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ON THE BEHAVIOR SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER DIVERGENT PARABOLIC EQUATIONS

Abstract

An article deals with the first boundary value problem for the second order divergent parabolic equations. The regularity of boundary point is considered in supposition that in some neighborhood of this point boundary of domain has some special symmetric.

Investigations for regularity of boundary point for the second order parabolic equations begins from well-known work of Petrowski [1], in which unclosing with one other necessary and sufficient conditions of regularity for one-dimensional heat equation for domains restricted by straights $t = t^1$, $t = t^2$ and curves $x = \varphi_1(t)$, $x = \varphi_2(t)$ have been established. In works of Landis [2], [3] have been received the criteria of regularity for many-dimensional heat equation. In work [4] Evans and Gariepy established the necessary and sufficient condition for heat equation in terms of divergence of series from heat capacity. Further these results have been used for more large class of parabolic equations in works [5], [6], [7], [8]. Also note the work of Garafolo and Lancanelli.

Setting of the problem. Let D be bounded domain in $R_{n+1}, \partial D$ and $\Gamma(D)$ are its Euclidean and parabolic domains respectively. We consider in D the first boundary value problem

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) - \frac{\partial u}{\partial t};\ (x,t) \in D, \tag{1}$$

$$u|_{\Gamma(D)} = \varphi; \quad \varphi \in C(\Gamma(D)), \tag{2}$$

where $\|a_{ij}(x,t)\|$ is real symmetric matrix with bounded in D elements, and for $(x,t) \in D$ and $\xi = E_n$ the next condition is fulfilled

$$\gamma|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \gamma^{-1}|\xi|^2; \quad \gamma \in (0,1] - const \tag{3}$$

Now we give the definition of generalized solution of the problem (1)-(2).

Through $W_2^{1,1}(D)$ we denote the Banach space of functions $u(x,t)$, which have at D, the finite norm

$$\|u\|_{W_2^{1,1}(D)} = \left(\int_D \left(u^2 + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right) dxdt\right)^{1/2},$$

and let $\mathring{W}_2^{1,1}(D)$- subdomain $W_2^{1,1}(D)$, the sense set in which is unity of all functions $u(x,t) \in C^\infty\left(\bar{D}\right)$, vanishes at $\Gamma(D)$.

Let $C_R^{t^1,t^2}\left(x^0\right)$ - cylinder $\left\{(x,t) : \left|x - x^0\right| < R,\ t^1 < t < t^2\right\}$, where $x^0 \in E_n$, $R > 0$; $B_R\left(x^0\right)$ open n dimensional ball of radius R with center in point x^0.

We denote through $\gamma^-(D)$ set of all points $(x,t) \in \gamma(D)$, for any of which there is $h > 0$ such that

$$C_h^{t,t+h}(x) \subset D,\quad C_h^{t-h,t}(x) \subset R_{n+1}\backslash D\ .$$

Let $S(D) = \Gamma(D)\backslash\gamma^-(D)$, $A(D)$ - unity of $u(x,t) \in C^\infty\left(\bar{D}\right)$, vanishes at $S(D)$.

At first we suppose that boundary fucntion $\varphi(x,t)$ of boundary value problem (1)-(2) permits continuous $\Phi(x,t)$ in D such that $\Phi(x,t) \in W_2^{1,1}(D)$. Fucntion $u(x,t) \in W_2^{1,1}(D)$ is called the generalized solution of the first boundary value problem (1)-(2), if $(u - \Phi) \in \mathring{W}_2^{1,1}(D)$ and for any fucntion $\psi(x,t) \in \mathring{W}_2^{1,1}(D)$ the following integral identity is true

$$\int_D \left(\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \psi}{\partial x_j} + \frac{\partial u}{\partial t}\psi \right) dxdt = 0\ .$$

In case when boundary fucntion is continuous, it is necessary to approximate in metric $C(\Gamma(D))$ fucntion $\varphi(x,t)$ by sequel of smooth functions $\varphi_m(x,t)$, permitting continuity $\Phi_m(x,t)$ in D from space $W_2^{1,1}(D)$. Let $u^m(x,t)$ be sequel of solutions of value problems $Lu^m(x,t) = 0$; $(x,t) \in D,\ u^m|_{\Gamma(D)} = \varphi_m$.

According to results of Nash and Moser functions $u^m(x,t)$ are continuous by Holder in every strictly interior subdomain of domain D. Pointwise limit of sequel $u^m(x,t)$ by $m \to \infty$ is called generated solution of boundary value problem (1)-(2).

Let $\left(x^0, t^0\right) \in \Gamma(D)$. This point is called regular according first boundary value problem (1)-(2), if for any $\varphi(x,t) \in C(\Gamma(D))$ for generalized solution $u(x,t)$ the following limit equality is true

$$\lim_{\substack{(x,t)\to\left(x^0,t^0\right)\\ (x,t)\in D}} u(x,t) = \varphi\left(x^0, t^0\right)$$

The aim of this work is to find necessary and sufficient conditions of regularity $\left(x^0, t^0\right)$ under condition that boundary of considered domain in some neighborhood of this point has some special symmetric.

Further recording $C(...)$ means that positive constant C depends only from contains of brackets.

