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# SOLUTION OF THE INVERSE PROBLEM FOR A SYSTEM OF EQUATIONS ON A SEGMENT

**Abstract**

*The inverse problem is completely solved for the Dirac operator with regular boundary conditions.*

**1. Introduction.** We consider the boundary value problems generated on the segment $[0,\pi]$ by a canonic system of Dirac equations

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \lambda \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \tag{1}$$

and separated boundary conditions

$$y_1(0) = y_1(\pi) = 0, \tag{2}$$

and also non-separated boundary conditions of the form

$$y_1(0) + \omega y_1(\pi) = \omega y_2(0) + \alpha y_1(\pi) + y_2(\pi) = 0, \tag{3}$$

where $p(x)$ and $q(x)$ are real functions from $L_2[0,\pi]$ and $\omega$ and $\alpha$ are real numbers where $|\omega| \neq 1$.

We shall denote problems (1), (2) and (1), (3) by $L_0$ and $L_1$, respectively.

At present the problems on the reconstruction of the system of equations (1) on the spectra of two boundary value problems with separated boundary conditions and also on spectral data of two problems with non-separated boundary conditions (see [1-7]) have been completely studied.

In this paper we investigate the problem on the reconstruction of problems $L_0$ and $L_1$. The necessary and sufficient conditions were found in order that two sequences of real numbers be the spectra of such problems. The uniqueness theorem of the solution of the inverse problem is proved.

In the sequel, for the definiteness we'll assume that $|\omega| > 1$ (the case $|\omega| < 1$ is considered quite similarly). Everywhere we'll assume that $k$ takes entire values.

Let $\begin{pmatrix} c_1(\lambda,x) \\ c_2(\lambda,x) \end{pmatrix}$ and $\begin{pmatrix} s_1(\lambda,x) \\ s_2(\lambda,x) \end{pmatrix}$ be the solutions of the system of equation (1) at initial data $c_1(\lambda,0) - 1 = c_2(\lambda,0) = s_1(\lambda,0) = s_2(\lambda,0) - 1 = 0$. Then the characteristic functions of boundary value problems $L_0$ and $L_1$ will have the form $\delta(\lambda) = s_1(\lambda,\pi)$ and

$$\Delta(\lambda) = U_+(\lambda) + \alpha\delta(\lambda) + 2\omega\ . \tag{4}$$

respectively, where

$$U_+(\lambda) = \omega^2 c_1(\lambda,\pi) + s_2(\lambda,\pi)\,. \tag{5}$$

It is known that [2], for the eigen-values $\lambda_k$ of the problem $L_0$ the asymptotic formula

$$\lambda_k = k + \alpha_k, \quad \{\alpha_k\} \in l_2 \tag{6}$$

is valid.

As in the paper [6] we can prove that the eigen-values $\mu_k^{\pm}$ of the problem $L_1$ are subjected to the asymptotics

$$\mu_k^{\pm} = 2k + a^{\pm} + \beta^{\pm}, \quad \left\{\beta_k^{\pm}\right\} \in l_2, \tag{7}$$

where $a^{\pm} = \dfrac{2}{\pi} arctg \dfrac{-\alpha \pm \sqrt{\alpha^2 + \left(\omega^2 - 1\right)^2}}{\left(\omega - 1\right)^2}$.

**2. Uniqueness theorem.**

**Theorem 1.** *The boundary value problems $L_0$ and $L_1$ are uniquely reconstructed if their spectra $\{\lambda_k\}$, $\left\{\mu_k^{\pm}\right\}$ and the sequence of signs $\sigma_k = sign\left[|\omega| - |s_2(\lambda_k, \pi)|\right]$ are known.*

**Proof.** By (7) we have

$$\left(\frac{\omega+1}{\omega-1}\right)^2 = -\lim_{k\to\infty} tg\frac{\pi}{2}\mu_k^- \cdot tg\frac{\pi}{2}\mu_k^+,$$

$$\frac{2\alpha}{\left(\omega-1\right)^2} = -\lim_{k\to\infty}\left(tg\frac{\pi}{2}\mu_k^- + tg\frac{\pi}{2}\mu_k^+\right).$$

The coefficients $\omega$ and $\alpha$ of boundary conditions (3) are determined from these relations. By sequences $\{\lambda_k\}$ and $\left\{\mu_k^{\pm}\right\}$ the functions $\delta(\lambda)$ and $\Delta(\lambda)$ are reconstructed in the form of infinite products. Knowing $\alpha, \omega, \delta(\lambda)$ and $\Delta(\lambda)$ we can define $U_+(\lambda)$ from (4).

