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ON SMOOTHNESS OF SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM FOR SECOND ORDER DEGENERATE ELLIPTIC-PARABOLIC EQUATIONS

Abstract

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Introduction. In this work the first boundary value problem is considered for second order degenerate elliptic-parabolic equation with, generally speaking, discontinuous coefficients. The matrix of senior coefficients satisfies the parabolic Cordes condition with respect to space variables. We'll prove that the generalized solution to the problem belongs to Holder space $C^{1+\lambda}$, when the right-hand side $f \in L_p, p > n$.

Investigations of boundary value problems for second order degenerate elliptic-parabolic equations ascend to the work by Keldysh [1], where correct statements for boundary value problems were considered for the case of one space variable as well as existence and uniqueness of solutions. In the work by Fichera [2] boundary value problems were given for multidimensional case. He proved existence of generalized solutions to these boundary value problems. In the work by Oleynik [3] existence and uniqueness of generalized solution to these problems were proved for smooth and piecewise smooth domains. In the case of smooth coefficients and some weighted functions the generalized solvability was studied in [4].

Let R_{n+1} be an $(n+1)$-dimensional Euclidian space of points $(x,t) = = (x_1, x_2, ..., x_n, t), \Omega-$ a bounded n-dimensional domain in R_n with the boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$- a cylinder in R_{n+1} and $T \in (0,\infty)$. $Q_0 = (x,t) : x \in \Omega, t = 0$ and $\Gamma(Q_T) = Q_0 \cup (\partial\Omega \times [0,T])$ - a parabolic boundary of the cylinder Q_T.

Let's consider in Q_T the first boundary value problem for second order degenerate elliptic-parabolic operator

$$Zu = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \psi(x,t)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + (x,t)u - \frac{\partial u}{\partial t} = f(x,t), \quad (1)$$

$$u|_{\Gamma(Q_T)} = 0. \quad (2)$$

Assume that, the coefficients of the operator Z satisfy the following conditions. $\|a_{ij}(x,t)\|$- a real symmetrical matrix with elements measurable in Q_T and for any $(x,t) \in Q_T$ and $\xi \in R_n$ the following inequalities are true

$$\gamma\left|\xi\right|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \gamma^{-1}\left|\xi\right|^2, \quad (3)$$

where $\gamma \in (0,1]$- a constant,

$$\sigma = \frac{\sup_{Q_T} \sum\limits_{i,j=1}^{n} a_{ij}^2(x,t)}{\inf_{Q_T}\left[\sum\limits_{i=1}^{n} a_{ii}(x,t)\right]^2} < \frac{1}{n - \dfrac{1}{2}}, \quad (4)$$

$$c(x,t) \le 0, c(x,t) \in L_{n+1}(Q_T), \tag{5}$$

$$|b(x,t)| \in L_{n+2}(Q_T), \left|b(x,t)^2\right| + Kc(x,t) \le 0. \tag{6}$$

Assume that, the following conditions are true for the weighted function:

$$\psi(x,t) = \lambda(\rho)w(t)\varphi(T-t),$$

where

$$\rho = \rho(x) = dist(x, \partial\Omega), \lambda(\rho) \ge 0, \lambda(\rho) \in C'[0, diam\Omega],$$

$$\left|\lambda'(\rho)\right| \le p\sqrt{\lambda(\rho)}, w(t) \ge 0, w(t) \in C'[0,T],$$

$$\varphi(z) \ge 0, \varphi'(z) \ge 0, \varphi(z) \in C'[0,T], \varphi(0) = \varphi'(0) = 0, \varphi(z) \ge \beta z\varphi'(z), \tag{7}$$

where p, β- positive constants.

The condition (4) is called the condition of Cordes type and is taken within the accuracy of a linear nonspecial transformation. Before we move to the proof of the basic result, let's consider some auxiliary problems.

Let

$$L'u = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} +$$

$$+\psi(x,t)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + b_0(x,t)\frac{\partial u}{\partial t} + c(x,t)u = f(x,t). \tag{8}$$

Wlog, assume that the coefficients are smooth in $\overline{Q_T}$ and their derivatives are bounded. For this purpose it's enough to consider averaged coefficients and a family of boundary value problems, where coefficients are smooth functions. Later we'll discuss all this in detail.

Let

$$L'_\varepsilon u = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} +$$

$$+\psi_\varepsilon(x,t)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + b_0(x,t)\frac{\partial u}{\partial t} + c(x,t)u = f(x,t), \tag{9}$$

where $\psi_\varepsilon(x,t)$ is defined so: for any fixed $\varepsilon \in (0,T)$ $\psi_\varepsilon(z) = \psi(\varepsilon) - \dfrac{\psi'(\varepsilon)\varepsilon}{m} +$ $+\dfrac{\psi'(\varepsilon)}{m\varepsilon^{m-1}}z^m$ at $z \in (0,\varepsilon]$, $\psi_\varepsilon(z) = \psi(z)$ at $z \in [\varepsilon, T]$, $m = \dfrac{2}{\beta}$.

Everywhere further we consider the case, when $\psi(z) > 0$, at $z > 0$. If $\psi(z) \equiv 0$, then the equation (1)- parabolic, and the corresponding result on smoothness of the solution ensues from [5]. But if $\psi(z) = 0$ at $z \in [0, z^0]$, then the solution to the problem (1)-(2) can be got by composition of the solution $u(x,t)$ to the first boundary value problem in the cylinder Q_{z^0} and the solution $v(x,t)$ to the first boundary value problem for the parabolic equation in the cylinder $\Omega \times (z^0, T)$ with the boundary conditions $v(x,z^0) = u(x,z^0), v\big|_{\partial\Omega\times[z^0,T]} = 0$. Note that under the conditions (3)-(6) for the coefficients, the smoothness of the solution results from [6]. Denote by $\sum^0$ the part of Q_T, where $\psi(x,t) = 0$, i.e. where the equation (8) degenerates; by Γ^0- the part of intersection of $\sum^0$ and the boundary Γ, where a tangent plane to the surface Γ is orthogonal to the axis t, i.e. has a characteristical direction.

