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STABILITY OF RECONSTRUCTION OF THE STURM-LIOUVILLE OPERATOR WITH MATRIX COEFFICIENTS ON SCATTERING DATA

Abstract

In this paper the stability of reconstruction of the Sturm-Liouville operator with hermitian matrix of coefficients on scattering data is considered.

The stability of inverse problems has been studied in many mathematical investigations. For self-adjoint Sturm-Liouville operator (in case of semi-axis) this problem has been solved by V.A.Marchenko [1]. The analogical problems for nonself-adjoint operators have been studied in [3],[4].

In this paper the stability of reconstruction of the Sturm-Liouville operator with hermitian matrix of coefficients on scattering data is considered.

1. Preliminary informations and notation. The operator L generated in $L^2_{(n)}[0,\infty)$ (Hilbert space of all vector-functions $f(x)=\{f_k(x)\}_1^n$ with summable in $[0,\infty)$ squares of all components, in which a scalar product is defined by the formula $(f,g)=\int\limits_0^\infty\sum\limits_{k=1}^n f_k(x)\times\bar{g}_k(x)\,dx$ by the expression

$$ly=-y''+V(x)\,y \tag{1.1}$$

with the hermitian matrix of coefficients $V(x)=(\upsilon_{jk}(x))_1^n$ and boundary condition

$$y(0)=0. \tag{1.1'}$$

Everywhere further we'll assume that the potential matrix $V(x)$ satisfies the condition

$$\int\limits_0^\infty x\,|V(x)|\,dx<\infty \tag{1.2}$$

(by $|V(x)|$ denote $\max\limits_j\sum\limits_k|\upsilon_{jk}(x)|$).

Boundary value problems (1.1)-(1.1') of which

$$\int\limits_x^\infty|V(t)|\,dt\le\alpha(x)\quad(0\le x<\infty), \tag{1.3}$$

where $\alpha(x)$ is a continuous nonincreasing function, we denote by $V\{\alpha(x)\}$.

As is known if condition (1.2) is satisfied, the operator L may have finite number of negative eigen values $\lambda_k^2<0\quad(Jm\lambda_k<0)$, and its continuous spectrum fill up all positive semi-axis. But its normed eigen vector-functions are the columns of the matrices $u(x,\lambda)$ $\left(\lambda>0;\ \lambda=\lambda_k,\ k=\overline{1,p}\right)$ and have as $x\to\infty$ the following asymptotics

$$u(x,\lambda)=e^{i\lambda x}I-e^{-i\lambda x}S(-\lambda)+o(1)\qquad(\lambda>0),$$

$$u(x,\lambda_k)=e^{-|\lambda_k|x}\left[M_k+o(1)\right]\qquad\left(k=\overline{1,p}\right),$$

where $S(-\lambda)=S^*(\lambda)$ is a unitary matrix (scattering matrix), M_k are hermitian nonnegative matrices (normed matrices), $o(1)$ is a matrix whose elements are of order $o(1)$, I is a unit matrix. The set of quantities $\{S(\lambda),\lambda_k,M_k\}$ is called the scattering data of the operator L. These data uniquely determine the operator L. The function $F(t)$

$$F(t)=\sum_k M_k^2e^{-|\lambda_k|t}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[I-S(\lambda)\right]e^{i\lambda t}d\lambda \tag{1.4}$$

is constructed on scattering data ([2]), and the basic equation is considered according to this function

$$F(x+y)+K(x,y)+\int_x^{\infty}K(x,t)F(t+y)\,dt=0\quad(0<x\le y)$$

from which $K(x,t)$ is defined. The kernel $K(x,t)$ is connected with the potential $V(x)$ by the equality

$$V(x)=-2\frac{d}{dx}K(x,x). \tag{1.5}$$

At fulfilling condition (1.2) the matrix differential equation

$$Y''+\lambda^2Y=V(x)Y,\quad 0<x<\infty$$

has the solution $E(x,\lambda)$ representable in the form

$$E(x,\lambda)=e^{-i\lambda x}I+\int_x^{\infty}K(x,t)e^{-i\lambda t}d\lambda\qquad(\operatorname{Im}\lambda\le 0). \tag{1.6}$$

