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ON SOLUTION OF ONE BOUNDARY VALUE PROBLEM FOR THE MIXED TYPE EQUATION WITH TWO DEGENERATION LINES BY THE FINITE DIFFERENCES METHOD

Abstract

Considered is a boundary value problem for a mixed type equation with two degeneration lines. Solution of the problem is brought to that of a boundary problem in the elliptic part of the domain under consideration, and the obtained problem is solved by the finite differences method. Proved are the existence of a unique solution and its convergence. The convergence rate of the difference problem is determined.

1. Problem statement and passage to the corresponding boundary value problem in the elliptic part of the considered domain.

Consider the equation of mixed type

$$u_{xx} + sign\,(xy)\, u_{yy} = 0. \tag{1.1}$$

Let Ω be a domain bounded by the arc $\sigma = AB : x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0$ and the characteristics $BC : y - x = 1,\ CD : x + y = 0,\ DA : x - y = 1$ of equation (1.1). Ω_1 and Ω_2 are hyperbolic parts at $x > 0$ and $x < 0$ respectively, and Ω_3 is an elliptic part of the domain Ω, $I_1 = OA$ and $I_2 = OB$ are unit intervals.

Problem. Find the function $u = u(x, y)$ satisfying the following conditions:

1) $u = u(x, y)$ satisfies equation (1.1) in the domain $\Omega \backslash I_1 \backslash I_2$;

2) $u \in C\left(\overline{\Omega} \backslash I_1 \backslash I_2\right) \cap C^{(1)}\left(\Omega \backslash I_1 \backslash I_2\right)$;

3) $u_x(x, 0),\ u_y(x, 0) \in C(I_1), u_x(0, y),\ u_y(0, y) \in C(I_2)$, moreover they can reduce to infinity of order below unit at the points $A(1, 0), O(0, 0), B(0, 1)$;

4) $u = u(x, y)$ satisfies the boundary conditions:

$$u|_{AB} = \varphi(\theta),\quad 0 \leq \theta \leq \frac{\pi}{2}, \tag{1.2}$$

$$u|_{OD} = \psi_1(x),\quad 0 \leq x \leq \frac{1}{2}, \tag{1.3}$$

$$u|_{OC} = \psi_2(y),\quad 0 \leq y \leq \frac{1}{2}, \tag{1.4}$$

where $\varphi \in C\left[0, \frac{\pi}{2}\right]$, $\psi_i(t) \in C\left[0, \frac{1}{2}\right] \cap C^{(1)}\left(0, \frac{1}{2}\right)$, $i = 1, 2, \psi_1(0) = \psi_2(0)$.

5) The functions $u(x, y)$, $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ satisfy the sewing conditions

6)

$$u(x, -0) = \alpha_{11}(x)\, u(x, +0) + \alpha_{12}(x), \tag{1.5}$$

$$u_y(x, -0) = \alpha_{21}(x)\, u_y(x, +0) + \alpha_{22}(x), \tag{1.6}$$

$$u(-0, y) = \alpha_{31}(y)\, u(+0, y) + \alpha_{32}(y), \tag{1.7}$$

$$u_x(-0,y) = \alpha_{41}(y)\, u_x(+0,y) + \alpha_{42}(y), \tag{1.8}$$

where $0 \le x \le 1,\ 0 \le y \le 1,\ \alpha_{ij} \in C^{(1)}[0,1],\ i = 1,2,3,4;\ j = 1,2.$

It should be noted that in [1] the following sufficient conditions of a unique solvability of the following problem were formulated:

$$\alpha'_{2i-1,2}(t) \equiv \alpha_{2i,2}(t), \quad \alpha'_{2i-1,1}(t) \cdot \alpha_{2i,1}(t) \ge 0, \quad i = 1,2. \tag{1.9}$$

