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ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH δ'-INTERACTION

Abstract

In the present paper some self-adjoint operator A that is one of possible realizations of the formal operator $-\frac{d^2}{dx^2}+\beta\delta'(x)$ is defined in the space $L_2(R)$. The spectrum of the operator A is investigated.

Spectral properties of one-dimensional Schrödinger operator with δ'-interactions and some of their generalizations are studied in [1-7].

Recall the definition of one-center δ'-interaction in one-dimensional case for the Schrödinger operator ([1], ch.I.4). In the space $L_2(R)$ it is considered the operator

$$E_\beta=-\frac{d^2}{dx^2},\quad \beta\in R=(-\infty,+\infty) \tag{1}$$

with dense domain of definition

$$D(E_\beta)=\left\{f\in W_2^2(R\backslash\{0\}): f'(+0)=f'(-0),\ f(+0)-f(-0)=\beta f'(0)\right\}\ . \tag{2}$$

In quantum mechanics ([1], ch.I.4) the self-adjoint operator E_β by definition describes the δ'-interaction with center at the point zero and corresponds to the formal operator $-\frac{d^2}{dx^2}+\beta\delta'(x)$ in the space $L_2(R)$, where $\delta'(x)$ is the derivative of Dirac's delta-fucntion. As is noted in [5], the self-adjoint operator E_β doesn't describe the δ'-interaction in an ordinary sense. In fact, we can show that the operator E_β coincides with the operator B:

$$Bf=-\frac{d^2f}{dx^2}+\beta f'(0)\,\delta'(x)\,,\quad D(B)=D(E_\beta)\ .$$

We can be convinced in it showing that the resolvents of the operators E_β and B coincide.

In fact the δ'-interaction of the Schrodinger operator may be connected namely with the operator

$$-\frac{d^2}{dx^2}+\beta\delta'(x)\ . \tag{3}$$

Therefore, naturally, there appears the question [5]: which self-adjoint operator in the space $L_2(R)$ is possible realization of formal operator (3)?

In the paper [5] the δ'-interaction of the Schrodinger operator is defined as a limit as $\varepsilon\to+0$ of the operators

$$\tilde{H}_{\alpha,\lambda,\varepsilon}=-\frac{d^2}{dx^2}\dot{+}\frac{\lambda}{\varepsilon^\alpha}\left[\delta(x-\varepsilon)-\delta(x+\varepsilon)\right],\quad \varepsilon>0,\ \alpha>0.$$

It happens that a strong resolvent limit of the operators $\tilde{H}_{\alpha,\lambda,\varepsilon}$ is not connected with operator (3). Therefore, this approach doesn't respond to the stated above question.

To our view, the construction of a selfadjoint operator corresponding to the formal operator (3) is connected with definition of the product $\delta' \cdot f$ where the fucntion $f(x)$ and its derivative $f'(x)$ have the first order discontinuity at the point $x = 0$. Note that for such functions $f(x)$ product $\delta' \cdot f$ is defined amibiguosly.

In the present paper the product $\delta' \cdot f$ is defined by the equality

$$\delta' \cdot f = -\frac{f'(+0) + f'(-0)}{2}\delta(x) + \frac{f(+0) + f(-0)}{2}\delta'(x). \tag{4}$$

In the case $f(x) \in C^1(R)$ formula (4) turns into the known formula

$$\delta' \cdot f = -f'(0)\,\delta(x) + f(0)\,\delta'(x)\ .$$

Now we state the method suggested in this paper for construction of self-adjoint operator A, corresponding to formal operator (3).

Consider in the space $L_2(R)$ the operator

$$Af = -\frac{d^2 f}{dx^2} + \beta\delta'(x)\, f\ , \quad \beta \in R \tag{5}$$

with dense domain of definition $D(A)$ consisting of the functions $f \in W_2^2(R\backslash\{0\})$ and satisfying the boundary conditions

$$\begin{cases} (\beta - 2)\, f(+0) + (\beta + 2)\, f(-0) = 0\ , \\ \\ (\beta + 2)\, f'(+0) + (\beta - 2)\, f'(-0) = 0\ . \end{cases} \tag{6}$$

It is easy to check that A is a closed symmetric operator in the space $L_2(R)$.

In this paper the self-adjointness of A is proved and the spectrum of the operator A is studied. Besides, integrability of the resolvent $R_z(A) = (A - zE)^{-1}$ is established in the space $L_2(R)$.

Theorem 1. *The operator A is self-adjoint in the space $L_2(R)$. The resolvent $R_z(A)$ is an integral operator in $L_2(R)$ with kernel*

$$G(x, y; z) = -\frac{1}{2i\sqrt{z}} e^{i\sqrt{z}|x-y|} +$$

$$+\frac{\beta^2 (1 - sign\ x \cdot sign\ y) - 2\beta (sign\ x + sign\ y)}{2i\sqrt{z}\left(4 + \beta^2\right)} e^{i\sqrt{z}(|x|+|y|)}\ , \tag{7}$$

$$(x, y \in R, \quad z \notin [0, +\infty))\,.$$

Proof. The operator A is close and symmetric. Therefore it suffices to show that if $Jmz \neq 0$ then $z \in \rho(A)$, where $\rho(A)$ is a resolvent set of the operator A.