The matrix $E(x,\lambda)$ satisfies the inequality

$$|K(x,t)|\le\frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\left\{\sigma_1(x)-\sigma_1\left(\frac{x+t}{2}\right)\right\}, \tag{1.7}$$

where

$$\sigma(x)=\int_x^{\infty}|V(t)|\,dt,\quad \sigma_1(x)=\int_x^{\infty}|\sigma(t)|\,dt. \tag{1.8}$$

2. The accuracy of reconstruction of special solutions. The problem on how strongly may differ two problems whose scattering data coincide with the given change of interval of the parameter λ^2, if for these problems the apriori estimates of the functions are known, is of great interest. First of all we consider a problem on stability of reconstruction of special solutions $E(x,\lambda)$ since they are reconstructed most stable. We derive the formula expressing the difference of such solutions by scattering data.

We consider two problems with the potentials $V_1(x), V_2(x)$ from the set $V[\alpha(x)]$.

We compose for the corresponding inverse problems the integral equations and then subtract one from the other. Passing to the adjoint matrices by virtue of hermiticity of $F(t)$ we obtain

$$\overline{K}_{1,2}(x,y)+\int_x^\infty F_1(t+y)\overline{K}_{1,2}(x,t)\,dt=$$
$$=-F_{1,2}(x+y)+\int_x^\infty F_{1,2}(t+y)\overline{K}_2(x,t)\,dt, \tag{2.1}$$

where ($\overline{K}(x,y)$ is a transposed matrix)

$$\overline{K}_{1,2}(x,y)=\overline{K}_1(x,y)-\overline{K}_2(x,y)\,,\quad F_{1,2}(x,y)=F_1(x,y)-F_2(x,y)\,.$$

We multiply equality (2.1) from the right by the constant vector-column a. At each fixed $x\geq 0$ the obtained equality is the equation with respect to the vector-function $\overline{K}_{1,2}(x,y)\,a$, solving of which we find

$$\overline{K}_{1,2}(x,y)\,a=-\left(\mathbf{I}+\mathbf{F}_{1x}\right)^{-1}\left\{F_{1,2}(x,y)\,a+\int_x^\infty F_{1,2}(t+y)\overline{K}_2(x,t)\,adt\right\}. \tag{2.2}$$

According to the basic equation

$$\left(\mathbf{I}+\mathbf{F}_{1x}\right)^{-1}=\left(\mathbf{I}+\mathbf{K}_{1x}^*\right)\left(\mathbf{I}+\mathbf{K}_{1x}\right), \tag{2.3}$$

where the operators $\mathbf{I}+\mathbf{F}_{1x},\ \mathbf{I}+\mathbf{K}_{1x}, \mathbf{I}+\mathbf{K}_{1x}^*$ are defined in $L^2_{(n)}(0,\infty)$ by the formulae

$$\left(\mathbf{I}+\mathbf{F}_{1x}\right)[f]=f(y)+\int_x^\infty F_1(x+y)\,f(t)\,dt,$$
$$\left(\mathbf{I}+\mathbf{K}_{1x}\right)[f]=f(y)+\int_y^\infty K_1(y,t)\,f(t)\,dt, \tag{2.4}$$
$$\left(\mathbf{I}+\mathbf{K}_{1x}^*\right)[f]=f(y)+\int_x^y \overline{K}_1(y,t)\,f(t)\,dt.$$

Let $\left\{S_j(\lambda)\,,\ \lambda_k^2,\ M_k(j)\right\}$ $(j=1,2)$ be scattering data and $E_j(x,\lambda)$ be solutions of the considered problems. Then

$$\varphi(x,y)=\left(\mathbf{I}+\mathbf{K}_{1x}\right)\left\{F_{1,2}(x+y)\,a+\int_x^\infty F_{1,2}(t+y)\overline{K}_2(x,t)\,adt\right\}=$$

$$= \sum_k E_1(y,\lambda_k)\left[M_k^2(1)-M_k^2(2)\right]\overline{E}_2(x,\lambda_k)\,a+$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} E_1(y,\lambda)\left[S_2(\lambda)-S_1(\lambda)\right]\overline{E}_2(x,\lambda)\,ad\lambda. \tag{2.5}$$