Denote

$$U_-(\lambda) = \omega^2 c_1(\lambda, \pi) - s_2(\lambda, \pi) \; . \tag{8}$$

Using the identity

$$c_1(\lambda, \pi) s_2(\lambda, \pi) - c_2(\lambda, \pi) s_1(\lambda, \pi) = 1, \tag{9}$$

we can easily establish

$$U_-(\lambda_k) = sign U_-(\lambda_k)\sqrt{U_+^2(\lambda_k) - 4\omega^2}. \tag{10}$$

By (8) and (9) we have

$$U_-(\lambda_k) = \frac{\omega^2}{s_2(\lambda_k, \pi)} - s_2(\lambda_k, \pi) = \frac{\omega^2 - s_2^2(\lambda_k, \pi)}{s_2(\lambda_k, \pi)}. \tag{11}$$

It is known [2], that sign $s_2(\lambda_k, \pi) = (-1)^k$. Therefore (10) and (11) imply the equality

$$U_-(\lambda_k) = (-1)^k \sigma_k \sqrt{U_+^2(\lambda_k) - 4\omega^2}.$$

Let's consider the fucntion

$$\varphi(\lambda) = \frac{1-\omega^2}{4\omega^2} U_+(\lambda) + \frac{1+\omega^2}{4\omega^2} U_-(\lambda) \; . \tag{12}$$

Using the representation for the functions $c_1(\lambda,\pi)$, $s_2(\lambda,\pi)$ and asymptotic formula (6) we can easily get $\{\varphi(\lambda_k)\}\in l_2$. Then by the known fact (see theorem 28 in [8]) it holds

$$\varphi(\lambda)=\delta(\lambda)\sum_{k=-\infty}^{\infty}\frac{\varphi(\lambda_k)}{(\lambda-\lambda_k)\,\delta'(\lambda_k)},$$

where $\varphi(\lambda_k)=\frac{1-\omega^2}{4\omega^2}U_+(\lambda_k)+\frac{1+\omega^2}{4\omega^2}(-1)^k\sigma_k\sqrt{U_+^2(\lambda_k)-4\omega^2}$.

Consequently, knowing $U_+(\lambda)$, $\delta(\lambda)$ and $\{\sigma_k\}$ we can uniquely define the fucntion $U_-(\lambda)$ from (12).

It follows from (5) and (8) that $s_2(\lambda,\pi)=\frac{1}{2}[U_+(\lambda)-U_-(\lambda)]$. It is known that by the zeros of this function and by $\{\lambda_k\}$ the functions $p(x)$ and $q(x)$ are uniquely reconstructed.

Thus, by sequence $\{\lambda_k\}$, $\{\mu_k^{\pm}\}$ and $\{\sigma_k\}$ the coefficients of system (1) and boundary conditions (3) are uniquely reconstructed. The theorem is proved.

**3. Necessary and sufficient solvability conditions of the inverse problem.**

**Theorem 2.** *In order the sequences of real numbers $\{\lambda_k\}$ and $\{\mu_k^{\pm}\}$ be the spectra of boundary value problems of the form $L_0$ and $L_1$ it is necessary and sufficient that the following conditions be fulfilled:*

*1) It holds asymptotic formulae (6) and (7) in which $a^{\pm}$ are real numbers;*

$$2)...\le\mu_k^-\le\lambda_{2k}<\mu_k^+\le\lambda_{2k+1}\le\mu_{k+1}^-\le.... \tag{13}$$

*where $\lambda_m<\lambda_{m+1}$, $m=0,\pm1,\pm2...$;*

$$3)\;|\Delta(\lambda_k)-2\omega|\ge2|\omega|, \tag{14}$$

*where $|\omega|>1$, $\left(\frac{\omega+1}{\omega-1}\right)^2=-tg\frac{\pi}{2}a^-\cdot tg\frac{\pi}{2}a^+$,*

$$\Delta(z)=(\omega+1)^2\prod_{k=-\infty}^{\infty}\frac{(\mu_k^--z)(\mu_k^+-z)}{(2k+a^-)(2k+a^+)}. \tag{15}$$

**Proof. Necessity.** Asymptotic formula (6) is in [1-2]. Formula (7) and inequality (13) are established as in theorems 2 and 3 of [6]. From equations (4) and (5) by (9) we get

$$|\Delta(\lambda_k)-2\omega|=|U_+(\lambda)|=\left|\omega^2c_1(\lambda,\pi)+\frac{1}{c_1(\lambda_k,\pi)}\right|\ge2|\omega|$$

i.e. it holds inequality (14). The representation of the entire fucntion $\Delta(z)$ in the form of infinite product (15) is established in standard way (see for example [4]).

**Sufficiency.** Let the conditions 1)- 3) be fulfilled.

For the function (15) it holds the representation

$$\Delta(z)=\left(\omega^2+1\right)\cos\pi z-\alpha\sin\pi z+f(z)+2\omega, \tag{16}$$

where $\alpha=-\frac{(\omega-1)^2}{2}\left(tg\frac{\pi}{2}a^-+tg\frac{\pi}{2}a^+\right)$, $f(z)=\int\limits_{-\pi}^{\pi}\tilde f(t)e^{izt}dt$, $\tilde f(t)\in L_2[-\pi,\pi]$ (see lemma in [6]).

