MECHANICS

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PULSATING FLOW OF BUBBLE LIQUID IN A VISCO-ELASTIC TUBE NON-HOMOGENEOUS IN LENGTH

Abstract

At present, the problems of mathematical physics, concerned with description of wave motions of different nature liquids, in particular, multiphase ones in deformed tubes draw great attention. This interest is stipulated not only by large applied significance of the indicated problems (transportation process in different chemical-engineering devices, hemodynamics, tubeline transportation), but also by their new theoretical and mathematical contents, which often don't have analogies in classical mathematical physics. The given paper, in development [1], in devoted to statement of the results, concerned with the problem of mathematical description of one class of motion of ideal compressible barotropic diphase bubble liquid , enclosed to semi-infinite viscous-elastic tube non-homogeneous by the length.

1. The liquids stream in deformed tubes in many cases may be described by the equations of hydraulic approximation. They can be obtained by averaging the corresponding linearized motion equations and continuity equation. Pass to description of this procedure, which permits significantly to simplify the initial simultaneous equation, not distorting at that the substance of effect.

Let's refer the consideration of small axisymmetric motions of diphase liquid in thin-walled tube of radius R and thickness h to the cylindrical coordinate system (x, φ, r).

Assume, that according to [2], the motion of mixture can be described by the model of ideal compressible medium. Uniting the axis x with the tubes axis and denoting by $v_x\,(x,r,t)$ and $v_r\,(x,r,t)$ the axis and radial vector component of velocity, respectively we'll write the impulse equations

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho_{f0}}\frac{\partial P}{\partial x}, \tag{1.1}$$

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho_{f0}}\frac{\partial P}{\partial r} \tag{1.2}$$

and continuity equation

$$\frac{1}{\rho_{f0}}\frac{\partial \rho}{\partial t} + \frac{\partial v_r}{\partial r} + \frac{1}{r}v_r + \frac{\partial v_x}{\partial x} = 0 \qquad (1.3)'$$

where [2]

$$\rho_{f0} = (1-\alpha_{20})\,\rho_{10}^0 + \alpha_{20}\rho_{20}^0$$

is density of gas-liquid mixture, α_{20} is a volume of bubble content, ρ_{10}^0 and ρ_{20}^0 are real densities of liquid and gas, respectively. Accepting that the mixture is barotropic $\rho = \rho(P)$ we have:

$$\frac{\partial \rho}{\partial t} = \frac{d\rho}{dP}\frac{\partial P}{\partial t} = \frac{1}{a^2}\frac{\partial P}{\partial t}.$$

Here, by [2]

$$a^2 = \frac{1}{\alpha_{20}(1-\alpha_{20})}\left\{\frac{\rho_{10}^0}{\rho_{10}^0 - \rho_{20}^0}\right\}^2 \frac{P_0}{\rho_{10}^0},$$

is a square of sounds velocity, and P_0 is a static pressure. Note, that the lower zero index corresponds to the values of the parameters in equilibrium position.

Now equation $(1.3)'$ we rewrite in the following way:

$$\frac{1}{a^2\rho_{f0}}\frac{\partial P}{\partial t} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_x}{\partial x} = 0. \qquad (1.3)$$

Assume, that the axial component of speed is larger than redial one and lower we accept $v_r \approx 0$. Hence, from equation (1.2) at once it follows that $\frac{\partial P}{\partial r} = 0$. In other words

$$P = P(x,t). \qquad (1.4)$$

Introduce the assumption according to which the tube is rigidly attached to the environment. At this the axial displacement is absent and the penetrability condition is determined by the equality

$$v_r = \frac{\partial w}{\partial t} \quad \text{at} \quad r = R \qquad (1.5)$$

where $w = w(x,t)$ is a redial displacement of tube wall. Multiplying the both parts of equation (1.3) by $2\pi r$ and integrating by the coordinate r from 0 to R. Then by virtue of condition (1.4) the first additive with (1.13) we give the form

$$\frac{2\pi}{\rho_{f0}a^2}\int_0^R r\frac{\partial P}{\partial r}dr = \frac{\pi R^2}{\rho_{f0}a^2}\frac{\partial P}{\partial t}.$$

