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WEAK SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR GILBARG-SERRIN PARABOLIC EQUATION IN CYLINDRICAL DOMAINS

Abstract

In the paper the first boundary value problem for Gilbarg-Serrin parabolic equation is considered. Its unique weak solvability in corresponding Sobolev weight spaces is established.

Introduction. The paper is devoted to the investigation of solvability of the first boundary value problem for parabolic equation of the form

$$\mathcal{L}u = \Delta u + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f + \sum_{i,j=1}^{n} \frac{\partial f^k}{\partial x^k}, \qquad (0.1)$$

where Δ is Laplace operator and $\lambda > -1$. It is easy to see that at $\lambda > -1$ operator $\mathcal{L}$ is uniformly parabolic. We'll keep the following notation: $\mathbf{E}_n$ and R_{n+1} are Euclidean spaces of the points $x = (x_1, ..., x_n)$ and $(t, x) = (t, x_1, ..., x_n)$ respectively; Ω is bounded domain in $\mathbf{E}_n$ with the boundary $\partial\Omega$ belonging to C; $Q_T = \Omega \times (0, T)$ is a cylinder, S_T is its lateral surface in $\partial\Omega \times [0, T]$, $\Gamma(Q_T) = \Omega \cup S_T$, $0 < T < \infty$.

$C_0^\infty(Q_T)$ is a space of all infinitely differentiable in Q_T functions vanishing near the parabolic boundary.

Let γ satisfy the following condition

$$\left.\begin{array}{ll} \gamma \in \left(-\frac{\lambda n-(n-2)}{1+\lambda};\ 2-n\right), & \text{if } \lambda > n-2 \\ \gamma \in \left(2-n;\ -\frac{\lambda n-(n-2)}{1+\lambda}\right), & \text{if } -1 < \lambda < n-2 \end{array}\right\} \qquad (0.2)$$

$A_0^\infty(Q_T)$ is a subspace of all infinitely differentiable finite functions for which the expression

$$\int_{Q_T} |x|^{\gamma-2} u^2 dx dt + \int_{Q_T} |x|^{\gamma} u_x^2 dx dt + \int_{Q_T} |x|^{\gamma+2} u_t^2 dx dt$$

is finite.

$\tilde{A}_0^\infty(Q_T)$ is a subspace $A_0^\infty(Q_T)$ vanishing on the upper cover of the cylinder Q_T.

Further, let $L_{2,\gamma}(Q_T)$ be Banach space of measurable functions $u(x,t)$ given on Q_T with the finite norm

$$\|u\|_{L_{2,\gamma}(Q_T)}=\left(\int_{Q_T}|x|^{\gamma}u^2dxdt\right)^{1/2}$$

$W^{1,0}_{2,\gamma}(Q_T)$ and $W^{1,1}_{2,\gamma}(Q_T)$ are Banach spaces of measurable functions $u(x,t)$ given on Q_T with finite norms

$$\|u\|_{W^{1,0}_{2,\gamma}(Q_T)}=\left(\int_{Q_T}|x|^{\gamma}\left(u^2+u_x^2\right)dxdt\right)^{1/2},$$

$$\|u\|_{W^{1,1}_{2,\gamma}(Q_T)}=\left(\int_{Q_T}|x|^{\gamma}\left(u^2+u_x^2+u_t^2\right)dxdt\right)^{1/2}$$

respectively, $\mathring{W}^{1,0}_{2,\gamma}(Q_T)$ and $\overset{\circ}{\widetilde{W}}{}^{1,1}_{2,\gamma}(Q_T)$ are subspaces of $W^{1,0}_{2,\gamma}(Q_T)$ and $W^{1,1}_{2,\gamma}(Q_T)$ respectively whose dense set are $A^{\infty}_0(Q_T)$, $\tilde{A}^{\infty}_0(Q_T)$ respectively.