It follows from formulae (2.2),(2.3) that

$$\overline{K}_2(x,y)\,a = -\ (I+K_{1x}^*)\,\varphi(x,y) \tag{2.6}$$

and so at $\operatorname{Im}\mu<0$

$$\left(\overline{E}_1(x,\mu)-\overline{E}_2(x,\mu)\right)a=\int_x^{\infty} e^{-i\mu y}\overline{K}_{1,2}(x,y)\,ady=-\int_x^{\infty}\overline{E}_1(y,\mu)\,\varphi(x,y)\,dy. \tag{2.7}$$

It follows form the equations that are satisfied by the functions $E_1(y,\lambda)$ that

$$\int_x^{\infty}\overline{E}_1(y,\mu)\,E_1(y,\lambda)\,dy=\frac{\overline{E}_1(x,\mu)\,E_1'(x,\lambda)-\overline{E}_1'(x,\mu)\,E_1(x,\lambda)}{\lambda^2-\mu^2}.$$

Using this equality and formula (2.5) defining the function $\varphi(x,y)$ (by virtue of arbitrariness we can omit vector a) the following lemma is proved.

Lemma 1. *At all values of μ from the open lower half-plane for which $\mu\neq\lambda_k$, the identity*

$$-\left\{\overline{E}_1(x,\mu)-\overline{E}_2(x,\mu)\right\}^2=A_{1,2}(x,\mu)-A_{2,1}(x,\mu), \tag{2.8}$$

where

$$A_{i,j}(x,\mu)=\overline{E}_j(x,\mu)\sum_k\frac{\overline{E}_i(x,\mu)\,E_i'(x,\lambda_k)-\overline{E}_i'(x,\mu)\,E_i(x,\lambda)}{|\lambda_k|^2+\mu^2}\times$$

$$\times\left[M_k^2(i)-M_k^2(j)\right]\overline{E}_j(x,\lambda_k)+\frac{1}{2\pi}\overline{E}_j(x,\mu)\times \tag{2.9}$$

$$\int_{-\infty}^{\infty}\frac{\overline{E}_i(x,\mu)\,E_i'(x,\lambda)-\overline{E}_i'(x,\mu)\,E_i(x,\lambda)}{\lambda^2-\mu^2}\left[S_i(\lambda)-S_j(\lambda)\right]\overline{E}_j(x,\lambda)\,d\lambda$$

is valid.

Let the scattering data $\left\{S_j(\lambda),\ \lambda_k^2,\ M_k(j)\right\}$ of the considered problems coincide at $\lambda^2\in(-\infty,N)$:

$$S_1(\lambda)=S_2(\lambda),\quad -\sqrt{N}<\lambda<\sqrt{N},\quad (N>0),$$

$$\lambda_k(1)=\lambda_k(2),\quad M_k(1)=M_k(2),\quad (k=\overline{1,n}).$$

We estimate the difference

$$\left\{\overline{E}_1(x,\mu)-\overline{E}_2(x,\mu)\right\}.$$

Theorem 1. *If the scattering data of two boundary value problems* $\{V_j(x)\} \in V\{\alpha(x)\}$ *coincide at all values* $\lambda^2 \in (-\infty, N)$, *then at* $\mu^2 \in [\_N, N]$ $(\operatorname{Im}\mu \le 0,\ N > 0)$

$$\left|\overline{E}_1(x,\mu) - \overline{E}_2(x,\mu)\right|^2 \le$$

$$\le \frac{4e^{3\alpha_1(x)}}{\pi}\left(\frac{1}{\sqrt{N}\left(1-\dfrac{|\mu|^2+\mu^2}{2N}\right)} + \frac{\alpha(x)\,e^{\alpha_1(x)}}{N\left(1-\dfrac{|\mu|^2+\mu^2}{2N}\right)}\right), \tag{2.10}$$

and at $\mu^2 < -N$

$$\left|\overline{E}_1(x,\mu) - \overline{E}_2(x,\mu)\right|^2 \le \frac{2e^{3\alpha_1(x)}}{\pi}\left[\frac{1}{\sqrt{N}\left(1-\dfrac{|\mu|^2+\mu^2}{2N}\right)}+\right. \tag{2.11}$$