We can show that the function $u = u(x,y)$ satisfying in hyperbolic domain Ω_1 equation (1.1) and boundary conditions (1.3), (1.5), (1.6) satisfies also the condition

$$\alpha_{11}(x)\frac{\partial u(x,+0)}{\partial x} - \alpha_{21}(x)\frac{\partial u(x,+0)}{\partial y} + \alpha'_{11}(x)\, u(x,+0) =$$

$$\psi'_1\left(\frac{x}{2}\right) - \alpha'_{12}(x) + \alpha_{22}(x). \tag{1.10}$$

Similarly the function $u = u(x,y)$ satisfying in hyperbolic domain Ω_2 equation (1.1) and boundary conditions (1.4) , (1.7), (1.8) satisfies also the condition

$$\alpha_{31}(y)\frac{\partial u(+0,y)}{\partial y} - \alpha_{41}(y)\frac{\partial u(+0,y)}{\partial x} + \alpha'_{31}(y)\, u(+0,y) =$$

$$= \psi'_2\left(\frac{y}{2}\right) - \alpha'_{32}(y) + \alpha_{42}(y). \tag{1.11}$$

Subject to conditions (1.10)-(1.11) for finding the exact solution, problems (1.1)-(1.8) in the elliptic part Ω_3 of domain Ω we have the following problem:

Find continuous in $\Omega_3 \cup \sigma \cup I_1 \cup I_2$ the function $u = u(x,y)$ satisfying the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } \quad \Omega_3 \tag{1.12}$$

and the boundary conditions

$$u|_\sigma = \varphi, \tag{1.13}$$

$$\alpha_{11}(x)\frac{\partial u(x,0)}{\partial x} - \alpha_{21}(x)\frac{\partial u(x,0)}{\partial y} + \alpha'_{11}(x)\, u(x,0) = \Phi_1(x), \ x \in I_1, \tag{1.14}$$

$$\alpha_{31}(y)\frac{\partial u(0,y)}{\partial y} - \alpha_{41}(y)\frac{\partial u(0,y)}{\partial x} + \alpha'_{31}(y)\, u(0,y) = \Phi_2(y), \ y \in I_2, \tag{1.15}$$

$$\phi_1(x) = \psi'_1\left(\frac{x}{2}\right) - \alpha'_{12}(x) + \alpha_{22}(x), \ \Phi_2(y) = \psi'_2\left(\frac{y}{2}\right) - \alpha'_{32}(y) + \alpha_{42}(y).$$

2. Approximation and determination of approximation error

Let's construct the net domain $\overline{\omega}_h$ in the closed domain $\overline{\Omega}_3 = \Omega_3 \cup I_1 \cup I_2 \cup \sigma$. Let $N \ge 2$ be a fixed natural number. Let's divide the segments $I_1 = OA$ and $I_2 = OB$ into N equal parts, and points of division we denote by x_i and y_j:

$$x_i = ih, \ i = 0,1,...,N, \ y_j = jh, \ j = 0,1,...,N, \ h = 1/N.$$

respectively.

Through the points of division $x = x_i$ and $y = y_j$ we draw the straight lines parallel to the coordinates axes Oy and Ox, respectively. Denote by σ_h the set of points of intersection of the straight lines $x = x_i$ and $y = y_j$ with the boundary σ. Let

$$\gamma_{h,1} = \{x_i = ih,\ i = 1, 2, ..., N-1,\ h = 1/N\},$$

$$\gamma_{h,2} = \{y_j = jh,\ j = 1, 2, ..., N-1, h = 1/N\},$$

$$\gamma_h = \gamma_{h,1} \cup \gamma_{h,2} \cup \sigma_h,\ \omega_h = \{(x_i, y_j) \in \Omega_3\},\ \overline{\omega}_h = \omega_h \cup \gamma_h.$$

As in [2] the set of internal nodes ω_h we divide into three sets of nodes ω_h^0, ω_h^* and ω_h^{**}, where ω_h^0 is a set of strongly internal, ω_h^* is a set of boundary, ω_h^{**} is a set of non-regular boundary nodes.