By maximum principle the solutions $u_\varepsilon(x,t)$ of the equation (9) in the domain satisfy the following estimate

$$|u_\varepsilon(x,t)| \leq \left|\frac{f(x,t)}{c(x,t)}\right|,$$

and that is $u_\varepsilon(x,t)$ are uniformly bounded with respect to ε.

Lemma 1. *The derivatives of the solution $u_\varepsilon(x,t)$ are uniformly bounded on a closed subset of the boundary Γ, that belongs to $\Gamma \backslash \Gamma_0$.*

Proof. Let's take a point $(x',t) \in \Gamma \backslash \Gamma_0$, such that, at the point a tangent plane to Γ is not orthogonal to the axis t, i.e. the surface Γ near the point has an equation of the kind $x_1 = \theta(x_2,...,x_n,t)$, where θ has derivatives up to second order. Let $\chi(x_2,...,x_n,t)$ be twice continuously differentiable function , equal to a positive constant β in some neighborhood of a projection (x',t) onto the plane $(x_2,...,x_n,t)$ and equal to zero in a little greater neighborhood $0 \leq \chi(x_2,...,x_n,t) \leq \beta$. We denote by Q_T^1 the part of Q_T being between the surfaces Γ and $\sigma\,\{x_1 = \theta + \chi\}$. Γ^1 - that part of Γ, where $\chi = \beta$. Consider a function $v = e^{\alpha(-x_1+\theta+\chi)}$. It's obvious, that on the surface σ $v = 1$. Then in Q_T^1 at sufficiently great α

$$L'_\varepsilon(v) \geq \alpha^2\gamma - \alpha\mu - \mu_1 > \frac{\alpha^2\gamma}{2}, \quad L'_\varepsilon(v \pm u_\varepsilon) > \frac{\alpha^2\gamma}{2} - \max_{\overline{Q_T}} |f(x,t)| > 0, \tag{10}$$

where μ, μ_1 are maximums of the modules of the solution itself and its first derivatives within Q_T . Then we choose α independent on ε so, that (10) is true and , besides, $e^{\alpha\beta} > 1+ \max_{\overline{Q_T}} |u_\varepsilon(x,t)|$. It means that on Γ^1 the values of functions $v \pm u_\varepsilon$, equal to $e^{\alpha\beta}$, are more than their values on σ, where $v = 1$ (taking into account that $u_\varepsilon(x,t)|_\Gamma = 0$). By maximum principle it results from the following estimate (10) that functions $v \pm u_\varepsilon$ within the domain Q_T^1 can't take maximal positive value. Whence, they reach maximum on the boundary Γ, i.e. on the part Γ^1 too, while on the other part of Γ $v \pm u_\varepsilon = e^{\alpha\chi} \leq e^{\alpha\beta}$. So, at points, that belong to Γ^1 $\dfrac{\partial(v \pm u_\varepsilon)}{\partial x_1} \leq 0$ or $\left|\dfrac{\partial u_\varepsilon(x,t)}{\partial x_1}\right|_{\Gamma^1} \leq -\left.\dfrac{\partial v}{\partial x_1}\right|_{\Gamma^1} = \alpha e^{\alpha\beta}$. That's on Γ^1 the derivatives $\dfrac{\partial u_\varepsilon(x,t)}{\partial x_1}$ are uniformly bounded. Besides, derivatives of $u_\varepsilon(x,t)$ with respect to directions, lying in a tangent plane, equal zero, as $u_\varepsilon(x,t)\,|_\Gamma = 0$. Thus, the derivatives $\dfrac{\partial u_\varepsilon(x,t)}{\partial x_i}$, $i = \overline{1,n}$ are uniformly bounded with respect to ε on Γ^1. Let's take a point $(x',t) \in \Gamma$
Gamma^0. Let a tangent plane to Γ at this point be orthogonal to the axis t. This case can be proved similarly.

Lemma has been proved.

Remark 1. *If the boundary does not contain points of Γ^0, then $\dfrac{\partial u_\varepsilon(x,t)}{\partial t}$ are uniformly bounded on the entire boundary .*

Lemma 2. *Suppose, that on $\sum^0$ the condition is true*

$$c(x,t) + \frac{\partial b_0(x,t)}{\partial t} < 0 \tag{11}$$

and $\overline{\Sigma^1}$ *is any closed domain with a boundary* σ_1*, which belongs to* $\overline{Q_T}$*. Then at* $(x,t) \in$

$$\overline{\Sigma^1} \sum_{i=1}^{n} \left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i}\right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 \leq$$

$$\leq C \max_{(x,t)\in\sigma_1} \left[\sum_{i=1}^{n} \left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i}\right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2\right] + C_1, \tag{12}$$

where C, C_1*- constants, depending on a structure of the equation.*

Proof. Introduce the denotation $(\overline{\Sigma^1} \cap \Sigma^0) \cap \Sigma^1 = \Sigma^2$. Let's prove the inequality in some neighborhood of closed domain Σ^2. The boundary of Σ^2 consists of the part σ_1 of the boundary Σ^1 and the surface σ_2 being in the part , where $\psi(x,t) > 0$. At points $(x,t) \in \sigma_2$ the inequality will be true

$$\sum_{i=1}^{n} \left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i}\right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 \leq$$

$$\leq C \max_{(x,t)\in\sigma_1} \left[\sum_{i=1}^{n} \left(\frac{\partial u_\varepsilon(x,t)}{\partial x_i}\right)^2 + \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2\right] + C_1. \tag{13}$$