$$\left.+\frac{2\alpha(x)e^{\alpha_1(x)}}{N\left(1-\dfrac{|\mu|^2+\mu^2}{2N}\right)} + \frac{\dfrac{\pi}{2}-arctg\dfrac{\sqrt{N}}{|\mu|}}{\sqrt{N}}\right],$$

where $\alpha_1(x) = \int\limits_x^\infty \alpha(t)\,dt.$

Proof. At first we'll assume that μ lies in the lower half-plane and $\mu \ne \lambda_k$. Then we can use formula (2.8) where in the present case

$$A_{i,j}(x,\mu) = \frac{1}{2\pi}\overline{E}_j(x,\mu)\int\limits_{|\lambda|>\sqrt{N}} \frac{\overline{E}_i(x,\mu)\,E_i'(x,\mu)\,[S_i(\lambda)-S_j(\lambda)]\,\overline{E}_j(x,\lambda)}{\lambda^2-\mu^2}d\lambda +$$

$$+\frac{1}{2\pi}\overline{E}_j(x,\mu)\int\limits_{|\lambda|>\sqrt{N}} \frac{\overline{E'}_i(x,\mu)\,E_i'(x,\mu)}{\lambda^2-\mu^2}\,[S_j(\lambda)-S_i(\lambda)]\,\overline{E}_j(x,\lambda)\,d\lambda, \tag{2.12}$$

since the scattering data of the considered problems coincide at all $\lambda^2 \in (-\infty, N)$. But formulae (2.11),(2.12) remain valid at $\mu^2 \in (-\infty, N)$, that we can be convinced having accomplished in them passage to the limit.

Denote the first and second addends in the right hand side of (2.12) by $B_1(x,\mu)$ and $B_2(x,\mu)$, respectively. From the estimate (at $\operatorname{Im}\nu \le 0$)

$$\left|\overline{E}_j(x,\nu)\right| \le e^{\sigma_1(x)},\quad \left|\overline{E'}_j(x,\nu)\right| \le |\nu| + \sigma(x)\,e^{\sigma_1(x)} \qquad (2.13),$$

from the relation $|S_j(\lambda) - S_i(\lambda)| = O\left(\dfrac{1}{|\lambda|}\right)$, $|\lambda| \to \infty$, and definition of the set

$V\left\{\alpha\left(x\right)\right\}$ at $\mu^2\in[-N,N]$ for sufficiently large N, we obtain

$$\left|B_1\left(x,\mu\right)\right|\leq\frac{e^{3\alpha_1(x)}}{2\pi}\int\limits_{|\lambda|>\sqrt{N}}\frac{|\lambda|+\alpha\left(x\right)e^{\alpha_1(x)}}{\left(\lambda^2-\mu^2\right)|\lambda|}d\lambda\leq$$
$$\leq\frac{e^{3\alpha_1(x)}}{2\pi}\left(\int\limits_{|\lambda|>\sqrt{N}}\frac{d\lambda}{\lambda^2-\mu^2}+\int\limits_{|\lambda|>\sqrt{N}}\frac{\alpha\left(x\right)e^{\alpha_1(x)}}{\left(\lambda^2-\mu^2\right)|\lambda|}d\lambda\right)\leq \tag{2.14}$$
$$\leq\frac{e^{3\alpha_1(x)}}{2\pi}\left(\frac{1}{\sqrt{N}\left(1-\frac{|\mu|^2+\mu^2}{2N}\right)}+\frac{\alpha\left(x\right)e^{\alpha_1(x)}}{N\left(1-\frac{|\mu|^2+\mu^2}{2N}\right)}\right).$$