Now we show some auxiliary results. We note that first three lemmas are well-known results, which we use during of the proof necessary conditions of regularity of boundary point.

Let for $k > 0$

$$G_k(x,t) = \begin{cases} t^{-\frac{n}{2}} \exp\left[-\frac{k|x|^2}{4t}\right], \ if \quad t > 0 \\ \\ 0, \quad if \quad t \le 0 \end{cases}$$

Lemma 1. *If according to coefficients of operator L the condition (3) is fulfilled, then there is fundamental solution $F(x,y;t,\tau)$ of equation (1), and there are constants $k_1(\gamma,n)$, $k_2(\gamma,n)$, $C_1(\gamma,n)$ and $C(\gamma,n)$ such that*

$$C_1 G_{k_1}(x-y,t-\tau) \le F(x,y;t,\tau) \le C_2 G_{k_2}(x-y,t-\tau) \ . \tag{4}$$

The proof of lemma 1 is shown in work [10], and this lemma gives estimation of fundamental solution of equation (1).

Note. It is easy to see $k_1 \ge k_2$.

Let E be compact in R_{n+1}. Measure μ at E is called k-feasible if

$$\int_E G_k(x-y,t-\tau)\, d\mu(y,\tau) \le 1 \quad for \quad (x,t) \notin E \ .$$

Number $P_k(E) = \sup \mu(E)$ where exact upper bound takes by all k-feasible measures is called k-capacity of compact E.

By the same way we define capacity created by fundamental solution $F(x,y;t,\tau)$. It is will be denote through $P_F(E)$.

In case of $k = 1$ capacity denote through $P(E)$ and called heat capacity.

Lemma 2. *For any compact $E \subset R_{n+1}$ the following estimations are true*

$$C_3(k,n)\, P(E) \le P_k(E) \le C_4 P(E) \ . \tag{5}$$

Lemma 2 is proved in work [12].

Corollary 1. *For any compact $E \subset R_{n+1}$ such inequalities are true*

$$C_5(\gamma,n)\, P(E) \le P_F(E) \le C_6(\gamma,n)\, P(E) \ . \tag{6}$$

In future we denote functions G_{k_1} and G_{k_2} through G^+ and G^- respectively and capacities $P_{k_1}(E)$ and $P_{k_2}(E)$ through $P^+(E)$ respectively.

We called domain $D \subset R_{n+1}$ step domain if there are numbers $t^0 < t^1 < \cdots < t^k$ and domains Ω_i; $i = 1, \cdots, k$; such that D is set of inner points of unity $\bigcup_{i=1}^{k} \bar{P}_i$ where

$$P_i = \Omega_i \times \left(t^{i-1}, t^i\right) \ ; \quad i = 1, ..., k;$$

Lemma 3. *Let D-step domain with sufficient smooth boundaries of bases of its cylinders. Then for any $k>0$ there is measure μ at $\bar{D}$ such that if*

$$U(x,t)=\int_{\bar{D}} G_k(x-y,t-\tau)\,d\mu(y,\tau) \quad \textit{then} \quad U|_{\Gamma^-(D)}=1 \textit{ and}$$

$$\mu\left(\bar{D}\right)=P_k\left(\bar{D}\right) \quad \textit{where} \quad \Gamma^-(D)=\partial D\backslash\gamma^-(D).$$

Lemma 3 is proved in work [12].

Note. The same result also is true for potentials created by fundamental solution $F(x,y;t,\tau)$.

We suppose that point $\left(x^0,t^0\right)$ coincide with point of origin and in some neighborhood of point $(0,0)$ D will be showed as

$$\left\{(x,t):|x|^2<-t\alpha(-t)\,,\ -d<t<0\right\}\ ,$$

where d sufficiently small is positive number, $\alpha(z)$ is positive nonincreasing fucntion $(0,d]$ and

$$\left|\alpha'(z)\right|\le\frac{K}{Z},\quad K<\frac{2n}{k_1};\quad z\in(0,d], \tag{7}$$

Moreover we suppose that there is finite or infinite limit

$$J=\lim_{z\to0+}\left|\alpha'(z)\right|z\ln\frac{1}{z}, \tag{8}$$

and if $J=\dfrac{4}{k_1}$, $d(z)=|\alpha'(z)|-\dfrac{4}{k_1 z\ln\frac{1}{z}}$, then $\displaystyle\int_0^d d^+(z)\,dz<\infty$ where $d^+(z)=\max\{d(z),0\}$.

Let for natural m

$$A_m^\pm=\left\{(y,\tau):e^{\frac{mn}{2}}\le G^\pm(-y,-\tau)\le e^{\frac{(m+1)n}{2}}\right\}\ ,$$

$$H_m^\pm=A_m^\pm\backslash D\ .$$

We denote through z_m nodule of temporary coordinate points of intersection ∂D with surface of level

$$\left\{(y,\tau):G(-y,-\tau)=e^{\frac{mn}{2}}\right\}\ ,$$

which belong to interval $\left(e^{-\beta m},e^{-m}\right)$ where $\beta=\dfrac{8n}{2n-k_1K}$, (For the ambiguity we denote through z_m the exact lower edge of these modules).