The following estimate (13) is obtained from the fact, that derivatives of the solution are bounded in any closed subdomain for the case of bounded derivatives up to the boundary of a domain. Now if we show, that the following estimate (13) is also true for the domain Σ^2, then from (13) and this following estimate we will get (12) for the domain Σ^1. Assume, that (11) is also fulfilled in Σ^2. For simplicity of calculations we will find following estimates for one space variable and in the end show the changes in calculations in the case of many space variables. Wlog, we take the coefficient at second derivative with respect to a space variable x equal to unit, as it can be easily obtained by division by terms by the coefficient

Denote by $z = \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^m + \alpha_1 \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^{m-2} u_\varepsilon^2(x,t)$. At first, we show, that at corresponding n, α_1 we have $L'_\varepsilon z > 0$ in Σ^2 , if $\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 > \mu_1$. Let n be a positive even number. We get

$$L'_\varepsilon \left[\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^m + \alpha_1 \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^{m-2} u_\varepsilon^2(x,t)\right] = L'_\varepsilon z > 0,$$

if $\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 > \mu_1$. Now if z takes its maximum within Σ^2, then at this point $L_\varepsilon z \leq 0$. So, either $\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 \leq \mu_1$, or the value of z within Σ^2 is not more than the maximum on the boundary Σ^2. As

$$\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 \leq z^{\frac{2}{m}} \leq C_2 \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 + C_3,$$

as

$$\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2\Bigg|_{\Sigma^2} \le z^{\frac{2}{m}}\Bigg|_{\Sigma^2} \le C_2 \max_{\sigma_2\cup\sigma_1} z^{\frac{2}{m}} + C_3 < C_4 \max_{\sigma_2\cup\sigma_1}\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^2 + C_5 \quad (14)$$

$\dfrac{\partial u_\varepsilon(x,t)}{\partial x}$ can be following estimated similarly.

Lemma has been proved.

Lemma 3. *Assume, that on the set Σ^0 the following condition is fulfilled*

$$\frac{\partial^2\psi(x,t)}{\partial t^2} + 2\frac{\partial b_0(x,t)}{\partial t} + c(x,t) < 0 \quad (15)$$

and first derivatives of $u_\varepsilon(x,t)$ are uniformly bounded in a closed domain $\overline{\Sigma^1}\subset\overline{Q_T}$ with the boundary σ. Then

$$\sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2 + \sum_{i,j=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2 + \left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^2 \le$$

$$\le C\max_{(x,t)\in\sigma}\left[\sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2 + \sum_{i,j=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2 + \left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^2\right] + C_1 \quad (16)$$

where C, C_1 do not depend on ε.

Proof. As $(x,t)<0$, $\dfrac{\partial^2\psi}{\partial t^2}\ge 0$ on Σ^0, so the statement of the lemma for first derivatives results form lemma 2. To prove the lemma we, as in the proof of lemma 2, have to show, that in some neighborhood of $\Sigma^0\cap\Sigma^1$: $L'_\varepsilon z_1 > 0$ at the corresponding m (an even number) and α_i. Here z_1 is the same as in lemma 2, but it contains additional terms. An element $\left(\dfrac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m$ is the main in it, so we've to estimate

$$L'_\varepsilon\left[\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m\right] = m\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-1.} L_\varepsilon\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right) +$$

$$+m(m-1)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2.}\sum_{i,j=1}^n a_{ij}(x,t)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_i}\right)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_j}\right) +$$

$$+m(m-1)\left(\frac{\partial u_\varepsilon(x,t)}{\partial t}\right)^{m-2.}\psi_\varepsilon(x,t)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2 - (m-1)(x,t)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m.$$

Taking into account (15) on Σ^0, at sufficiently great m, we've

$$-m\left(\frac{\partial^2\psi_\varepsilon}{\partial t^2} + 2\frac{\partial b_0}{\partial t} + c - \frac{c}{m}\right) > \mu_1 m$$

in some neighborhood of Σ^0. Now we choose $\beta < \mu_1 - \mu_2$, where $\mu_2 > 0$, and fix β. Then

$$L_\varepsilon\left[\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m\right] >$$

$$> m(m-1)\mu\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2.}\sum_{i=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_i}\right)^2+m\mu_2\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m+$$

$$+m(m-1)\mu\psi_\varepsilon(x,t)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2.}\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2-$$

$$-m\mu_3\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2.}\left[\psi_\varepsilon(x,t)\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2+\right.$$

$$\left.+\sum_{i+j>0;i\neq j}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2+\sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2+\sum_{i,j=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial x_i\partial x_j\partial t}\right)^2+1\right].$$

Let's choose sufficiently great m, so that $-m\mu_3+m(m-1)\mu>\mu_3>0$ and fix m. Under this condition

$$L_\varepsilon\left[\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m\right]\geq\mu_4\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2.}\sum_{i=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^2\partial x_i}\right)^2+$$

$$+\mu_5\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^m+\mu_3\psi_\varepsilon(x,t)\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2.}\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2-$$

$$-\mu_4\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial t^2}\right)^{m-2.}\left[\sum_{i+j>0;i\neq j}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial x_j}\right)^2+\sum_{i=1}^n\left(\frac{\partial^2 u_\varepsilon(x,t)}{\partial x_i\partial t}\right)^2+\right.$$

$$\left.+\sum_{i,j=1}^n\left(\frac{\partial^3 u_\varepsilon(x,t)}{\partial t^3}\right)^2+1\right]$$

Having obtained the other estimates similarly to lemma 2, we get the statement of the lemma.