For the estimate $B_2\left(x,\mu\right)$ we also use the inequality $\left|\frac{\mu}{\lambda}\right|<1$ valid at the considered values μ and λ

$$\left|B_2\left(x,\mu\right)\right|\leq\frac{e^{3\alpha_1(x)}}{2\pi}\int\limits_{|\lambda|>\sqrt{N}}\frac{|\mu|+\alpha\left(x\right)e^{\alpha_1(x)}}{\left(\lambda^2-\mu^2\right)|\lambda|}d\lambda\leq$$
$$\leq\frac{e^{3\alpha_1(x)}}{2\pi}\left(\frac{1}{\sqrt{N}\left(1-\frac{|\mu|^2+\mu^2}{2N}\right)}+\frac{\alpha\left(x\right)e^{\alpha_1(x)}}{N\left(1-\frac{|\mu|^2+\mu^2}{2N}\right)}\right). \tag{2.15}$$

Inequality (2.10) immediately follows from (2.8), (2.14) and (2.15). We now consider the case $\mu^2<-N$. $B_1\left(x,\mu\right)$ is estimated as in (2.14). We cite computations for $B_2\left(x,\mu\right)$. Denote the second addend in the right hand side of (2.15) by B_{21} :

$$\left|B_2\left(x,\mu\right)\right|\leq\frac{e^{3\alpha_1(x)}}{2\pi}\left(\int\limits_{|\lambda|>\sqrt{N}}\frac{|\mu|\,d\lambda}{\left(\lambda^2-\mu^2\right)|\lambda|}+2B_{21}\right)\leq\frac{e^{3\alpha_1(x)}}{2\pi}\left(2B_{21}+\right.$$

$$\left.+\int\limits_{|\lambda|>\sqrt{N}}\frac{|\mu|\,d\lambda}{-\mu^2\left(\frac{\lambda^2}{\mu^2}+1\right)|\lambda|}d\lambda\right)\leq\frac{e^{3\alpha_1(x)}}{2\pi}\left(\frac{1}{\sqrt{N}}\int\limits_{|\lambda|>\sqrt{N}}\frac{d\lambda}{|\mu|\left(\frac{\lambda^2}{-\mu^2}+1\right)}+2B_{21}\right)=$$

$$=\frac{e^{3\alpha_1(x)}}{2\pi}\left(\frac{1}{\sqrt{N}}\left(\frac{\pi}{2}-arctg\frac{\sqrt{N}}{|\mu|}\right)+B_{21}\right). \tag{2.16}$$

From (2.14) and (2.16) we obtain (2.11)

3. Estimate of difference of potentials. We pass to the estimate of difference of the considered boundary value problems. For this we appeal to formula (2.6) and assume in it $y = x$. Then by virtue of (2.5) and (1.5) we obtain

$$\frac{1}{2}\int_x^\infty [V_1(t) - V_2(t)]\,dt = \sum_k E_1(x,\lambda_k)\,E_2(x,\lambda_k)\left[M_k^2(2) - M_k^2(1)\right] + \\ +\frac{1}{2\pi}\int_{-\infty}^{\infty} E_1(x,\lambda)\,E_2(x,\lambda)\,[S_1(\lambda) - S_2(\lambda)]\,d\lambda. \tag{3.1}$$

In particular, if the conditions of theorem 1 are satisfied, then

$$\frac{1}{2}\int_x^\infty [V_1(t) - V_2(t)]\,dt = \frac{1}{2\pi}\int_{|\lambda|>\sqrt{N}} E_1(x,\lambda)\,E_2(x,\lambda)\,[S_1(\lambda) - S_2(\lambda)]\,d\lambda. \tag{3.2}$$

It fails immediately to estimate the right hand side in (3.2). Therefore we choose the sufficiently smooth matrix-function $G(x)$ equal to zero outside of the interval $(x_0, x_0 + h)$, we multiply from the left the both sides of (3.2) by $G\prime(x)$ and integrate. After integration by parts the left hand side we obtain

$$\frac{1}{2}\int_{x_0}^{x_0+h} G(t)\,[V_1(t) - V_2(t)]\,dt =$$

$$= \frac{1}{2\pi}\int_{|\lambda|>\sqrt{N}}\int_{x_0}^{x_0+h} G'(t)\,E_1(t,\lambda)\,E_2(t,\lambda)\,[S_1(\lambda) - S_2(\lambda)]\,d\lambda. \tag{3.3}$$

The following lemma helps to choose the function $G(x)$ so that the right hand side in (3.3) by modulus was small as far as possible.

