the sum $\tilde{U}=W+V+\eta$, which following (24), (26), (28) vanishes on the boundary Γ.

Denoting $U-\tilde{U}=z$, we'll obtain the following asymptotic expansion in small parameter of solution of problem (1), (2):

$$U=\sum_{i=0}^{n}\varepsilon^i W_i+\sum_{j=0}^{n+1}\varepsilon^j V_j+\sum_{j=0}^{n+1}\varepsilon^j\eta_j+z, \tag{54}$$

where z is a remainder term.

Estimate the remainder term z.

Adding (6), (19), (25) we have that $\tilde{U}$ satisfies the equation

$$L_\varepsilon \tilde{U} = 0\left(\varepsilon^{n+1}\right). \tag{55}$$

It is obvious, that z satisfies the homogeneous boundary condition

$$z|_\Gamma = 0. \tag{56}$$

Subtracting (55) from (1), having multiplied the both parts of the obtained equality by $z = U - \tilde{U}$, and integrating by parts the both parts, taking into account conditions (56), after definite transformations we obtain the estimation

$$\varepsilon^{2k+1}\int\int_D \left[\left(\frac{\partial z}{\partial x}\right)^{2k+2} + \left(\frac{\partial z}{\partial y}\right)^{2k+2}\right] dxdy + \varepsilon\int\int_D \left[\left(\frac{\partial z}{\partial x}\right)^{2} + \left(\frac{\partial z}{\partial y}\right)^{2}\right] dxdy +$$

$$+C_1\int\int_D z^2 dxdy \le C_2\varepsilon^{2(n+1)}, \tag{57}$$

where $C_1 > 0,\ C_2 > 0$ are constants, that don't depend on ε.

Unifying the obtained results we come to the following statement.

Theorem 3. *Let $F(x, y, U) \subset C^{2n+2}\left(\bar{D} \times (-\infty; +\infty)\right)$ satisfy conditions (3), (4), and (11) at $k = 2n+2$, (48), (52) in case, when F depends on U linearly, condition (12) at $k = 2n+2$, (50), (53), in case, when F depends on U nonlinearly. Then for the generalized solution of problem (1), (2) asymptotic representation (54) holds, where the functions W_i are defined by the first iteration process, V_j, η_j are the functions of type of boundary layer near the boundaries $x = 1$ and $y = 1$, respectively, which are defined by the appropriate iteration processes, z is a remainder term, and estimation (57) is true for it.*

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