Lemma has been proved.

Lemma 4. *Let the condition (15) be fulfilled on the set Σ^0 and the boundary of Q_T have no points of γ^0. Then in the closed domain $\overline{Q_T}$ derivatives of $u_\varepsilon(x,t)$ with respect to space up to the second order variables are uniformly bounded .*

Proof. Let's take a point $(x^*,t^*)\in\Gamma$, and let in its neighborhood the boundary Γ be presented in the form $x_1=\varphi(x_2,...,x_n,t)$. By means of change of variables $t=t^*,\xi_1=x_1-\varphi(x_2,...,x_n,t)$, $\varsigma_2=x_2,...,\xi_n=x_n$ in the neighborhood of (x^*,t^*) the equation (9) is reduced to the form

$$\begin{gathered}L^*_\varepsilon u_\varepsilon=\sum_{i,j=1}^n a^*_{ij}(\xi,t^*)\frac{\partial^2 u_\varepsilon}{\partial\xi_i\partial\xi_j}+\psi^*_\varepsilon(\xi,t^*)\frac{\partial^2 u_\varepsilon}{\partial(t^*)^2}+\\+\sum_{i=1}^n b^*_i(\xi,t^*)\frac{\partial u_\varepsilon}{\partial\xi_i}+b^*_0(\xi,t^*)\frac{\partial u_\varepsilon}{\partial t^*}+c^*(\xi,t^*)u_\varepsilon=f^*(\xi,t^*),\end{gathered}\tag{17}$$

where $a^*_{11}(\xi,t^*)\geq\mu>0,c^*(\xi,t^*)<0$, and due to assumptions on smoothness of the coefficients and boundary, the coefficients of (17) have uniformly bounded derivatives. The boundary Γ will have the equation $\xi_1=0$ in the neighborhood of (x^*,t^*). For clarity we'll take the axis ξ_1 to be pointed into Q_T. As in lemma 1 we denote by $\chi(\xi_2,...,\xi_n,t^*)$ a nonnegative twice continuously differentiable function, equal to the constant β in some neighborhood Γ^1 of the point (x^*,t^*) on the boundary Γ and equal to zero out of a little greater neighborhood $0 \leq \chi \leq \beta$. The part of the domain Q_T lying between the boundary $\Gamma\{\xi_1 = 0\}$ and $\sigma\left\{\xi_1 = \dfrac{\gamma}{\alpha}\chi(\xi_2,...,\xi_n,t^*)\right\}$, will be denoted by Q_T^ε. Further, α will be chosen as depending on ε, and γ -not depending on ε. In $\overline{Q_T}$ the uniform boundedness results from lemma 2 for first derivatives of $u_\varepsilon(x,t)$ with respect to x_i and t, and hence with respect to ξ_i, t^* in a neighborhood of (x^*,t^*). By lemma 3 second derivatives of $u_\varepsilon(x,t)$ are estimated via their values on the boundary , and as second derivatives with respect to x_i and t, as well as with respect to ξ_i, t^* , are mutually expressed by each other and by first derivatives in a neighborhood of (x^*,t^*) in a uniformly bounded way, so

$$\left|\frac{\partial^2 u_\varepsilon(\xi,t^*)}{\partial\xi_i\partial\xi_j}\right| + \left|\frac{\partial^2 u_\varepsilon(\xi,t^*)}{\partial\xi_i\partial t}\right| + \left|\frac{\partial^2 u_\varepsilon(\xi,t^*)}{\partial t^2}\right| < \mu H(\varepsilon) + \mu_1 \tag{18}$$

at $(\xi,t^*) \in \overline{Q_T}$, $i,j = \overline{1,n}$. Here a maximum of second derivatives on the boundary Γ is denoted by $H(\varepsilon)$.

If at the point (x^*,t^*) a tangent plane to Γ is orthogonal to the axis t, then by definition of $\sum^0$ at the point, and that's in some its neighborhood $\psi_\varepsilon(x,t) > \mu_1 > 0$. Thus, for each point $(x^*,t^*) \in \Gamma$ such a neighborhood exists on the boundary, that

$$\left|\frac{\partial^2 u_\varepsilon}{\partial\xi_i\partial\xi_j}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial\xi_i\partial t}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial t^2}\right| < \mu_6\sqrt{H(\varepsilon)} + \mu_7, \quad i,j = \overline{1,n}.$$

Taking a finite number of such neighborhoods, covering Γ, and taking into account the smoothness of change of coordinates in each of these neighborhoods, we get

$$\left|\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial x_i\partial t}\right| + \left|\frac{\partial^2 u_\varepsilon}{\partial t^2}\right| < \mu_8\sqrt{H(\varepsilon)} + \mu_9,$$

on entire boundary Γ or, due to definition of $H(\varepsilon)$, $H(\varepsilon) \leq \mu_{10}\sqrt{H(\varepsilon)} + \mu_{11}$. Whence $H(\varepsilon) < \mu_{12}$, i.e. we've boundedness of second derivatives on the boundary , and by lemma 3, in the whole domain $\overline{Q_T}$. Here we used only boundedness of first derivatives of coefficients of the equation (17).

Lemma has been proved.

Now we can move to the proof of existence and uniqueness theorem for the first boundary value problem for the equation (8).