Integrating by parts the second additive we'll write:

$$2\pi\int_0^R r\frac{\partial v_r}{\partial r}dr = 2\pi\left\{ r v_r|_0^R - \int_0^R v_r dr\right\}.$$

Allowing for equality (1.5) we have:

$$2\pi\int_0^R r\frac{\partial v_r}{\partial r}dr = 2\pi\left\{ R\frac{\partial w}{\partial t} - \int_0^R v_r dr\right\}.$$

Determining by $u(x,t)$ the mean speed of mixture in section of tube by the dependence

$$u(x,t) = \frac{1}{\pi R^2}\int_0^R 2\pi r v_x dr,$$

We'll write the last additive by means of the expression

$$\frac{\partial}{\partial x}\int_0^R 2\pi r v_x dr = \pi r^2 \frac{\partial u}{\partial x}.$$

Adding the last relations we'll finally obtain:

$$\frac{1}{a^2 \rho_{f0}} \frac{\partial P}{\partial x} + \frac{\partial u}{\partial x} + \frac{2}{R}\frac{\partial w}{\partial t} = 0. \tag{1.6}$$

Transforming equation (1.1) by the analogous way we'll have:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_{f0}}\frac{\partial P}{\partial x}. \tag{1.7}$$

To equations (1.6) and (1.7) we must add the equation describing the connection between the pressure P and the flexure w. This equation has different form depending on accepted precondition simulating the mechanical behavior of tube wall. The wall's material we'll accept linearly visco-elastic and non-homogeneous in length.

Neglecting the dynamic effects for long waves we have:

$$P = \frac{h}{R^2} E^v w, \tag{1.8}$$

where the visco-elastic properties of material are described by Yu.N.Rabotnov [3] hereditary elasticity theory:

$$E^v w = E(x)\left\{ w - \int_{-\infty}^{t} \Gamma(t-\tau, x)\, w(\tau)\, d\tau \right\}. \tag{1.9}$$

In formula (1.9) $E(x)$ is elasticity module, and $\Gamma(t-\tau, x)$ is a difference relaxation kernel.

Combining equations (1.6) and (1.7) we'll have:

$$\frac{1}{\rho_{f0}}\frac{\partial^2 P}{\partial x^2} - \frac{1}{a^2 \rho_{f0}}\frac{\partial^2 P}{\partial t^2} - \frac{2}{R}\frac{\partial^2 w}{\partial t^2} = 0.$$

Here using equalities (1.8) and (1.9) for the function w we obtain

$$\begin{gathered} \frac{\partial^2 w}{\partial t^2} + \frac{1}{a^2}\frac{hE(x)}{2\rho_{f0}R}\frac{\partial^2}{\partial t^2}\left\{ w - \int_{-\infty}^{t} \Gamma(t-\tau, x)\, w(\tau)\, d\tau \right\} - \\ -\frac{h}{2\rho_{f0}R}\frac{\partial^2}{\partial x^2}\left\{ E(x)\left[w - \int_{-\infty}^{t} \Gamma(t-\tau, x)\, w(\tau)\, d\tau \right] \right\} = 0. \end{gathered} \tag{1.10}$$

Without loss of generality we'll assume that the functions $E(x)$ and $\Gamma(t-\tau, x)$ have the special form:

$$E(x) = g_1(x) E_\infty, \ \Gamma(t-t, x) = g_2 \Gamma_\infty (t-\tau), \tag{1.11}$$

where $g_1(x)$ and $g_2(x)$ are positive constants that have smoothness conditions necessary for legitimacy of carrying out corresponding operations. Besides, we'll assume the tube as homogeneous at infinity.