Lemma 2. *Let $V_1(x)$, $V_2(x)$ be potentials of the problems from $V\{\alpha(x)\}$, bounded in the interval $(x_0, x_0 + h)$ and*

$$Q(x) = \int_x^\infty [V_1(t) + V_2(t)]\,dt.$$

Then for any continuously-differentiable matrix-function (in terms of continuously-differentiability of each of its element), equal to zero out of the interval $(x_0, x_0 + h)$, the following identity is valid

$$\int_{x_0}^{x_0+h} G'(t)\,E_1(t,\lambda)\,E_2(t,\lambda)\,dt = \int_{x_0}^{x_0+h}\left\{G'(t) + G(t)\,Q(t)\right\}e^{-2i\lambda t}dt + r(\lambda, x_0, h)\,, \tag{3.4}$$

where

$$|r(\lambda, x_0, h)| \le \frac{\alpha^2(x_0)\,m^2(x_0,\lambda)}{4\lambda^2}\left\{3\,|G'(2\lambda)| + \widetilde{G}'(-2\lambda)\right\} + \\ + \frac{4\alpha(x_0)\,m^2(x_0,\lambda)\,\beta(x_0,h)\,h}{4\lambda^2\lambda h2} = \int_{x_0}^{x_0+h} |G'(t)|\,dt, \tag{3.5}$$

$$m(x,\lambda)=\max_{j=1,2}\left\{\sup_{x\le t<\infty}|E_j(t,\lambda)|\right\},\quad \beta(x,h)=\max_{j=1,2}\left\{\sup_{x<t<x+h}|V_j(t)|\right\}$$

We now choose the matrix function $G(x)$. Let

$$\delta_0(t)=\frac{k}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^k e^{2ik\lambda t}d\lambda,\quad (k>3)$$

$$\delta(t)=\frac{1}{h}\delta_0\left(-\frac{1}{2}+\frac{t-x_0}{h}\right).$$

As $G(x)$ we take the solution of matrix differential equation

$$G'(x)+G(x)Q(x)=\delta'(x)\ I+\delta(x)C \tag{3.6}$$

(C -is a constant matrix) vanishing at $x\le x_0$

$$G(x)=\int_{x_0}^{x}\left\{\delta'(t)I+\delta(t)C\right\}\Phi^{-1}(t)\Phi(x)dt=$$

$$=\delta(x)I-\int_{x_0}^{x}\{Q(t)-C\}\Phi^{-1}(t)\Phi(x)\delta(x)dt, \tag{3.7}$$

where $\Phi(x)$ is a fundamental matrix corresponding to the homogeneous equation

$$G'(x)+G(x)Q(x)=0.$$

We choose the matrix-constant C such that $G(x)$ vanishes at $x\ge x_0+h$, i.e. we define it from the equality

$$\int_{x_0}^{x_0+h}\{Q(t)-C\}\Phi^{-1}(t)\Phi(x)\delta(x)dt=0. \tag{3.8}$$

Since $\Phi(x)$ is a fundamental matrix, from (3.8)

$$\int_{x_0}^{x_0+h}\{Q(t)-C\}\Phi^{-1}(t)\delta(x)dt=0. \tag{3.8$\prime$}$$

Applying the mean value theorem to each element of a matrix being in the left hand side of (3.8') we find

$$c_{ij}=q_{ij}(t_i)\quad \left(i,j=\overline{1,n}\right) \tag{3.9}$$

where

$$(c_{ij})_1^n=C,\quad (q_{ij}(t))_1^n=Q(t),\ t_i\in(x_0,x_0+h)\ .$$

From (3.7)

$$G'(x) = \delta'(t) I - \delta(t)\{Q(t) - C\} + \int_{x_0}^{x} \{Q(t) - C\} \Phi^{-1}(t) \Phi(x) Q(x)\, dt. \quad (3.10)$$