Theorem 1. *Let the equation (8), defined in a cylindrical domain Q_T with the boundary Γ, degenerate on the set $\Sigma^0 \subset \overline{Q_T}$ to a parabolic one and assume, that the condition (3) is fulfilled and all the coefficients and the right-hand side of the equation (8) have bounded derivatives up to the first order, satisfying the Hölder condition. Assume, that in a cylindrical domain $Q_T \supset \overline{Q_T}\psi(x,t) \geq 0$ and the conditions (7) are fulfilled. If the boundary Γ has no points of Γ^0 and on Σ^0 the following condition is fulfilled*

$$\frac{\partial^2\psi(x,t)}{\partial t^2} + 2\frac{\partial b_0(x,t)}{\partial t} + c(x,t) < 0, \tag{19}$$

then in Q_T there exists a unique solution of the equation (8), satisfying the condition (2) and having in $\overline{Q_T}$ derivatives of the first order , satisfying the Hölder condition; and the following estimate is true

$$\|u\|_{C^{1+\lambda}(Q_T)} \leq K_1(\|f\|_{C^\lambda(Q_T)} + \sup|u|_{Q_T}). \tag{20}$$

Proof. From lemma 4 it results , that solutions of the equation

$$L_\varepsilon u_\varepsilon(x,t) = f(x,t), \tag{21}$$

vanishing on Γ, are uniformly bounded in the closed domain $\overline{Q_T}$ along with their derivatives up to the second order. That is, it's possible to find a sequence $u_\varepsilon(x,t)$ such, that at $\varepsilon \to 0$ it uniformly converges to some function $u(x,t)$ along with its derivatives up to the first order in the closed domain $\overline{Q_T}$. And it's clear, that these derivatives of $u(x,t)$ will be Hölder derivatives and the function $u(x,t)$ equals zero on the boundary Γ. Besides, for such solutions the estimate (20) is true (See [5], Chapter 3, p. 235). Passing to the limit in the equation (21) at $\varepsilon \to 0$, we get, that $u(x,t)$ satisfies the equation (8) and the estimate (20) is true. Uniqueness of the solution results directly from maximum principle.

Remark 2. *From the proof of theorem1 also results the convergence of the solutions of the equation (21) to the solution of the equation (1) at* $\varepsilon \to 0$

Remark 3. *The condition (19) can't be omitted. It's an essential difference from existence theorems that are for smooth solution of the Dirichlet problem for elliptic equation. Let's give an example.*

Example. Let's consider the equation

$$t^2\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} + \beta t\frac{\partial u(x,t)}{\partial t} + cu = 0 \tag{22}$$

with sufficiently smooth coefficients, β, c -constants, $c \le 0$. It's easy to check, that the equation has a solution

$$u(x,t) = t^\gamma \sin px, \tag{23}$$

$$\gamma(\gamma-1) + \beta\gamma + c = p^2. \tag{24}$$

The equation degenerates on the axis x. The condition (19) for the equation means, that $2 + 2\beta + c < 0$. Let the condition be not fulfilled, f.e. $2 + 2\beta + c > 0$. Then such $p, \gamma < 2$ exist, that they satisfy (24). Let's consider the domain Q_T, containing a segment of the axis x, whose boundary near the axis x consists of straight lines $x = 0$ and $x = \dfrac{\pi}{p}$ and everywhere is sufficiently smooth. Then the solution (23) will be sufficiently smooth on the boundary (near the axis $x = 0$ it's zero), but, nevertheless, its first order derivatives will not satisfy the Hölder condition at $t = 0$, $0 < x < \dfrac{\pi}{p}$.

Let's give the scheme of proof of solvability when passing from smooth coefficients to coefficients satisfying (3)-(6),(8).

Let at first $f(x,t)$ be sufficiently smooth in $\overline{Q_T}$. Denote by $v(x,t)$ a classical solution of the first boundary value problem

$$\Delta v - v_t = f(x,t), (x,t) \in Q_T$$

$$v|_{\Gamma(Q_T)} = 0. \tag{25}$$

It's known, that the solution $v(x,t)$ to the problem exists, and $v(x,t) \in^{2,1} (\overline{Q_T})$. Now we take an operator L_ε and let $u_\varepsilon(x,t)$ be a classical solution of the Dirichlet problem

$$L_\varepsilon u_\varepsilon(x,t) = f(x,t), \quad (x,t) \in Q_T$$

$$u_\varepsilon|_{\Gamma(Q_T)} = 0, \quad u_\varepsilon|_{t=T} \; = v|_{t=T}$$

Such a solution $u_\varepsilon(x,t)$ exists due to smoothness of $\psi_\varepsilon(t)$ and $f(x,t)$. As we've shown, $\{u_\varepsilon(x,t)\}$ are uniformly bounded with respect to ε in $C_0^{2,1}(Q_T)$. Therefore, it's compact in this space, i.e. there exist such a function $u(x,t) \in C_0^{2,1}(Q_T)$ and a sequence $\varepsilon_k \to 0$ at $k \to \infty$, that the corresponding sequence $\{u_{\varepsilon_k}(x,t)\}$ converges to the function $u(x,t) \in C_0^{2,1}(Q_T)$ at $k \to \infty$. Further we can obtain, that $L_0 u = f$ in Q_T. Now let $f(x,t)$ $L_p(Q_T), p > n+2$. Then such a sequence exists $\{f_m(x,t)\}$, $f_m(x,t) \in C^\infty(\overline{Q_T})$, that

$$\lim_{m\to\infty} \|f_m - f\|_{L_p(Q_T)} = 0.$$

For natural m denote by $u_m(x,t)$ the sequence of solutions of the first boundary value problem for

$$u_m(x,t) \in C_0^{2,1}(Q_T)$$

$$L_0 u_m(x,t) = f_m(x,t), \quad (x,t) \in Q_T.$$

It's proved, that the limit $u(x,t)$ of the sequence $\{u_m(x,t)\}$ in $C_0^{2,1}(Q_T)$ $m \to \infty$ satisfies in Q_T the equation $L_0 u(x,t) = f(x,t)$.