Denote by W_{ij} the value of the net function $W = W(x, y)$ in nodes (x_i, y_j) of the net $\overline{\omega}_h$.

Associate the following difference problem to problem (1.12)-(1.15):

$$L^{(1)}W_{ij} \equiv 4W_{ij} - W_{i+1,j} - W_{i+1,j} - W_{i,j-1} - W_{i,j+1} = 0,\ (x_i y_j) \in \omega_h \backslash \omega_h^{**}, \quad (2.1)$$

$$L^{(2)}W_{ij} \equiv W_{ij} - \frac{h_2}{(h+h_1)(h_1+h_2)}\left(hW_{i+\frac{1}{2},j} + h_1 W_{i-1,j}\right) -$$

$$-\frac{h_1}{(h+h_2)(h_1+h_2)}\left(hW_{i,j+\frac{1}{2}} + h_2 W_{i,j-1}\right) = 0,\ (x_i, y_j) \in \omega_h^{**}, \quad (2.2)$$

$$L^{(3)}W_{i0} \equiv \alpha_{11}(x_i)(W_{i0} - W_{i-1,0}) - \alpha_{21}(x_i)(W_{i1} - W_{i0}) +$$

$$+h\alpha_{11}'(x_i) W_{i0} = h\Phi_1(x_i),\quad i = 1, 2, ..., N-1, \quad (2.3)$$

$$L^{(4)}W_{0j} \equiv \alpha_{31}(y_j)(W_{0j} - W_{0,j-1}) - \alpha_{41}(y_j)(W_{1j} - W_{0j}) +$$

$$+h\alpha_{31}'(y_j) W_{0j} = h\Phi_2(y_j),\quad j = 1, 2, ..., N-1, \quad (2.4)$$

$$W_{ij}\big|_{\sigma_h} = \varphi. \quad (2.5)$$

Here $W_{i\pm\frac{1}{2},j}$ or $W_{i,j\pm\frac{1}{2}}$ is a value of the net function $W = W(x, y)$ in the node $\left(x_{i\pm\frac{1}{2}} y_j\right) \in \sigma_h$ or $\left(x_i, y_{j\pm\frac{1}{2}}\right) \in \sigma_h$ whose distance from the non-regular neighboring node (x_i, y_j) is smaller than h. In equation (2.2) these distances in directions Ox and Oy are denoted by the h_1 and h_2, respectively. Note that this difference equation has most general form. In this equation in particular h_1 or h_2 can be equal to h.

For determining the error of approximation of difference problem (2.1)-(2.5) write the difference problem relative to the net function $Z_{ij} = W_{ij} - u(x_i, y_j)$:

$$L^{(1)}Z_{ij} = \psi_{ij}, \qquad (x_i, y_j) \in \omega_h \backslash \omega_h^{**}, \quad (2.6)$$

$$L^{(2)}Z_{ij} = \psi_{ij}^{**}, \qquad (x_i, y_j) \in \omega_h^{**}, \quad (2.7)$$

$$L^{(3)}Z_{i0} = \psi_{i0}, \quad i = 1, 2, ..., N-1, \quad (2.8)$$

$$L^{(4)}Z_{0j} = \psi_{0j}, \quad i = 1, 2, ..., N-1, \quad (2.9)$$

$$Z_{ij|\sigma_h} = 0. \quad (2.10)$$

Here $\psi_{ij}, \psi_{ij}^{**}, \psi_{i0}, \psi_{0j}$ determine the error of approximation in the nodes $\psi_h \backslash \psi_h^{**}, \psi_h^{**}, \gamma_{h,1}$ and $\gamma_{h,2}$,, respectively. Using Taylor formula for these errors we can easily obtain the corresponding expressions