It is easy to see that set $A_m^\pm$ will be

$$A_m^\pm=\left\{(y,\tau):\frac{2n}{k_1}(-\tau)\ln\frac{e^{-m-1}}{-\tau}\le|y|^2\le\frac{2n}{k_1}(-\tau)\ln\frac{e^{-m}}{-\tau}\right\}.$$

Then z_m is root of equation

$$d\left(z_m\right)=\frac{2n}{k_1}\ln\frac{e^{-m}}{z_m}. \tag{9}$$

We need to show that for sufficiently big m root z_m of equation (9) at the $\left(e^{\beta m},e^{-m}\right)$ exist. Let

$$f\left(z\right)=\alpha\left(z\right)-\frac{2n}{k_1}\ln\frac{e^{-m}}{z}.$$

It is clear that $f\left(e^{-m}\right)=\alpha\left(e^{-m}\right)>0$. But

$$f\left(e_m^{-\beta m}\right)=\alpha\left(e_m^{-\beta_m}\right)-\frac{2n\left(\beta-1\right)}{k_1}m\ . \tag{10}$$

From (7) we conclude that for $t^1<t^2$

$$\alpha\left(t^1\right)-\alpha\left(t^2\right)=-\int\limits_{t^1}^{t^2}\alpha'\left(z\right)dz\leq K\int\limits_{t^1}^{t^2}\frac{dz}{z}=K\ln\frac{t^2}{t^1}\ . \tag{11}$$

In particular

$$\alpha\left(e_m^{-\beta m}\right)\leq\alpha\left(d\right)+K\ln d+K\beta m\leq\alpha\left(d\right)+K\beta m\leq\frac{K+\frac{2n}{R_1}}{2}\beta m\ ,$$

and for validity $f\left(e^{-\beta m}\right)<0$ according to (10) will be enough that

$$\frac{K+\frac{2n}{R_1}}{2}\beta m<\frac{2n}{R_1}\left(\beta-1\right),$$

$$\frac{\beta}{\beta-1}<\frac{\frac{4n}{k_1}}{K+\frac{2n}{k_1}}=\frac{4n}{k_1K+2n},$$

$$1-\frac{1}{\beta}>\frac{k_1K+2n}{4n},\quad\frac{1}{\beta}<\frac{2n-k_1K}{4n},$$

finally $\beta>\dfrac{4}{2n-k_1K}$, because of choosing β. So root z_m of equation (9) at the given interval exists. We show that $z_{m+1}<z_m$. Really we suppose reverse, that is $z_{m+1}\geq z_m$. Then according to (9)

$$z_m=e^{-m}\exp\left[-\frac{k}{2n}\alpha\left(z_m\right)\right]\ ,$$

so

$$\frac{z_m}{z_{m+1}}=e\exp\left[-\frac{k_1}{2n}\left(\alpha\left(z_m\right)-\alpha\left(z_{m+1}\right)\right)\right]\geq$$

$$\geq e\exp\left[-\frac{k_1}{2n}K\ln\frac{z_{m+1}}{z_m}\right]=e\exp\left[\ln\left(\frac{z_m}{z_{m+1}}\right)^{\delta}\right]\ ,$$

where $\delta = \dfrac{Kk_1}{2n} < 1$. (We used (11)).

Now
$$\frac{z_m}{z_{m+1}} \geq e\left(\frac{z_m}{z_{m+1}}\right)^{\delta}; \quad \left(\frac{z_m}{z_{m+1}}\right)^{1-\delta} \geq e$$

and finally $\dfrac{z_m}{z_{m+1}} \geq e^{\frac{1}{1-\delta}} > 1$, which contradict to our supposition.

So, $z_{m+1} < z_m$. Moreover because of
$$\frac{z_m}{z_{m+1}} = e \exp\left[-\frac{k_1}{2n}\left(\alpha\left(z_m\right) - \alpha\left(z_{m+1}\right)\right)\right] \geq e,$$

that is $z_m \geq e\ z_{m+1}$.

But
$$\frac{z_m}{z_{m+1}} = e \exp\left[\frac{k_1}{2n}\left(\alpha\left(z_{m+1}\right) - \alpha\left(z_m\right)\right)\right] \leq e \exp\left[\frac{k_1}{2n} K \ln \frac{z_m}{z_{m+1}}\right] = e\left(\frac{z_m}{z_{m+1}}\right)^{\delta},$$

and follows
$$\left(\frac{z_m}{z_{m+1}}\right)^{1-\delta} \leq e\ , \quad that\ \ is\ \ z_m \leq e^{\frac{1}{1-\delta}} z_{m+1}\ .$$

Let now $V_m = \dfrac{z_m}{e^{-m}}$. We have $V_m = \exp\left[-\dfrac{k_1}{2n}\alpha\left(z_m\right)\right]$. So
$$\frac{V_{m+1}}{V_m} = \exp\left[-\frac{k_1}{2n}\left(\alpha\left(z_{m+1}\right) - \alpha\left(z_m\right)\right)\right] \leq 1$$

because of $z_{m+1} < z_m$. So the following is proved

Lemma 4. *For sufficiently big m the following estimations are true*
$$e z_{m+1} \leq z_m \leq e^{\delta_1} z_m, \tag{12}$$

$$V_{m+1} \leq V_m, \tag{13}$$

where $\delta_1 = \dfrac{2n}{2n - Kk_1}$. So there is nonnegative limit $V_0 = \lim\limits_{m\to\infty} V_m$.