Note, that as we said above, $\psi(x,t) > 0$. If $\psi(x,t) \equiv 0$, then the equation (1) is parabolic and that's why under the conditions (3)-(6) and $f(x,t) \in L_p(Q_T), \;\; p > n+2$, for the bounded solution of the equation (1) the following estimate is true

$$\|u\|_{C^{1+\lambda}(Q_T^\rho)} \le K_1(\|f\|_{L_p(Q_T)} + \sup_{Q_T} |u|), \tag{26}$$

If $\psi(x,t) > 0$ and the condition of theorem 1 is fulfilled for the coefficients, then for the bounded solution of the equation (1) the estimate (26) is true. The estimate (26) can be obtained by composition of the solution $u(x,t)$ to the problem in the cylinder Q_{z^0}, where $\psi(z) = 0$ at $z \in [0, z^0]$, and the solution $v(x,t)$ to the first boundary value problem for parabolic equation in the cylinder $\Omega \times (z^0, T)$ with boundary conditions $v(x,z^0) = u(x,z^0), v\mid_{\partial\Omega\times[z^0,T]} = 0$. It must be noted, that the theorem has been obtained for smooth coefficients, but we can pass to $f(x,t)$ $L_p(Q_T)$ by means of the abovementioned scheme.

Further to prove the estimate (26) under the conditions (3)-(7) we'll apply the method of continuation by parameter.

Theorem 2. *Suppose, that the equation (8) defined in Q_T degenerates on the set $\Sigma^0 \subset \overline{Q_T}$ to parabolic and the conditions (3)-(7) are fulfilled for the coefficients, and the right-hand side of the equation $f(x,t)$ $L_p(Q_T), p > n+2$. If the boundary Γ has no points of Γ^0 and on Σ^0 the condition (19) is fulfilled, then for the bounded solution $u(x,t)$ of the equation (8) the following estimate is true*

$$\|u\|_{C^{1+\lambda}(Q_T^\rho)} \le K_1(\|f\|_{L_p(Q_T)} + \sup |u|_{Q_T}),$$

where $\lambda > 0$ depends only on coefficients of the operator L and n; and K_1, besides, on $p, \rho, diam Q_T$.

Remark 4. *Theorem 2 in this formulation is also true for the equation (1), just in the condition (19) instead of $b_0(x,t)$ will be taken $b_1(x,t)$.*

Proof of Theorem 2. To prove it, we'll consider a family of operators $Z^{(\tau)} = (1-\tau)L' + \tau Z$ for $\tau \in [0,1]$, where L' - a model operator, defined from the equation

(8) with Laplacian main part and smooth coefficients, and the operator Z is defined from the equation (1). Let's show, that the set E of points τ, at which for solutions of the problem

$$Z^{(\tau)}u = f(x,t),\ (x,t) \in Q_T \tag{27}$$

$$u|_{\Gamma(Q_T)} = 0 \tag{28}$$

the estimate (26) is true at $f(x,t)\ L_p(Q_T), p > n+2$, is nonempty, and open and closed at one and the same time with respect to the segment $[0,1]$. Hence, $E = [0,1]$ and, in particular, for the solution of the problem (27)-(28) the estimate (26) is true at $\tau = 1$, i.e. when $Z^{(1)} = Z$. The set is nonempty by theorem 1. Let's show, that it's open. For this purpose we'll prove that for solutions of the problem (27)-(28) the estimate (26) is true for all such $\tau \in [0,1]$, that $|\tau - \tau_0| < \varepsilon$ (here $\tau_0 \in E$, and $\varepsilon > 0$ will be chosen later). Rewrite the problem (27)-(28) in the equivalent form

$$Z^{(\tau_0)}u = f(x,t) - \left(Z^{(\tau)} - Z^{(\tau_0)}\right)u, \quad (x,t) \in Q_T, \tag{29}$$

$$u(x,t) \in C_0^{2,1}(Q_T).$$

We introduce an arbitrary function $v(x,t) \in C_0^{2,1,\lambda}(Q_T)$ and consider the first boundary value problem

$$Z^{(\tau_0)}u = f(x,t) - (Z^{(\tau)} - Z^{(\tau_0)}v, \quad (x,t) \in Q_T, \tag{30}$$

$$u(x,t) \in C_0^{2,1}(Q_T).$$

It's clear , that $\left(Z^{(\tau)} - Z^{(\tau_0)}\right)v \in C^{2,1,\lambda}(Q_T)$. Indeed, note, that for all operators $Z^{(\tau)}$ the conditions (3) and (4) are fulfilled with constants $\gamma^0_{(\tau)} \geq \min\{\gamma, n\}$ and $\sigma_{(\tau)} \leq \sigma$ respectively. Let's show that. Denote by $a_{ij}^{(\tau)}(x,t)$, $i = \overline{1,n}$ the coefficients of the operator $Z^{(\tau)}$ at higher derivatives with respect to space variables. Let

$$\bar{\iota} = \sup_{Q_T} \frac{\sum\limits_{i,j=1}^n a_{ij}^2(x,t)}{g^2(x,t)},\ \ \iota^{(\tau)} = \sup_{Q_T} \frac{\sum\limits_{i,j=1}^n [\ a_{ij}^{(\tau)}(x,t)\ ]\,2}{\left[\sum\limits_{i=1}^n a_{ii}^{(\tau)}(x,t)\right]^2},\ \ \overline{\iota^{(\tau)}} = \sup_{Q_T} \iota^{(\tau)}(x,t),$$

where $g(x,t) = \sum\limits_{i=1}^n a_{ii}(x,t)$. Taking into account (4) and the fact, that for any operator of Z- type the inequality $\bar{\iota} \geq 1$ is true, we conclude

$$\iota^{(\tau)}(x,t) = \frac{n(1-\tau)^2 + 2\sigma(1-\tau)g(x,t) + \tau^2 \sum\limits_{i,j=1}^n a_{ij}^2(x,t)}{n^2(1-\tau)^2 + 2\tau(1-\tau)ng(x,t) + \tau^2 g^2(x,t)} \leq$$

$$\leq \frac{1}{n} + \frac{\tau^2(\bar{\iota} - \frac{1}{n})g^2(x,t)}{\tau^2 g^2(x,t)} = \bar{\iota} \tag{31}$$