The first boundary value problem

$$\mathcal{L}u=\Delta u+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{|x|^2}\cdot\frac{\partial^2u}{\partial x_i\partial x_j}-\frac{\partial u}{\partial t}=f+\sum_{k=1}^{n}\frac{\partial f^k}{\partial x^k},\tag{0.3}$$

$$u|_{\Gamma(Q_T)}=0,\tag{0.4}$$

is considered in the domain, where $f\in L_{2,\gamma}(Q_T)$, $f^k\in L_{2,\gamma}(Q_T)$, $k=\overline{1,n}$..

Definition. *The function $u(x,t)\in\mathring{W}^{1,0}_{2,\gamma}(Q_T)$ is called a weak solution of problem (0.3)-(0.4) in the domain Q_T if at any function $\upsilon(x,t)\in\overset{\circ}{\widetilde{W}}{}^{1,1}_{2,\gamma}(Q_T)$ the integral identity*

$$\int_{Q_T}|x|^{\gamma}\left(\sum_{i,k=1}^{n}\left(\delta_{ik}+\lambda\frac{x_ix_k}{|x|^2}\right)\right)u_k\upsilon_idxdt+((1+\lambda)\gamma+\lambda(n-1))\times$$

$$\times\int_{Q_T}|x|^{\gamma-2}\upsilon\sum_{i=1}^{n}x_iu_idxdt+\int_{Q_T}|x|^{\gamma}u\upsilon_tdxdt=-\int_{Q_T}|x|^{\gamma}f\upsilon dxdt+$$

$$+\int_{Q_T}|x|^{\gamma}\sum_{i=1}^{n}\upsilon_if^idxdt-\gamma\int_{Q_T}|x|^{\gamma-2}\upsilon\sum_{i=1}^{n}x_if^idxdt\tag{0.5}$$

is fulfilled.

2. Main a priori estimation

Let's represent the Gilbarg-Serrin parabolic operator in the form of divergent operator with unbounded minor coefficients.

We have

$$\mathcal{L}u=\Delta u+\lambda\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{|x|^2}u_j\right)_i-(n-1)\,\lambda\sum_{i=1}^{n}\frac{x_i}{|x|^2}u_i-u_t. \tag{2.1}$$

For any function $u\left(x,t\right)\in A_0^{\infty}\left(Q_T\right)$ we have

$$-\int\limits_{Q_T}|x|^{\gamma}\,u\mathcal{L}udxdt=-\int\limits_{Q_T}|x|^{\gamma}\,u\left\{\sum_{i=1}^{n}u_{ii}+\lambda\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{|x|^2}u_j\right)_i-\right.$$

$$\left.-(n-1)\,\lambda\sum_{i=1}^{n}\frac{x_i}{|x|^2}u_i-u_t\right\}dxdt=-\int\limits_{Q_T}|x|^{\gamma}\,u\sum_{i=1}^{n}u_{ii}dxdt-$$

$$-\lambda\int\limits_{Q_T}|x|^{\gamma}\,u\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{|x|}u_j\right)_i dxdt+(n-1)\,\lambda\int\limits_{Q_T}|x|^{\gamma}\,u\sum_{i=1}^{n}\frac{x_i}{|x|^2}u_idxdt+$$

$$+\int\limits_{Q_T}|x|^{\gamma}\,u\cdot u_tdxdt. \tag{2.2}$$

$$J_1=-\int\limits_{Q_T}|x|^{\gamma}\,u\sum_{i=1}^{n}u_{ii}dxdt=\int\limits_{Q_T}\sum_{i=1}^{n}u_i\left(|x|^{\gamma}\,u\right)_i dxdt=\int\limits_{Q_T}\sum_{i=1}^{n}|x|^{\gamma}\,u_i^2dxdt+$$

$$+\gamma\int\limits_{Q_T}|x|^{\gamma-2}\sum_{i=1}^{n}x_iu_iudxdt=\int\limits_{Q_T}|x|^{\gamma}\,u_x^2dxdt+\frac{\gamma}{2}\int\limits_{Q_T}|x|^{\gamma-2}\sum_{i=1}^{n}x_i\left(u^2\right)dxdt=$$