Lemma 5. *For sufficiently big m the following estimation is true*
$$P\left(H_m^+\right) \leq C_7\left(\gamma, n\right)\left(z_m \ln \frac{1}{V_m}\right)^{n/2}. \tag{14}$$

Proof. We need to find that value of z^0, for which "width" of surface of level $\left\{(y,\tau) : G_k^+\left(-y,-\tau\right) = e^{\frac{mn}{2}}\right\}$ will be increased. For this we find maximum of fucntion $\dfrac{2n}{k_1} z \ln \dfrac{e^{-m}}{z}$. We have
$$\left(z \ln \frac{e^{-m}}{z}\right)' = \ln \frac{e^{-m}}{z} - 1 = 0\ .$$

Then $z^0 = e^{-m-1}$. We consider only the most interesting case, when $V_0 = 0$. Then for sufficiently big m $z_m < z^0$. So

$$H_m^+ \subset C^{-z_m,0}_{\sqrt{\frac{2n}{k_1} z_m \ln \frac{1}{V_m}}} (0) .$$

For the same reason $z_m \leq \dfrac{2n}{k_1} z_m \ln \dfrac{1}{V_m}$ and we obtain

$$H_m^+ \subset C^{-\frac{2}{k_1} z_m \ln \frac{1}{V_m},0}_{\sqrt{\frac{2n}{k_1} z_m \ln \frac{1}{V_m}}} .$$

Otherwise according to theorem of Landis [3]

$$P\left(C_R^{-R^2,0}(0)\right) = C_8(n) R^n .$$

So

$$P\left(H_m^+\right) \leq C_8 \left(\frac{2n}{k_1}\right)^{n/2} \left(z_m \ln \frac{1}{V_m}\right)^{n/2},$$

and lemma is proved.

Lemma 6. *For sufficiently big m the following estimation is true*

$$P\left(H_m^+\right) \leq C_9 (\gamma, n)^{n/2} \left(z_m \ln \frac{1}{V_m}\right)^{n/2} . \tag{15}$$

Let $T_m = H_m^+ \cap \{(y,\tau) : -z_m \leq \tau \leq -z_{m+1}\}$.

Lemma 7. *For sufficiently big m the following estimation is true*

$$P(T_m) \geq C_{13} (\gamma, n) \left(z_m \ln \frac{1}{V_m}\right)^{n/2} .$$

Proof. We act as in proof of previous lemma and see that measure $\dfrac{1}{C_{12}} dS(y,\tau)$ is k_1-admissible at T_m. So

$$P(T_m) \geq \frac{1}{C_{12}} \int\limits_{T_m} dS(y,\tau) = \frac{mes(T_m)}{C_{12}} \geq \frac{mes\left(\tilde{T}_m\right)}{C_{12}}, \tag{16}$$

where $\tilde{T}_m$-projection T_m at the hyper plane $\tau = 0$. It is easy to see that $\tilde{T}_m$ is ball layer

$$\bar{B}_{\sqrt{\frac{2n}{k_1} z_m \ln \frac{1}{V_m}}} (0) \Big/ B_{\sqrt{\frac{2n}{k_1} z_{m+1} \ln \frac{1}{V_{m+1}}}} (0) .$$

So

$$mes\, \tilde{T}_m = \left(\frac{2n}{k_1}\right)^{n/2} \omega_n \left(\left(z_m \ln \frac{1}{V_m}\right)^{n/2} - \left(z_{m+1} \ln \frac{1}{V_{m+1}}\right)^{n/2}\right) =$$

$$= \left(\frac{2n}{k_1}\right)^{n/2} \omega_n (z_m)^{n/2} \left[\left(\ln \frac{1}{V_m}\right)^{n/2} - \left(\frac{z_{m+1}}{z_m} \ln \frac{1}{V_{m+1}}\right)^{n/2}\right] .$$

We use lemma 1 and obtain

$$mes\ \tilde{T}_m \geq \left(\frac{2n}{k_1}\right)^{n/2} \omega_n (z_m)^{n/2} \left[\left(\ln \frac{1}{V_m}\right)^{n/2} - \left(\frac{z_{m+1}}{z_m}\right)^{n/2} \left(\ln \frac{1}{V_{m+1}}\right)^{n/2}\right] =$$

$$= \left(\frac{2n}{k_1}\right)^{n/2} \omega_n (z_m)^{n/2} \left[\left(\ln \frac{1}{V_m}\right)^{n/2} - \left(\frac{z_{m+1}}{z_m} \ln \frac{1}{V_m} \frac{V_m}{V_{m+1}}\right)^{n/2}\right] \geq$$

$$\geq \left(\frac{2n}{k_1}\right)^{n/2} \omega_n (z_m)^{n/2} \left[\left(\ln \frac{1}{V_m}\right)^{n/2} - \left(\frac{z_{m+1}}{z_m} \left(\ln \frac{1}{V_m} + \ln \frac{V_m}{V_{m+1}}\right)\right)^{n/2}\right] =$$

$$= \left(\frac{2n}{k_1}\right)^{n/2} \omega_n (z_m)^{n/2} \left[\left(\ln \frac{1}{V_m}\right)^{n/2} - \right.$$