Let now

$$\lambda^- = \inf_{Q_T} g(x,t),\ \lambda^+ = \sup_{Q_T} g(x,t),\ \overline{\lambda}(\tau) = \inf_{Q_T} \sum_{i=1}^n a_{ii}^{(\tau)}(x,t) \sup_{Q_T} \sum_{i=1}^n a_{ii}^{(\tau)}(x,t).$$

Calculations we made before show, that $\overline{\lambda}(\tau)=\dfrac{(1-\tau)n+\tau\lambda^{-}}{(1-\tau)n+\tau\lambda^{+}}$. But on the other hand, $\overline{\lambda}(\tau)=\dfrac{\lambda^{-}-\lambda^{+}}{\left[(1-\tau)n+\tau\lambda^{+}\right]^2}\leq 0$. That's why

$$\overline{\lambda}(\iota)\geq\overline{\lambda}(1)=\lambda. \tag{32}$$

From (31) and (32) it results, that $\sigma_{(\tau)}=\overline{\iota}^{(\tau)}-\dfrac{1}{n-\overline{\lambda}^2(\tau)}\leq\overline{\iota}-\dfrac{1}{n-\lambda^2}=\sigma$, that's the needed statement is obtained.

Let's note, that from all we said above and lemma 4 it results, that at $T\leq T^0$ for any $\tau\in[0,1]$ and any function $u(x,t)\in C_0^{2,1,\lambda}(Q_T)$ the following estimate is true

$$\|u\|_{C^{2,1,\lambda}(Q_T)}\leq K_2(\left\|Z^{(0)}u\right\|_{C^{0,\lambda}(Q_T)} \tag{33}$$

For the solution $u(x,t)$ of the boundary value problem (30) due to the assumption made, the estimate (26) is true for any $v(x,t)\in C_0^{2,1,\lambda}(Q_T)$. Thus, an operator Φ is defined from $C_0^{2,1,\lambda}(Q_T)$ to $C_0^{2,1,\lambda}(Q_T)$, and $u=\Phi v$. This operator is compressing at ε, chosen in an appropriate way. Indeed, let $v^{(i)}(x,t)\in C_0^{2,1,\lambda}(Q_T)$, $u^{(i)}=\Phi v^{(i)}, i=1,2$. Then taking into account, that $\left(Z^{(\tau)}-Z^{(\tau_0)}\right)=(\tau-\tau_0)(Z-L')$, we conclude, that $u^{(1)}(x,t)-u^{(2)}(x,t)$ is a classical solution of the first boundary value problem

$$Z^{(}\tau_0)\left(u^{(1)}(x,t)-u^{(2)}(x,t)\right)=(\tau-\tau_0)(Z-L')\left(v^{(1)}(x,t)-v^{(2)}(x,t)\right),$$

$$\left(u^{(1)}(x,t)-u^{(2)}(x,t)\right)\in C_0^{2,1,\lambda}\left(Q_T\right).$$

Using (33), we get

$$\left\|u^{(1)}(x,t)-u^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)}\leq K_2\left|\tau-\tau_0\right|v^{(1)}(x,t)-v^{(2)}(x,t)\Big\|_{C^0\lambda(Q_T)} \tag{34}$$

On the other hand,

$$v^{(1)}(x,t)-v^{(2)}(x,t)\Big\|_{C^0\lambda(Q_T)}\leq$$

$$\leq K_3(Z,n,\Omega,T)\left\|v^{(1)}(x,t)-v^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)} \tag{35}$$

So,

$$\left\|u^{(1)}(x,t)-u^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)}\leq K_2K_3\varepsilon\left\|v^{(1)}(x,t)-v^{(2)}(x,t)\right\|_{C^{2,1,\lambda}(Q_T)}.$$

Now taking $\varepsilon=1/2\ K_2\ K_3$, we prove, that the operator Φ is compressing. Whence, it has a stationary point $u=\Phi u$, that is a classical solution of the boundary value problem (29), and of (27)-(28) as well, and for the solution the estimate (26) is true. So, we've proved, that the set E is open.

Let's show, that the set E is closed. Let $\tau_k\in E, k=1,2,...,\lim_{k\to\infty}\tau_k=\tau$. For natural k we denote by $u_{[k]}(x,t)$ the solution of the first boundary value problem

$$Z^{(\tau_k)}u_{[k]}(x,t)=f(x,t),(x,t)\in Q_{T,}u_{[k]}(x,t)\Big|_{\Gamma(Q_T)}=0,$$

for which the following estimate takes place

$$\left\| u_{[k]}(x,t) \right\|_{C^{2,1}(Q_T)} \le K_3 \left\| f \right\|_{L_p(Q_T)} \tag{36}$$