Hence it follows that

$$\lim_{x\to\infty} g_1(x) = \lim_{x\to\infty} g_2(x) = 1. \tag{1.12}$$

Using (1.11) in (1.10) and allowing for the notation $c_0^2 = \frac{hE_\infty}{2R\rho_{f0}}$ we come to the following integro-differential equation with variable coefficients

$$\begin{gathered} \frac{\partial^2 w}{\partial t^2} + \frac{c_0^2}{a^2} g_1(x) \left\{ \frac{\partial^2 w}{\partial t^2} - g_2(x) \frac{\partial^2}{\partial t^2} \int\limits_{-\infty}^{t} \Gamma(t-\tau, x)\, w(\tau)\, d\tau \right\} - \\ -c_0^2 \frac{\partial^2}{\partial x^2} \left\{ g_1(x) \left[w - g_2(x) \int\limits_{-\infty}^{t} \Gamma(t-\tau, x)\, w(\tau)\, d\tau \right] \right\} = 0. \end{gathered} \tag{1.13}$$

Equation (1.13) describes the longitudinal vibrations of described above system.

2. Study the case of distribution of harmonic waves with given frequency ω on the semi-infinite interval $0 \le x \le +\infty$. For this using variable separation method we'll write the solution of equation (1.13) in the following form:

$$w(x,t) = w_0(x) \exp(i\omega t). \tag{2.1}$$

Here $w_0(t)$ is the unknown complex function coordinates of position coordinates. At first we'll use the form of solution (2.1) and we'll calculate the integral incoming in equation (1.13):

$$\int\limits_{-\infty}^{t} \Gamma_\infty(t-\tau)\, w(\tau)\, d\tau = w_0 \int\limits_{-\infty}^{t} \Gamma_\infty(t-\tau) \exp(i\omega\tau)\, d\tau.$$

Making the substitution $t - \tau = \theta$ we find that

$$w_0 \int\limits_{-\infty}^{t} \Gamma_\infty(t-\tau) \exp(i\omega\tau)\, d\tau = w_0 \mu \exp(i\omega\tau),$$

where the complex value

$$\mu = \int\limits_{0}^{\infty} \Gamma_\infty(\theta) \exp(-i\omega\tau)\, d\theta = \mu_0 + i\mu_1 \tag{2.2}$$

can be determined analytically or numerically for arbitrary kernels of relaxation.

Substituting (2.1) in (1.13) allowing for formula (2.2) and introducing a new variable

$$y(x) = w_0 g_1(x) \left[1 + \mu g_2(x)\right] = w_0 v(x) \tag{2.3}$$

we'll obtain the equation

$$y'' + \varphi(x)\, y = 0 \tag{2.4}$$

in which for brevity of writing we'll denote:

$$\varphi(x) = \omega^2 \left\{ \frac{1}{a^2} + \frac{1}{C_0^2 v(x)} \right\}. \tag{2.5}$$

at this square of speed of wave propagation

$$c^2 = \frac{\omega^2}{\{\varphi(x)\}}$$

Lets' calculate the limit of the function $\varphi(x)$ as x$\rightarrow \infty$. For this expression we find:

$$\lim_{x\to\infty} \varphi(x) = \omega^2 \left\{ \frac{1}{a^2} + \frac{1}{c_0^2 (1-\mu)} \right\} = \delta^2, \tag{2.6}$$

or conditions (1.12) are fulfilled.

Analysis of roots of dispersion equation (2.6) doesn't represent principal difficulties and therefore later we'll use the root where $\operatorname{Im}\delta < 0$. Introduce the function $q(x)$ by substitution

$$q(x) = 1 - \frac{\varphi(x)}{\delta^2},$$

which transforms the differential equation (2.4) onto the equation

$$y'' + \delta^2 y = \delta^2 q(x)\, y. \tag{2.7}$$

We'll assume that the potential $q(x)$ is integrable. Thus,

$$\int_0^\infty |q(x)|\, dx < +\infty. \tag{2.8}$$

We'll complete equation (2.8) by the following boundary conditions

$$y(0) = y_0, \quad y \to 0 \ \text{ as } \quad x \to \infty. \tag{2.9}$$

The value y_0 we can calculate depending on concrete assignment of a boundary condition on the end-wall of tube ($x = 0$).

The second condition (2.9) provides the boundedness and uniqueness of the desired solution. As a result we succeeded to lead to the solution of problem the Stourm-Liouville singular boundary value problem (2.7) and (2.9).