Equalities (3.7),(3.9),(3.10) lead to the following estimates at $x_0 \le x \le x_0 + h$

$$|G(x) - \delta(x) I| \le \frac{h}{2} \delta(x) \omega(h, x_0) \nu(h, x_0), \quad (3.11)$$

$$\left|G'(x) - \delta'(x) I\right| \le \delta(x) \omega(h, x_0) (1 + h\alpha(x_0) \nu(h, x_0)), \quad (3.12)$$

where

$$\omega(h, x_0) = \max_{x_0 \le x, y \le x_0 + h} |Q(x) - Q(y)|, \quad \nu(h, x_0) = \max_{x_0 \le x, y \le x_0 + h} \left|\Phi^{-1}(t) \Phi(x)\right|.$$

These inequalities together with lemma 2 and equation (3.6) lead to the estimate

$$\left|\int_{x_0}^{x_0+h} G'(t) E_1(t, \lambda) E_2(t, \lambda)\, dt\right| \le 2\left(\frac{k}{h}\right)^k |\lambda|^{-k+1} \left\{1 + |\lambda|^{-1} \alpha(x_0)\right\} +$$

$$+\frac{2\alpha^2(x_0) m^2(x_0, \lambda)}{\lambda^2} \left(\frac{k}{h}\right)^k |\lambda|^{-k+1} +$$

$$+\frac{\alpha^2(x_0) m^2(x_0, \lambda) \omega(h, x_0)}{\lambda^2} \{1 + h\alpha(x_0) \nu(h, x_0)\} +$$

$$+\frac{4\alpha(x_0) \beta(x_0, h) m^2(x_0, \lambda)}{\lambda^2} \{2k + h\omega(h, x_0)[1 + h\alpha(x_0) \nu(h, x_0)]\}.$$

Further, taking into account that $S_1(\lambda)$, $S_2(\lambda)$ are unitary matrices of order $(n \times n)$

$$|S_1(\lambda) - S_2(\lambda)| \le 2h \quad (3.13)$$

(since the elements of these matrices on absolute values don't exceed unit). Using (3.13) as well as the estimations

$$|\omega(h, x_0)| \le 2h\beta(x_0, h), \quad |\omega(h, x_0)| \le 4\alpha(x_0)$$

from (3.3) we obtain

$$\frac{1}{2}\left|\int_{x_0}^{x_0+h} G(t)\{V_1(t) - V_2(t)\}\, dt\right| \le$$

$$\le \frac{4}{\pi} n \left(\frac{k}{h}\right)^k \frac{N^{-\frac{k-2}{2}}}{k-2} \left\{1 + \frac{\alpha(x_0)}{\sqrt{N}} + \frac{\alpha^2(x_0) m_N^2(x_0, \lambda)}{N}\right\} + \quad (3.14)$$

$$+\frac{4n\alpha(x_0) \beta(x_0, h) m_N^2(x_0)}{\pi\sqrt{N}} \{4k + 9h\alpha(x_0)[1 + h\alpha(x_0) \nu(h, x_0)]\},$$

where

$$m_N(x_0) = \sup_{|\lambda|>\sqrt{N}} m(x_0, \lambda).$$

Besides, by virtue of (3.11)

$$\left| \frac{1}{2} \int_{x_0}^{x_0+h} \{G(t) - \delta(t) I\} \{V_1(t) - V_2(t)\} dt \right| \leq 2h\alpha(x_0) \gamma(h, x_0) \beta(x_0, h). \quad (3.15)$$

Using (3.14), (3.15) the following theorem is proved.

Theorem 2. If the scattering data of two boundary value problems from $V\{\alpha(x)\}$ coincide at all values $\lambda^2 \in (-\infty, N)$ and $N \geq 1$, then in the domain, where

$$\frac{5\{[\ln N] + 1\}}{\sqrt{N}} \alpha(x) < 1$$

the inequality

$$|V_1(x) - V_2(x)| \leq \frac{4n}{\sqrt{N}\{3[\ln N] + 1\}} + \left\{ \frac{2\{[\ln N](10\nu(h, x) + 48n) +}{\sqrt{N}} \right.$$

$$+ \frac{(36n + 10)\nu(h, x) + 84n}{\sqrt{N}} \{\beta(x, h)\alpha(x) + \gamma(x, h)\}$$

is valid.

Here

$$h = 5N^{-\frac{1}{2}}\{[\ln N] + 1\}, \ \gamma(x, h) = \max_{j=1,2} \sup_{x<t<x+h} |V_j'(t)|.$$

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