So, from (35) we get , that the family of functions $\left\{ u_{[k]}(x,t) \right\}$ is compact in $C_0^{2,1}(Q_T)$, i.e. there exists such a subsequence of natural numbers $\{k_l\}$, $\lim_{l\to\infty} k_l = \infty$ and a function $u(x,t)$, that for any $\varphi(x,t) \in C_0^\infty(\overline{Q_T})$

$$\lim_{l\to\infty} (Z^{(\tau_{k_l})} u_{[k_l]}, \varphi) = (Z^{(\tau)} u, \varphi) \tag{37}$$

But

$$(Z^{(\tau)} u_{[k_l]}, \varphi) = ((Z^{(\tau)} - Z^{(\tau_{k_l})}) u_{[k]}, \varphi) + (f, \varphi) = J_1(l) + (f, \varphi) \tag{38}$$

Besides, taking into account (34), (35), we have

$$|J_1(l)| \le |\tau - \tau_{kl}| \left| (Z - L') u_{[k_l]}, \varphi \right| \le |\tau - \tau_{kl}| \, K_4 \left\| u_{[k_l]} \right\|_{C^{2,1}(Q_T)} \|\varphi\|_{0,\lambda(Q_T)} \le$$

$$\le K_3 K_4 \, |\tau - \tau_{kl}| \, \|f\|_{L_p(Q_T)} \|\varphi\|_{C^{0,\lambda}(Q_T)} \tag{39}$$

It results from (38), that $\lim_{l\to\infty} J_1(l) = 0$. From (37) and (38) we get, that $(Z^{(\tau)} u, \varphi) = (f, \varphi)$, i.e. $Z^{(\tau)} u = f(x,t)$ everywhere in Q_T, that's we showed, that $\tau \in E$, i.å. the set E is closed.

Theorem has been proved.

Now we'll prove some estimate for the solution, which can also be taken as an independent result.

Theorem 3. *Let the conditions (3)-(7) be fulfilled for the coefficients of the operator (1). Then for any function* $u(x,t) \in \overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$ *the following estimate is true*

$$\|u(x,t)\|_{C(Q_T)} \le k \, \|f\|_{L_{n+1}(Q_T)} , \tag{40}$$

where $k = k(\gamma, n)$ *.*

Proof. Suppose, that $(x^0, t^0) \in Q_T$, and at this point $\sup_{Q_T} u = u(x^0, t^0) = \mu > 0$. Let's take an auxiliary function $z = u^m, m \ge 2$- a natural number, which will be chosen later. Denote by A_z the set $\{(x,t) : (x,t) \in Q_T, u(x,t) \ge 0, z_t(x,t)\} \ge 0$, $z_{tt}(x,t) \le 0$, $\|z_{ij}(x,t)\|$ - a positively defined matrix}. We've

$$\mu^{m(n+1)} \le K_1 \int\limits_{A_z} (z_t - \sum_{i,j=1}^n a_{ij} z_{ij})^{n+1} dxdt \le$$

$$\le K_1 \int\limits_{A_z} (z_t - \sum_{i,j=1}^n a_{ij} z_{ij} - \psi(x,t) z_{tt})^{n+1} dxdt \le K_2 \int\limits_{A_z} [m u^{m-1} (-Zu) +$$

$$+ \mu^{m-2} (u(x,t) (\sum_{i=1}^n b_i(x,t))^2)^{\frac{1}{2}} \left| \nabla_x u(x,t) \right| + c(x,t) u^2 -$$

$$- (m-1) \gamma \left| \nabla_x u(x,t) \right|^2]^{n+1} dxdt. \tag{41}$$

If $(x,t) \in A_z$ is so, that $|\nabla_x u(x,t)| \geq \dfrac{|b(x,t)|}{(m-1)\gamma} u(x,t)$, then

$$u\,|b|\,|\nabla_x u(x,t)|\,cu^2 - (m-1)\gamma\,|\nabla_x u(x,t)|^2 \leq 0$$

But if for $(x,t) \in A_z$ $|\nabla_x u(x,t)| \leq \dfrac{|b(x,t)|}{(m-1)\gamma} u(x,t)$, then

$$u\,|b|\,|\nabla_x u(x,t)| + cu^2 - (m-1)\gamma\,|\nabla_x u(x,t)|^2 \leq \frac{u^2}{(m-1)\gamma}(|b|^2 + (m-1)\gamma c).$$

Now we take $\max\left\{2, 1+\dfrac{m}{\gamma}\right\}$ as m. Then from (14) we get, that

$$\mu^{m(n+1)} \leq K_2 m^{n+1} \mu^{(m-1)(n+1)} \int\limits_{Q_{T\,z}} |f|^{n+1}\, dxdt.$$

Hence, the estimate (39) with $K = K_2^{\overline{\frac{1}{n+1}}} m.$ is obtained in a standard way. The case, when $(x^0,t^0) = (x^0,T)$, $x^0 \in \Omega$ is considered similarly.

Theorem 4. *Let conditions of theorem 2 be fulfilled and in the cylinder Q_T the solution be defined to the first boundary value problem (1),(2), $f \in L_p(Q_T), p > n+2$. Then the following estimate is true*

$$\|u(x,t)\|_{C^{1+\lambda}(Q_T)} \leq K_4 \|f\|_{L_p(Q_T)}\,. \tag{42}$$

Proof. To prove it, we should use the estimate (26) from theorem 2 and the estimate (39) from theorem 3. From where results the estimate (41).

As a consequence of the estimate (41) we put the theorem on classical solvability of the first boundary value problem for the operator Z, which can be proved by the standard Lere-Shauder [5].

Theorem 5. *Let conditions of theorem 2 be fulfilled. Then the problem (1),(2) has a classical solution $u(x,t) \in C^{2,1,\lambda}(Q_T)$, and $\lambda > 0$ depends only on σ, n.*

Note, that classical solvability could be proved analogously to theorem 2.

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