Find the resolvent of the operator A. In the space $L_2(R)$ solve the equation

$$Af + \lambda^2 f = g\ , \quad g(x) \in L_2(R)\,, \quad \lambda > 0\ .$$

By (4) we can write this equation in the following form

$$-\frac{d^2f}{dx^2}-\beta\frac{f'(+0)+f'(-0)}{2}\delta(x)+\beta\frac{f(+0)+f(-0)}{2}\delta'(x)+\lambda^2 f=g. \qquad (8)$$

Adding Fourier transformation to equation (8) and considering that

$$F\left[\frac{d^2f}{dx^2}\right]=\xi^2F[f]\ ,\quad F[\delta(x)]=1\ ,\quad F[\delta'(x)]=-i\xi\ ,$$

we get

$$F[f]=\frac{f'(+0)+f'(-0)}{2}\frac{\beta}{\xi^2+\lambda^2}+\frac{f(+0)+f(-0)}{2}\frac{i\beta\xi}{\xi^2+\lambda^2}+\frac{1}{\xi^2+\lambda^2}F[g]\ .$$

Now, we apply the inverse Fourier transformation and use the known formulae

$$F^{-1}\left[\frac{\xi}{\xi^2+\lambda^2}\right]=-\frac{i\ sign\ x}{2}e^{-\lambda|x|},\quad F^{-1}\left[\frac{1}{\xi^2+\lambda^2}\right]=\frac{1}{2\lambda}e^{-\lambda|x|}\ ,$$

$$F^{-1}\left[\frac{1}{\xi^2+\lambda^2}F[g]\right]=F^{-1}\left[\frac{1}{\xi^2+\lambda^2}\right]*g\ .$$

Then we find

$$f(x)=\frac{f'(+0)+f'(-0)}{4\lambda}\beta e^{-\lambda|x|}+$$

$$+\frac{f(+0)+f(-0)}{4}\beta\ sign\ x\cdot e^{-\lambda|x|}+\frac{1}{2\lambda}\int\limits_R e^{-\lambda|x-y|}g(y)\,dy. \qquad (9)$$

Find the quantities $f(+0)$, $f(-0)$, $f'(+0)$ and $f'(-0)$. Pass in (9) to limit as $x\to+0$. Then

$$f(+0)=\frac{f'(+0)+f'(-0)}{4\lambda}\beta+\frac{f(+0)+f(-0)}{4}\beta+\frac{1}{2\lambda}\int\limits_R e^{-\lambda|y|}g(y)\,dy\ . \qquad (10)$$

Further we calculate the derivative $f'(x)$ and in the obtained equality we pass to limit as $x\to+0$:

$$f'(+0)=-\frac{f'(+0)+f'(-0)}{4\lambda}\beta-$$

$$-\frac{f(+0)+f(-0)}{4}\beta\lambda+\frac{1}{2}\int\limits_R e^{-\lambda|y|}sign\ y\cdot g(y)\,dy\ . \qquad (11)$$

Solving the system composed of equations (6), (10) and (11) with respect to $f(+0)$, $f(-0)$, $f'(+0)$ and $f'(-0)$ we find

$$f(+0)=\frac{\beta(2+\beta)}{2(4+\beta^2)\lambda}\int\limits_R e^{-\lambda|y|}sign\ y\cdot g(y)\,dy+\frac{2+\beta}{(4+\beta^2)\lambda}\int\limits_R e^{-\lambda|y|}g(y)\,dy\ ,$$

$$f(-0)=\frac{\beta(2-\beta)}{2(4+\beta^2)\lambda}\int\limits_R e^{-\lambda|y|}sign\ y\cdot g(y)\,dy+\frac{2-\beta}{(4+\beta^2)\lambda}\int\limits_R e^{-\lambda|y|}g(y)\,dy\ ,$$

$$f'(+0)=\frac{2-\beta}{4+\beta^2}\int\limits_R e^{-\lambda|y|}sign\ y\cdot g(y)\,dy-\frac{(2-\beta)\beta}{2\left(4+\beta^2\right)}\int\limits_R e^{-\lambda|y|}g(y)\,dy\ ,$$

$$f'(-0)=\frac{2+\beta}{4+\beta^2}\int\limits_R e^{-\lambda|y|}sign\ y\cdot g(y)\,dy-\frac{(2+\beta)\beta}{2\left(4+\beta^2\right)}\int\limits_R e^{-\lambda|y|}g(y)\,dy\ .$$

Taking the obtained expressions into account in (9), after simple calculations we get

$$f(x)=\int\limits_R G\left(x,y;-\lambda^2\right)g(y)\,dy\ , \tag{12}$$

where

$$G\left(x,y;-\lambda^2\right)=\frac{1}{2\lambda}e^{-\lambda|x-y|}-$$
$$-\frac{\beta^2\left(1-sign\ x\cdot sign\ y\right)-2\beta\left(sign\ x+sign\ y\right)}{2\left(4+\beta^2\right)\lambda}e^{-\lambda(|x|+|y|)}\ . \tag{13}$$

If $\lambda\in(0,+\infty)$ and $\beta\in R$ then integral operator (12) is bounded in $L_2(R)$, and consequently, the inverse operator $\left(A+\lambda^2E\right)^{-1}$ exists and bounded in $L_2(R)$. Therefore

$$\left(A+\lambda^2E\right)^{-1}g(x)=\int\limits_R G\left(x,y;-\lambda^2\right)g(y)\,dy,\quad g\in L_2(R)\ ,$$

and $-\lambda^2\in\rho(A)$.