$$\left. - \left(\frac{z_{m+1}}{z_m} \left(\ln \frac{1}{V_m} + \ln \exp\left[\frac{2n}{k_1} (\alpha (z_{m+1}) - \alpha (z_m))\right]\right)\right)^{n/2}\right] \geq$$

$$\geq \left(\frac{2n}{k_1}\right)^{n/2} \omega_n (z_m)^{n/2} \left[\left(\ln \frac{1}{V_m}\right)^{n/2} - \left(\frac{z_{m+1}}{z_m} \left(\ln \frac{1}{V_m} + \frac{k_1 K}{2n} \ln \frac{z_m}{z_{m+1}}\right)\right)^{n/2}\right] =$$

$$= \left(\frac{2n}{k_1}\right)^{n/2} \omega_n \left(z_m \ln \frac{1}{V_m}\right)^{n/2} \left[1 - \left(\frac{z_{m+1}}{z_m} \left(1 + \frac{k_1 K}{2n} \frac{\ln \frac{z_m}{z_{m+1}}}{\ln \frac{1}{V_m}}\right)\right)^{n/2}\right] .$$

Because of $\frac{z_m}{z_{m+1}} \leq e^{\beta}$, $V_m \to 0$ when $m \to \infty$, so for sufficiently big m

$$\frac{z_{m+1}}{z_m} \left(1 + \frac{k_1 K}{2n} \frac{\ln \frac{z_m}{z_{m+1}}}{\ln \frac{1}{V_m}}\right) \leq e^{-\beta} (1 + O(1)) \leq C_{14}(n) < 1 .$$

So

$$mes\ \tilde{T}_m \geq \left(\frac{2n}{k_1}\right)^{n/2} \omega_n \left(z_m \ln \frac{1}{V_m}\right)^{n/2} \left[1 - C_{14}^{n/2}\right] .$$

Now the statement of lemma follows from (16).

Now we will prove one of the main lemmas, which used under proved of sufficiently conditions of regularity of boundary point. This lemma is about increasing of positive solutions.

Lemma 8. **(about increasing of positive solutions)** *Let $D_m = D \cap \{(y,\tau) : -z_m < \tau < 0\}$ and in D_m defined the positive solution $u(x,t)$ of equation (1), continuous in $\bar{D}_m$ and in vanishes at $P(D_m) = \partial D \cap \{(y,\tau) : -z_m \leq \tau \leq 0\}$.*

Then there is $C_{15}(\gamma,n)$ *such that if* $l(m)\geq m+C_{15}\ln b$ *then for sufficiently big* m

$$\sup_{D_m} u \geq \left(1+\eta\, e^{\frac{mn}{2}} P\left(H_m^+\right)\right) \sup_{D_m+l_m} u \ , \tag{17}$$

where $\eta=\eta(\gamma,n)$.

Lemma 9. *Let for natural* $i=3$ $m_i=[4C_{15}i\ln\ln i]+1$. *Then for sufficiently big* i

$$m_{i+1}-m_i\geq C_{15}\ln\ln m_i \ .$$

Proof. We have

$$m_{i+1}-m_i=[4C_{15}(i+1)\ln\ln(i+1)]-[4C_{15}i\ln\ln i]\geq$$

$$\geq 4C_{15}(i+1)\ln\ln(i+1)-4C_{15}i\ln\ln i-1=$$

$$=4C_{15}(i+1)\ln\ln i-4C_{15}i\ln\ln i-1=4C_{15}\ln\ln i-1$$

and we need to show that for sufficiently big i

$$4C_{15}\ln\ln i-1\geq C_{15}\ln\ln\left\{[4C_{15}i\ln\ln i]+1\right\} \ .$$

But $4C_{15}\ln\ln i-1\geq 3C_{15}\ln\ln i$, that is will be enough to show

$$3\ln\ln i\geq \ln\ln\left\{[4C_{14}i\ln\ln i]+1\right\}$$

or $3\ln\ln i\geq\ln\ln\left\{5C_{15}i\ln\ln i\right\}$,

$$3\ln\ln i\geq\ln\ln i+\ln\ln 5C_{15}+\ln\ln\ln\ln i \ ,$$

and the lemma is proved.

Corollary. *If the conditions of lemma 8 are fulfilled then for sufficiently big* i

$$\sup_{D_{m_i}} u\geq\left(1+\eta e^{\frac{m_i n}{2}}P\left(H_{m_i}^+\right)\right)\sup_{D_{m_{i+1}}} u \ .$$

We will prove the sufficient condition of regularity.