Let $\varepsilon > 0-$ be a sufficiently small number and

$$\overline{\Omega}_0(\varepsilon) = \left\{(x,y) \in \overline{\Omega}_3,\ x^2+y^2 \le \varepsilon^2\right\},\ \overline{\Omega}_1(\varepsilon) = \left\{(x,y) \in \overline{\Omega}_3,\ (x-1)^2+y^2 \le \varepsilon^2\right\}$$

$$\overline{\Omega}_2(\varepsilon) = \left\{(x,y) \in \overline{\Omega}_3,\ x^2+(y-1)^2 \le \varepsilon^2\right\},\ \overline{\Omega}_3(\varepsilon) = \overline{\Omega}_3 \backslash \overline{\Omega}_0(\varepsilon) \backslash \overline{\Omega}_1(\varepsilon) \backslash \overline{\Omega}_2(\varepsilon)$$

Assume that solution of equation (1.12) the function $u = u(x,y)$ in $\Omega_3(\varepsilon)$ has bounded partial derivatives till the third order inclusively and

$$\left|\frac{\partial^k u}{\partial x^k}\right| \le M, \quad \left|\frac{\partial^k u}{\partial y^k}\right| \le M, \qquad k = 1,2,3. \tag{2.11}$$

We simultaneously assume that in domains $\overline{\Omega}_m(\varepsilon)$, $m = 0,1,2,$ the solution $u = u(x,y)$ satisfies the conditions

$$\left|\frac{\partial^k u}{\partial x^k}\right| \le \frac{C}{\varepsilon^{k-\alpha}}, \quad \left|\frac{\partial^k u}{\partial y^k}\right| \le \frac{C}{\varepsilon^{k-\alpha}}, \qquad k = 1,2,3, \tag{2.12}$$

where $M > 0,\ C > 0,\ 0 < \alpha \le 1$ are constants.

Subject to conditions (2.11) and (2.12) after the elementary transformations for error approximation we have:

a) Let $(x_i, y_j) \in \Omega_3(\varepsilon)$. Then

$$|\psi_{ij}| \le \frac{2M}{3}h^3, \quad |\psi_{ij}^{**}| \le \frac{2h^2 h_1 h_2}{3(h_1+h_2)}M,$$

$$|\psi_{i0}| \le h^2 LM, \quad |\psi_{0j}| \le h^2 LM, \tag{2.13}$$

where

$$L = \max_{0 \le t \le l} |\alpha_{i1}(t)|, \qquad i = 1,2,3,4.$$

b) Let $(x_i, y_j) \in \overline{\Omega}_m(\varepsilon)$. Then

$$|\psi_{ij}| \le \frac{2C}{3}\frac{h^3}{\varepsilon^{3-\alpha}}, \quad |\psi_{ij}^{**}| \le \frac{2C}{3}\frac{h^2 h_1 h_2}{(h_1+h_2)\varepsilon^{3-\alpha}},$$

$$|\psi_{i0}| \le CL\frac{h^2}{\varepsilon^{2-\alpha}}, \quad |\psi_{0j}| \le CL\frac{h^2}{\varepsilon^{2-\alpha}}. \tag{2.14}$$

3. Extremum principle and convergence of difference problem.

Consider the difference problem

$$L^{(1)}W_{ij} = g_{ij}, \quad (x_i, y_j) \in \omega_h \backslash \omega_h^{**}, \tag{3.1}$$

$$L^{(2)}W_{ij} = g_{ij}^{**}, \quad (x_i, y_j) \in \omega_h^{**}, \tag{3.2}$$

$$L^{(3)}W_{i0} = g_{i0}, \quad i = 1,2,...,N-1, \tag{3.3}$$

$$L^{(4)}W_{0j} = g_{0j}, \quad j = 1, 2, ..., N-1, \tag{3.4}$$

$$W_{ij}\left|\sigma_h\right. = \varphi. \tag{3.5}$$

Assume that the coefficients $\alpha_{im}(t)$ contained in expressions $L^{(3)}W_{i0}$ and $L^{(4)}W_{oj}$ satisfy the conditions

$$\alpha_{i1}(t) > 0, \quad \alpha_{i+1,1}(t) > 0, \quad \alpha'_{i1}(t) \geq 0, \quad 0 < t < 1, \quad i = 1, 3. \tag{3.6}$$