3. Considering the addend $\delta^2 q(x)$ as external source and applying the variation method of arbitrary constants we can show that the solution of equation (2.7) satisfies the integral equation [4]

$$y(x,-\delta) = Ce^{-i\delta x} + \delta \int_x^\infty \sin\delta(\xi - x)\, q(\xi)\, y(\xi - \delta)\, d\xi. \tag{3.1}$$

Here

$$C=\frac{y_0}{f(0,-\delta)},\quad y=y_0\frac{f(x,-\delta)}{f(0,-\delta)},$$

and the function $f(x,-\delta)$ should be determine from the solution of the integral equation.

$$f(x,-\delta)=e^{-i\delta x}+\delta\int_x^\infty \sin\delta(\xi-x)\,q(\xi-\delta)\,d\xi. \tag{3.2}$$

Equation (3.2) is a Volterra type integral equation of the and is solved by the successive approximations method of the form

$$f(x,-\delta)=\sum_{n=0}^{\infty}\delta^n f_n(x,-\delta), \tag{3.3}$$

where

$$f_0(x,-\delta)=e^{-i\delta x},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3.4}$$

$$f_n(x,-\delta)=\int_x^\infty \sin\delta(\xi-x)\,q(\xi)\,f_{n-1}(x,-\delta)\,d\xi \qquad (n=1,2,...)$$

By virtue of inequalities (2.8) and choice of the sign $\operatorname{Im}\delta$ by the Weierstrass sign it follows from uniform convergence of successive approximations that a unique solution (3.2) is determined by formula (3.3). On the other hand by the immediate checking it is easy to establish that this solution satisfies equation (2.7). It appropriate to note, that the series obtained by the term wise differentiation (3.3) also uniformly converge.

Now from formula (1.8) and (2.3) we obtain:

$$w=\frac{y_0}{v(x)}\frac{f(x,-\delta)}{f(0,-\delta)}\exp(i\omega t), \tag{3.5}$$

$$P=y_0\frac{hE_\infty}{R^2}\frac{f(x,-\delta)}{f(0,-\delta)}\exp(i\omega t). \tag{3.6}$$

Then, from equation (1.7) it follows that

$$u=iy_0\frac{hE_\infty}{\omega\rho_{f0}R^2}\frac{f'(x,-\delta)}{f(0,-\delta)}\exp(i\omega t). \tag{3.7}$$

It remains to determine the value y_0 proceeding from regime of system functioning. For this on end-walls of tube we'll give:

$$P=P_{00}\exp(i\omega t). \tag{3.8}$$

Comparing (3.8) with (3.6) at $x=0$ we have:

$$y_0=\frac{R^2}{hE_\infty}P_{00},$$

Substituting of the last equality in (3.5)-(3.7) leads to the following relations:

$$P = P_{00}\frac{f(x,-\delta)}{f(0,-\delta)}\exp(i\omega t),$$

$$u = i\frac{P_{00}}{\omega\rho_{f0}}\frac{f'(x,-\delta)}{f(0,-\delta)}\exp(i\omega t),$$

$$w = \frac{P_{00}}{v(x)}\frac{R^2}{hE_\infty}\frac{f(x,-\delta)}{f(0,-\delta)}\exp(i\omega t).$$

The second type of regime is in giving liquid at the end-wall of speed

$$u = u_0\exp(i\omega t).$$

Thereby we determine

$$y_0 = -iu_0\frac{\omega\rho_{f0}R^2}{hE_\infty}\frac{f(0,-\delta)}{f'(0,-\delta)}.$$

The final result is obtained by the analogy with previous considerations. It has the form:

$$P = -iu_0\rho_{f0}\omega\frac{f(x,-\delta)}{f'(0,-\delta)}\exp(i\omega t),$$

$$u = u_0\frac{f'(x,-\delta)}{f'(0,-\delta)}\exp(i\omega t),$$

$$w = -iu_0\frac{\omega\rho_{f0}R^2}{hE_\infty v(x)}\frac{f(x,-\delta)}{f'(0,-\delta)}\exp(i\omega t).$$

Thus, series (3.3) in conjunction with equalities (3.4) represents the constructive solution of the stated problem. Note that the physical value represent the real parts of obtained solutions, and the result of paper [1] is automatically follows at $\mu = 0$.

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