Theorem 1. *If relative to coefficients of operator* L *the condition (3) is fulfilled, then for regularity of point* $(0,0)$ *relative to boundary value problem (1)-(2) will be enough that*

$$\sum_{m=3}^{\infty}\frac{1}{\ln\ln m}e^{\frac{mn}{2}}P\left(H_{m_i}^+\right)=\infty \ . \tag{18}$$

Proof. At first we show that if condition (30) is fulfilled, then

$$\sum_{m=3}^{\infty}e^{\frac{m_i n}{2}}P\left(H_{m_i}^+\right)=\infty \ . \tag{19}$$

Because of for any continuous and monotonously decreasing at $[1,\infty)$ positive fucntion $f(z)$, tending to zero, when $z\to\infty$, for any natural $i>1$

$$\sum_{k=2}^{i} f(k) \le \int_{1}^{i} f(z)\,dz \le \sum_{k=2}^{i-1} f(k) \ ,$$

then for sufficiently big natural q

$$\sum_{i=3}^{q} e^{\frac{m_i n}{2}} P\left(H_{m_i}^{+}\right) \ge C_{28}(\gamma,n) \int_{3}^{q+1} e^{\frac{m(z)n}{2}} P\left(H_{\{m(z)\}}^{+}\right) dz \ , \tag{20}$$

where $m(z)=4C_{15}z\ln\ln z$, $\;H_{\{z\}}^{+}=\left\{(y,\tau): e^{\frac{zn}{2}} \le G(-y,-\tau) \le e^{\frac{(z+1)n}{2}}\right\}\Big/ D$.

But otherwise after changing

$$4C_{15}z\ln\ln z = t;\ 4C_{15}z\left(\ln\ln z+\frac{1}{\ln z}\right)dz = dt;$$

$$dz=\frac{dt}{4C_{15}\ln\ln z+\dfrac{1}{\ln z}} \ge \frac{1}{5C_{15}}\frac{dt}{\ln\ln z} \ge \frac{1}{5C_{15}}\frac{dt}{\ln\ln t}$$

(because of $z\le t$) we obtain

$$\int_{3}^{q+1} e^{\frac{m(z)n}{2}} P\left(H_{\{m(z)\}}^{+}\right) dz \ge \frac{1}{5C_{15}} \int_{12C_{15}\ln\ln 3}^{4C_{15}(q+1)\ln\ln(q+1)} e^{\frac{tn}{2}} P\left(H_{\{t\}}^{+}\right) \frac{dt}{\ln\ln t} \ge$$

$$\ge \frac{1}{5C_{15}} \sum_{m=[12C_{15}\ln\ln 3]+2}^{[4C_{15}(q+1)\ln\ln(q+1)]} \frac{1}{\ln\ln m} e^{\frac{mn}{2}} P\left(H_{\{t\}}^{+}\right) \ .$$

Now it is easy to see that from (18) follows (19).

In order to prove regularity of point $(0,0)$ will be enough to show following: for any $\varepsilon_1>0$ and $\varepsilon_2>0$ there is $V>0$ such that for any L-subparabolic in D fucntion $u(x,t)\le 1$, vanishes at $\partial D\cap\{(x,t): -\varepsilon_1<t<0\}$, for $(x,t)\in D\cap\{(x,t): V<t<0\}$ the following inequality is fulfilled $u(x,t)<\varepsilon_2$. From the j we denote least natural number, for which $z_{m_j}<\varepsilon_1$ and let natural number $k>j+1$ will be such that in D_{m_k} will be point $\left(x^0,t^0\right)$, in which

$$u\left(x^0,t^0\right) \ge \varepsilon_2 \ .$$

We respectively use corollary from lemma 9 and obtain

$$1 \ge M_{m_j} \ge \prod_{i=j}^{k-1}\left(1+\eta e^{m_i\frac{n}{2}} P\left(H_{m_i}^{+}\right)\right)\varepsilon_2 \ , \ i.e.$$

$$\sum_{i.j=1}^{k-1} \ln\left(1+\eta e^{m_i \frac{n}{2}} P\left(H_{m_i}^+\right)\right) \le \ln\frac{1}{\varepsilon_2}\ .$$

It is easy to see that $P\left(H_{m_i}^+\right) \le C_{29}(\gamma, n)$.

Let

$$C_{30}(\gamma,n) = \inf_{t\in\{0,C_{29}]} \frac{\ln(1+\eta t)}{t}\ .$$

Then

$$\sum_{i=1}^{k-1} e^{\frac{\mu_i n}{2}} P\left(H_{m_i}^+\right) \le \frac{1}{C_{30}} \ln\frac{1}{\varepsilon_2}\ .$$

But because of (19) the last inequality is true only when $k = k_0(\varepsilon_1, \varepsilon_2, \gamma, n)$. Now will be enough to choose $V = z_{m_{k_0}+1}$ and theorem is proved.

Lemma 10. *If respectively to domain D the conditions (7)-(8) are fulfilled, then from*

$$\sum_{m=1}^{\infty} e^{\frac{mn}{2}} P\left(H_{m_1}^+\right) = \infty \tag{21}$$

follows condition (18).

Proof. We put more mild condition to $\alpha(z)$, than (8): if $J = \dfrac{4}{k_1}$, then

$$d^+(z) = \left\{\left|\alpha'(z)\right| z \ln\frac{1}{z} - \frac{4}{k_1}\right\}^+ \le \frac{K_1}{\ln\ln\dfrac{1}{z}}, \quad K_1 < \frac{2n}{k_1}\ ,$$

only when z is sufficiently small.