Theorem 1 (Extremum principle). *Let W_{ij} be unequal constant net function satisfy the difference problem (3.1)-(3.5). Assume that the coefficients $\alpha_{im}(t)$ satisfy conditions (3.6). Then if $g_{ij} \leq 0,\ g_{ij}^{**} \leq 0,\ g_{0j} \leq 0$ $\left(g_{ij} \geq 0,\ g_{ij}^{**} \geq 0,\ g_{0j} \geq 0\right)$, then the net function W_{ij} can't take on largest positive value in net point nodes $\omega_h \cup \gamma_{h,1} \cup \gamma_{h,2}$.*

Theorem 2. *Let conditions (3.6) be fulfilled. Then if $g_{ij} \geq 0,\ g_{ij}^{**} \geq 0,\ g_{0j} \geq 0,\ \varphi \geq 0$ $\left(g_{ij} \leq 0,\ g_{ij}^{**} \leq 0,\ g_{0j} \leq 0,\ \varphi \leq 0\right)$, then for the net function W_{ij} satisfying problem (3.1)-(3.5) it holds the inequality $W_{ij} \geq 0$ $(W_{ij} \leq 0)$ in $\overline{\omega_h}$.*

Corollary. If $g_{ij} \equiv 0,\ g_{ij}^{**} \equiv 0,\ g_{i0} \equiv 0,\ \varphi \equiv 0$, then at fulfilling conditions (3.6) the difference scheme (3.1)-(3.5) has only trivial solution $W_{ij} \equiv 0$.

From this corollary it follows the existence of a unique solution of difference problem (3.1)-(3.5).

Theorem 3 (Comparison theorem). *Let W_{ij} be a solution of difference problem (3.1)-(3.5), and $\overline{W_{ij}}$ be a solution of difference problem obtained from (3.1)-(3.5) at substituting the functions g_{ij}, g_{ij}^{**}, g_{i0}, g_{0j} and φ respectively by $\overline{g_{ij}}$, $\overline{g_{ij}^{**}}$, $\overline{g_{i0}}$, $\overline{g_{0j}}$ and $\overline{\varphi}$. Let conditions (3.6) be fulfilled. Then if $|g_{ij}| \leq \overline{g_{ij}}$, $\left|g_{ij}^{**}\right| \leq \overline{g_{ij}^{**}}$, $|g_{i0}| \leq \overline{g_{i0}}$, $|g_{0j}| \leq \overline{g_{0j}}$ and $|\varphi| \leq \overline{\varphi}$ then it holds the inequality*

$$|W_{ij}| \leq \overline{W}_{ij} \quad \text{in} \quad \overline{\omega}_h.$$

Using the comparison theorem we prove the convergence of difference problem (2.1)-(2.5) and determine the convergence rate.

Let the numbers a and ε_1 be determined by the equalities

$$a^2 = \begin{cases} 1 \ \text{ at } \ (x_i, y_j) \in \Omega_3(\varepsilon) \setminus \gamma_{h,1} \setminus \gamma_{h,2}, \\ 1 + bh - h^2 \ \text{ at } \ (x_i, y_j) \in \gamma_{h,1} \cup \gamma_{h,2} \setminus \overline{\Omega_0}(\varepsilon), \end{cases}$$

$$\varepsilon_1^2 = \begin{cases} \varepsilon^2 \ \text{ at } \ (x_i, y_j) \in \overline{\Omega_0}(\varepsilon) \setminus \gamma_{h,1} \setminus \gamma_{h,2}, \\ \varepsilon^2 + bh - h^2 \ \text{ at } \ (x_i, y_j) \in \left(\gamma_{h,1} \cup \gamma_{h,2}\right) \cap \overline{\Omega_0}(\varepsilon), \end{cases}$$

where $b > 0$ is some number.