By the sequence $\{\lambda_k\}$ we construct the fucntion

$$\delta(z)=\pi(z-\lambda_0)\prod_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\frac{\lambda_k-z}{k}\ .$$

By lemma 4 of the paper [2] this fucntion supposes the representation of the form

$$\delta(z)=-\sin\pi z+f_1(z), \tag{17}$$

where $f(z)=\int\limits_{-\pi}^{\pi}\tilde f_1(t)e^{izt}dt,\quad \tilde f_1(t)\in L_2[-\pi,\pi]$.

Assume

$$U_1(z)=\Delta(z)-\alpha\delta(z)-2\omega\ . \tag{18}$$

By representations (16) and (17) we have

$$U_1(z)=\left(\omega^2+1\right)\cos\pi z+f_2(z), \tag{19}$$

where $f_2(z)=f(z)-\alpha f_1(z)$. From (18) allowing for (14) we get $|U_1(\lambda_k)|\geq 2|\omega|$, i.e. $U_1(\lambda_k)\leq -2|\omega|$ or $U_1(\lambda_k)\geq 2|\omega|$. The system of inequalities (13) shows that the signs of the terms of the sequence $\{U_1(\lambda_k)\}$ alternate. By relations (16) and (19) we get $signU_1(\lambda_k)=(-1)^k$ for all sufficiently large values of $|k|$. Then there exists such a number $h_k$ that

$$U_1(\lambda_k)=2|\omega|(-1)^k\,ch\ h_k. \tag{20}$$

Consider the function

$$U_2(z)=\frac{\omega^2-1}{\omega^2+1}U_1(z)+\frac{4\omega^2}{\omega^2+1}\tilde\varphi(z), \tag{21}$$

where $\tilde\varphi(z)=\delta(z)\sum\limits_{k=-\infty}^{\infty}\frac{\tilde\varphi(\lambda_k)}{(z-\lambda_k)\delta'(\lambda_k)}$,

$$\tilde\varphi(\lambda_k)=\frac{1-\omega^2}{4\omega^2}U_1(\lambda_k)+\frac{1+\omega^2}{4\omega^2}(-1)^k\sigma_k'\sqrt{U_1^2(\lambda_k)-4\omega^2}=$$
$$=\frac{1-\omega^2}{2|\omega|}(-1)^k\,ch\ h_k+\frac{1+\omega^2}{2|\omega|}(-1)^k\sigma_k'\,|sh\ h_k|, \tag{22}$$

and $\{\sigma_k'\}$ is the sequence of signs satisfying the following condition: $\sigma_k'=0$ if $|U_1(\lambda_k)|=2|\omega|$ and $\sigma_k'=\pm1$ otherwise, moreover there exists such a natural number $N$ that $\sigma_k'=1$ at $|k|>N$. By (6) and (19) $\{\tilde\varphi(\lambda_k)\}\in l_2$ and consequently by theorem 28 of [8] and Paley-Wiener theorem [8, p.47] it holds the representation

$$\tilde\varphi(z)=\int\limits_{-\pi}^{\pi}h(t)e^{izt}dt,\quad h(t)\in L_2[-\pi,\pi]\ . \tag{23}$$

Since the function $s(z)=\frac{1}{2}[U_1(z)-U_2(z)]$ according to (19), (21) and (23) is representable in the form

$$s(z)=\cos\pi z+g(z)\quad\left(g(z)=\int_{-\pi}^{\pi}\tilde{g}(t)e^{izt}dt,\quad \tilde{g}(t)\in L_2[-\pi,\pi]\right)$$

then by lemma 4 of [2] its zeros $\theta_k$ are subjected to the asymptotics

$$\theta_k=k-\frac{1}{2}+\tau_k,\quad \{\tau_k\}\in l_2\ . \tag{24}$$

Then, allowing for (20)-(23) we have

$$s(\lambda_k)=\frac{1}{2}[U_1(\lambda_k)-U_2(\lambda_k)]=|\omega|(-1)^k ch\ h_k\left(1-\sigma_k'|th\ h_k|\right)$$

and since $|th\ h_k|<1$, then $sign\ s(\lambda_k)=(-1)^k$. Hence, it follows that only one and by asymptotic equality (24) only one zero of the function $s(z)$ lies on each interval $(\lambda_k,\lambda_{k+1})$.

Consequently, the zeros of the functions $\delta(z)$ and $s(z)$ alternate

$$...<\lambda_k<\theta_{k+1}<\lambda_{k+1}<\theta_{k+2}<...$$

Thus, the sequences $\{\lambda_k\}$ and $\{\theta_k\}$ satisfy all the conditions of theorem 2 of [2]. According to this theorem there exist the functions $p(x)$ and $q(x)$ such that $\delta(x)$ and $s(z)$ are the characteristic functions of boundary value problems, generated by the system of Dirac equations with these coefficients and boundary conditions (2) and $y_1(0)=y_2(\pi)=0$.

It is easily established that the spectrum of the constructed problem $L_1$ coincides with $\{\mu_k^{\pm}\}$.

The theorem is proved.

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Received June 08, 2004; Revised October 12, 2004.
Translated by Nazirova S.H.