Continuing analytically $G\left(x,y;-\lambda^2\right)$ to a complex plane with crack along a positive semi-axis we get that $R_z(A)$ is an integral operator and the kernel $G(x,y;z)$ has representation (7). As a result, we get that if $Jmz\neq 0$, then $z\in\rho(A)$. Consequently, A is a self-adjoint operator in the space $L_2(R)$. The theorem is proved.

The structure of the spectrum of the operator A is described by the following theorem.

Theorem 2. *The essential spectrum of the operator A coincides with absolute continuous part of the spectrum A, and*

$$\sigma(A)=\sigma_{ess}(A)=\sigma_{ac}(A)=[0,+\infty)\ . \tag{14}$$

Proof. Let A_0 be a minimal operator in $L_2(R)$ generated by the expression $-\dfrac{d^2f}{dx^2}$. It is known that A_0 is a nonnegative self-adjoint operator and the resolvent $R_z(A_0)$, $z\notin[0,+\infty)$ is an integral operator in $L_2(R)$ with kernel

$$G_0(x,y;z)=-\frac{1}{2i\sqrt{z}}e^{i\sqrt{z}|x-y|}\ .$$

The equalities

$$\sigma(A_0)=\sigma_{ess}(A_0)=\sigma_{ac}(A_0)=[0,+\infty)$$

are valid.

Denote

$$B = \left(A_0 + \lambda_0^2 E\right)^{-1} - \left(A + \lambda_0^2 E\right)^{-1}, \quad \lambda_0 > 0, \ -\lambda_0^2 \in \rho\left(A\right) \cap \rho\left(A_0\right) \ .$$

The operator B is an integral operator in $L_2\left(R\right)$ with kernel

$$K\left(x,y\right) = \frac{\beta^2\left(1 - sign\ x \cdot sign\ y\right) - 2\beta\left(sign\ x + sign\ y\right)}{2\left(4+\beta^2\right)\lambda_0} e^{-\lambda_0(|x|+|y|)} \ .$$

Since $K\left(x,y\right) \in L_2\left(R \times R\right)$, then β is a Hilbert-Schmidt operator and consequently is compact. By Weyl's theorem ([8], theorem XIII.14) the essential spectra of the operators A and A_0 coincide:

$$\sigma_{ess}\left(A\right) = \sigma_{ess}\left(A_0\right) = [0, +\infty) \ . \tag{15}$$

It follows from representation (13) that B is a finitedimensional operator. According to the known theorem ([9], ch. X, theorem 4.2) absolutely continuous parts of the operators A and A_0 are unitary equivalent and in particular, absolutely continuous parts of the spectra A and A_0 coincide

$$\sigma_{ac}\left(A\right) = \sigma_{ac}\left(A_0\right) = [0, +\infty) \ .$$

Hence and from (15) we get (14). The theorem is proved.

References

[1]. Albeverio S., Gesztesy F., Hoegh-Krohn R., Holden H. *Solvable models in quantum mechanics.* M.: "Mir", 1991. (Russian)

[2]. Grossmann A., Hoegh-Krohn R.,Mebkhout M. *A class of explicitly soluble, local, many-center Hamiltonians for one-particle quantum mechanics in two and three dimensions.* I. Journ. Math. Phys., 1980, v.21, pp.2376-2385.

[3]. Gesztesy F., Holden H. *A new class of solvable models in quantum mechanics describing point interactions on the line.* Journ. Phys. A. 1987, v.20, pp.5157-5177.

[4]. Šeba P. *The generalized point interaction in one-dimension.* Czech. J. Phys. B. 1986, v.36, pp.667-673.

[5]. Šeba P. *Some remarks on the δ'-interaction in one dimension.* Rep. Math. Phys. 1986, v.24, No1, pp.111-120.

[6]. Mikhailets V.A. *On Schrödinger operator with point δ'-interactions.* Russian Doklady, 1996, v.348, No6, pp.727-730. (Russian)

[7]. Kadiev R.I. (junior) *On a spectrum of a class of Schrodinger operators with finite number distributions.* Izv. vish. ucheb. zavedeniy. Matematika, 1998, No7(434), pp.26-31. (Russian)

[8]. Reed M., Simon B. *Methods of modern mathematical physics.* V.4, Analysis of operators. M.: "Mir", 1982. (Russian)

[9]. Kato T. *Perturbation theory for linear operators.* M.: "Mir", 1972. (Russian)

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Received June 20, 2003; Revised October 16, 2003.
Translated by Nazirova S.H.