Determine the net function $\overline{Z_{ij}}$ by the equality

$$\overline{Z}_{ij} = \begin{cases} Kh\left(a^2 - x_i^2 - y_j^2\right) \quad \text{at} \quad (x_i, y_j) \in \Omega_3(\varepsilon), \\ K\dfrac{h^\alpha}{\varepsilon^2}\left(\varepsilon_1^2 - x_i^2 - y_j^2\right) \ \text{at} \ (x_i, y_j) \in \overline{\Omega_0}(\varepsilon), \\ K\dfrac{h^\alpha}{\varepsilon^2}\left(a^2 - x_i^2 - y_j^2\right) \ \text{at} \ (x_i, y_j) \in \overline{\Omega_m}(\varepsilon), \quad m = 1, 2, \end{cases}$$

where $K > 0$ is some constant.

If the number b is determined by the equality

$$b = \max (b_1, b_3), \ b_i = \max_{0\le t\le 1} \frac{1+2\alpha_{i1}(t)}{\alpha_{i+1,1}(t)}, \ i = 1, 3,$$

then after the elementary transformations we'll obtain:

$$L^{(1)}\overline{Z_{ij}} = 4Kh^3, \ L^{(2)}\overline{Z_{ij}} = \frac{2h^2 h_1 h_2}{h_1+h_2}K, \ L^{(3)}\overline{Z_{i0}} \ge Kh^2,$$

$$L^{(4)}\overline{Z_{0j}} \ge Kh^2 \ \text{ at } \ (x_i, y_j) \in \Omega_3(\varepsilon),$$

$$L^{(1)}\overline{Z_{ij}} = 4K\frac{h^{2+\alpha}}{\varepsilon^2}, \ \ L^{(2)}\overline{Z_{ij}} = \frac{2Kh^{1+\alpha}h_1 h_2}{\varepsilon^2(h_1+h_2)}, \ L^{(3)}\overline{Z_{i0}} \ge K\frac{h^{1+\alpha}}{\varepsilon^2},$$

$$L^{(4)}\overline{Z_{0j}} \ge K\frac{h^{1+\alpha}}{\varepsilon^2} \ \text{ at } \ (x_i, y_j) \in \overline{\Omega_0}(\varepsilon) \cup \overline{\Omega_2}(\varepsilon),$$

$$\overline{Z_{ij}}\,|\sigma_h \ge 0.$$

Compare the right hand-sides of the obtained relations for $\overline{Z_{ij}}$ with right hand-sides of corresponding equations in (2.6)-(2.10). Subject to inequalities (2.13)-(2.14) it is easy to be sure that the right parts (2.6)-(2.10) by module will be no more than correspoding expressions in the right hand-sides of obtained relations for $\overline{Z_{ij}}$ if the constants K are determined by the equality

$$K = \max\left(\frac{M}{3}, \ LM\right). \tag{3.7}$$

Then by virtue of comparison theorem we obtain the validity of the inequality $|Z_{ij}| \le \overline{Z_{ij}}$ in $\overline{\omega_h}$.

Thus it holds the following theorem:

Theorem 4. *Let the solutions $u = u(x, y)$ of equation (1.12) satisfy conditions (2.11)-(2.12). Assume that conditions (3.6) are fulfilled. Then the solution of difference problem (2.1)-(2.5) is converges to the solution of problem (1.12)-(1.15) and at this the estimation*

$$|W_{ij} - u(x_i, y_j)| \le \begin{cases} Kh \ \text{ at } \ (x_i, y_j) \in \Omega_3(\varepsilon), \\ 2Kh^\alpha \ \text{ at } \ (x_i, y_j) \in \Omega_m(\varepsilon), \ m = 0, 1, 2, \end{cases}$$

is true, where K is determined by equality (3.7).

References

[1]. Bitsadze A.V. *On some problems of mixed type.* DAN SSSR, 70, 4, 1950, pp.561-564. (Russian)

[2]. Samarskii A.A. *Introduction to theory of difference scheme.* M.: "Nauka", 1971, 552p. (Russian)

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