Let at first $J = \dfrac{4}{k_1}$. We will show that there is constant $C_{31}(n)$ such that for sufficiently big m

$$z_m \ge C_{31} \frac{e^{-m}}{m^{n/2} (\ln m)^{2/n+1}}\ . \tag{22}$$

Then

$$\alpha(z) = \alpha(d) - \int_z^d \alpha'(t)\,dt \le \alpha(d) + \frac{4}{k_1}\int_z^d \frac{dt}{t\ln\dfrac{1}{t}} +$$

$$+K_1 \int_z^d \frac{dt}{t\ln\dfrac{1}{t}\ln\ln\dfrac{1}{t}} = \alpha(d) + \frac{4}{k_1}\ln\ln\frac{1}{z} - \frac{4}{k_1}\ln\ln\frac{1}{d} +$$

$$+K_1\ln\ln\ln\frac{1}{z} - K_1\ln\ln\ln\frac{1}{d} \le \alpha(d) + \frac{4}{k_1}\ln\ln\frac{1}{z} + K_1\ln\ln\ln\frac{1}{z}\ .$$

From the proof of lemma 4 we conclude that will be enough to show the validity of the inequality

$$\alpha\left(\frac{C_{31}e^{-m}}{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}\right) < \frac{2n}{k_1}\ln\frac{e^{-m}m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}{C_{31}e^{-m}}\ ,$$

or

$$\alpha\left(\frac{C_{31}e^{-m}}{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}\right)<\frac{2n}{k_1}\frac{2}{n}\ln m+\frac{2n}{k_1}\ln\frac{1}{C_{31}}+\frac{2n}{k_1}\frac{2+n}{n}\ln\ln m,\ i.e.$$

$$\alpha\left(\frac{C_{31}e^{-m}}{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}\right)<\frac{4}{k_1}\ln m+\frac{2n}{k_1}\ln\frac{1}{C_{31}}+\frac{2(2+n)}{k_1}\ln\ln m\ . \tag{23}$$

Otherwise

$$\alpha\left(\frac{C_{31}e^{-m}}{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}\right)<\alpha(d)+\frac{4}{k_1}\ln\ln\frac{e^m m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}{C_{31}}+$$

$$+K_1\ln\ln\ln\frac{e^m m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}{C_{31}}=\alpha(d)+\frac{4}{k_1}\ln m+\frac{4}{k_1}\ln\left(\frac{2}{n}\ln m\right)+$$

$$+\frac{4}{k_1}\ln\ln\frac{1}{C_{31}}+\frac{4}{k_1}\ln\ln\left(\frac{2}{n}+1\right)\ln m+g_\alpha\alpha(d)+\frac{4}{k_1}\ln m+\frac{4}{k_1}\ln\frac{2}{n}+$$

$$+\frac{4}{k_1}\ln\ln m+\frac{4}{k_1}\ln\ln\frac{1}{C_{31}}+\frac{4}{k_1}\ln\ln\left(\frac{2}{n}+1\right)\ln m+K_1\ln\ln m+\theta(\ln\ln m)\ . \tag{24}$$

Now from (23) follows that for validity of (22) will be enough that $\frac{2(2+n)}{k_1}>\frac{4}{k_1}+K_1$ that is $K_1<\frac{2n}{k_1}$, which is fulfilled according to (8). So

$$V_m\geq\frac{C_{31}}{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}},\quad i.e.$$

$$V_m\ln\frac{1}{V_m}\geq\frac{C_{31}}{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}\ln\frac{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}+1}}{C_{31}}\geq C_{38}(n)\frac{1}{m^{\frac{2}{n}}(\ln m)^{\frac{2}{n}}}\ ,$$

only when m is sufficiently big. From here follows that

$$\left(V_m\ln\frac{1}{V_m}\right)^{\frac{n}{2}}\geq C_{33}(n)\frac{1}{m\ln m}\ .$$

Then

$$\sum_{m=2}^{\infty}e^{\frac{mn}{2}}P\left(H_m^+\right)\geq C_{34}(\gamma,n)\sum_{m=2}^{\infty}\left(V_m\ln\frac{1}{V_m}\right)^{\frac{n}{2}}\geq C_{33}C_{34}\sum_{m=2}^{\infty}\frac{1}{m\ln m}=\infty\ .$$

But otherwise

$$\sum_{m=3}^{\infty}\frac{1}{\ln\ln m}e^{\frac{mn}{2}}P\left(H_m^+\right)\geq C_{34}\sum_{m=3}^{\infty}\frac{1}{\ln\ln m}\left(V_m\ln\frac{1}{V_m}\right)^{\frac{n}{2}}\geq$$

$$\geq C_{33}C_{34}\sum_{m=3}^{\infty}\frac{1}{m\ln m\ln\ln m}=\infty,$$

and lemma is proved in case of $J = \frac{4}{k_1}$.

By the same way will be shown that both conditions (21) and (18) are fulfilled when $J < \frac{4}{k_1}$. Let now $J > \frac{4}{k_1}$. Then there is J', $J > J' > \frac{4}{k_1}$ such that

$$d(z) \geq \alpha(d) + J' \ln\ln\frac{1}{z} - J' \ln\ln\frac{1}{d} = J' \ln\ln\frac{1}{z} - C_{35}(\alpha) \ .$$

So

$$V_m = \exp\left[-\frac{k_1}{2n}\alpha(z_m)\right] \leq \exp\left[\frac{k_1}{2n}J' \ln\ln\frac{1}{z_m}\right] C_{36}(\alpha,n) =$$

$$= \left(\ln\frac{1}{z_m}\right)^{-\frac{k_1 J'}{2n}} C_{36} \leq C_{37}(\alpha,n)\, m^{-\frac{k_1 J'}{2n}}, \quad i.e.$$

$z_m > e^{-\beta m}$. From here we conclude

$$V_m \ln\frac{1}{V_m} \leq C_{37} m^{-\frac{k_1 J'}{2n}} \ln\left(\frac{1}{C_{37}} m^{\frac{k_1 J}{2n}}\right) \leq C_{38}(\alpha,n)\, m^{-\frac{k_1 J'}{2n}} \ln m$$

Then

$$\left(V_m \ln\frac{1}{V_m}\right)^{n/2} \leq C_{39}(\alpha,n)\, m^{-\frac{k_1 J'}{4}} (\ln m)^{\frac{n}{2}}$$

because of $\frac{k_1 J'}{4} > 1$, from here follows that both conditions (21) and (18) are fulfilled.

Lemma is proved.

Corollary. *For the regularity of point* $(0,0)$ *according to boundary value problem (1)-(2) will be enough that fulfilled condition (18).*

Now we will prove the necessary condition of regularity.

Theorem 2. *If according to coefficients of operator* L *will be fulfilled condition (3), then for regularity of point* $(0,0)$ *respectively to boundary value problem (1)-(2) will be necessary that*

$$\sum_{m=1}^{\infty} e^{\frac{mn}{2}} P\left(H_m^-\right) = \infty \tag{25}$$

Proof. Let condition (24) is not fulfilled. We denote through m_1 the least natural number for which

$$\sum_{m=m_1}^{\infty} e^{\frac{mn}{2}} P\left(H_m^-\right) \leq \frac{1}{4C_2C_6e^n} \ .$$

We choose continuous boundary function $\varphi(x,t)$ of first boundary value problem (1)-(2) such that $\varphi(0,0) = 1$, $\varphi(x,t) = 0$ when $t \leq \inf\limits_{(y,\tau)\in H_{m_1}^-} \{\tau\}$, $0 \leq \varphi(x,t) \leq 1$. According to lemma 3 for any natural $m \geq m_1$ when $\varepsilon_m > 0$ there is step domain

$Q_m \supset H^-_{m_1}$ with sufficiently smooth boundaries bases of its component cylinders and measure μ_m with support in $\bar{Q}_m$ such that if

$$U_m(x,t) = \int_{\bar{Q}_m} F(x,y;t,\tau)\, d\mu_m(y,\tau) \ ,$$

then

$$U_m|_{\Gamma^-(Q_m)} = 1, \tag{26}$$

$$\mu_m\left(\bar{Q}_m\right) \le P_F\left(H^-_m\right) + \varepsilon_m \ . \tag{27}$$

From lemma 1 and 2 follows that if condition (24) is not fulfilled then

$$\sum_{m=1}^{\infty} e^{\frac{mn}{2}} P_F\left(H^-_m\right) < \infty \ .$$

We put $\varepsilon_m = \dfrac{C_2^{-1} e^{-n} e^{-\frac{mn}{2}} 2^{-m}}{4}$. Let

$$V_m(x,t) = \sum_{m=m_1}^{\infty} \int_{\bar{Q}_m} G^-(x-y, t-\tau)\, d\mu_m(t,\tau) \ ,$$

$$U(x,t) = \sum_{m=m_1}^{\infty} U_m(x,t) \ ,$$

We denote $D^1 = D \backslash \overline{\bigcup_{m=m_1}^{\infty} Q_m}$. It is clear that fucntion $u(x,t) - V(x,t)$ is solution of equation (1) in D^1 and $(u-U)|_{\Gamma(D^1)} \le 0$ (according to (25)). According to principle of maximum $u(x,t) \le U(x,t)$ in D^1 and in particular

$$I = \lim_{\substack{(x,t)\to(0,0)\\ (x,t)\in D^1}} u(x,t) \le U(0,0) \le C_2 V(0,0) =$$

$$= C_2 \sum_{m=m_1}^{\infty} G^-(-y,-\tau)\, d\mu_m(y,\tau) \ .$$

We could suppose that $\bar{Q}_m \subset \left\{(y,\tau) : G^-(-y,\tau) \le e^{\frac{(m+2)n}{2}}\right\}$. So

$$I \le C_2 e^n \sum_{m=m_1}^{\infty} e^{\frac{mn}{2}} \mu_m\left(\bar{Q}_m\right) \le C_2 e^n \sum_{m=m_1}^{\infty} e^{\frac{mn}{2}} \left(P_F\left(H^-_m\right) + \varepsilon_m\right) \le$$

$$\le C_2 e^n \sum_{m=m_1}^{\infty} e^{\frac{mn}{2}} P_F\left(H^-_m\right) + \frac{1}{4} \ .$$

We used (26). Otherwise according to (6)

$$I \leq C_2 e^n C_6 \sum_{m=m_1}^{\infty} e^{\frac{mn}{2}} P_F \left(H_m^-\right) + \frac{1}{4} \ \leq \frac{1}{2} \, .$$

So the point $(0,0)$ is irregular and theorem is proved.

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