$$=\int\limits_{Q_T}|x|^{\gamma}\,u_x^2dxdt-\frac{\gamma}{2}\int\limits_{Q_T}u^2\sum_{i=1}^{n}\left(x_i\,|x|^{\gamma-2}\right)_i dxdt=\int\limits_{Q_T}|x|^{\gamma}\,u_x^2dxdt-$$

$$-\frac{\gamma n}{2}\int\limits_{Q_T}u^2\,|x|^{\gamma-2}\,dxdt-\frac{\gamma\,(\gamma-2)}{2}\int\limits_{Q_T}u^2\,|x|^{\gamma-2}\,dxdt=\int\limits_{Q_T}|x|^{\gamma}\,u_x^2dxdt-$$

$$-\frac{\gamma\,(n+\gamma-2)}{2}\int\limits_{Q_T}u^2\,|x|^{\gamma-2}\,dxdt. \tag{2.3}$$

Besides,

$$J_2=-\lambda\int\limits_{Q_T}|x|^{\gamma}\,u\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{|x|^2}u_j\right)_i dxdt=\lambda\int\limits_{Q_T}\sum_{i,j=1}^{n}\left(|x|^{\gamma}\,u\right)_i\frac{x_ix_j}{|x|^2}u_jdxdt=$$

$$= \lambda \int_{Q_T} |x|^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j dxdt + \lambda\gamma \int_{Q_T} |x|^{\gamma-2} u \sum_{i,j=1}^{n} \frac{x_i^2 x_j}{|x|^2} u_j u dxdt =$$

$$= \lambda \int_{Q_T} |x|^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j dxdt + \frac{\lambda\gamma}{2} \int_{Q_T} |x|^{\gamma-2} \sum_{j=1}^{n} x_j \left(u^2\right)_j dxdt =$$

$$= \lambda \int_{Q_T} |x|^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j dxdt - \frac{\lambda\gamma}{2} \int_{Q_T} \sum_{j=1}^{n} \left(|x|^{\gamma-2} x_j\right)_j u^2 dxdt =$$

$$= \lambda \int_{Q_T} |x|^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j dxdt - \frac{\lambda\gamma n}{2} \int_{Q_T} |x|^{\gamma-2} u^2 dxdt -$$

$$- \frac{\lambda\gamma(\gamma-2)}{2} \int_{Q_T} |x|^{\gamma-2} u^2 dxdt = \lambda \int_{Q_T} |x|^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j dxdt -$$

$$- \frac{\lambda\gamma^2 + \lambda\gamma(n-2)}{2} \int_{Q_T} |x|^{\gamma-2} u^2 dxdt. \tag{2.4}$$

$$J_3 = (n-1)\lambda \int_{Q_T} |x|^{\gamma} u \sum_{i=1}^{n} \frac{x_i}{|x|^2} u_i dxdt = \frac{(n-1)\lambda}{2} \int_{Q_T} |x|^{\gamma-2} \sum_{i=1}^{n} x_i \left(u^2\right)_i dxdt =$$

$$= -\frac{(n-1)\lambda}{2} \int_{Q_T} u^2 \sum_{i=1}^{n} \left(|x|^{\gamma-2} x_i\right)_i dxdt = -\frac{(n-1)\lambda n}{2} \int_{Q_T} |x|^{\gamma-2} u^2 dxdt -$$

$$- \frac{(\gamma-2)(n-1)}{2} \int_{Q_T} |x|^{\gamma-2} u^2 dxdt =$$

$$= -\frac{(n-1)\lambda n + (\gamma-2)(n-1)\lambda}{2} \int_{Q_T} |x|^{\gamma-2} u^2 dxdt \tag{2.5}$$

$$J_4 = \int_{Q_T} |x|^{\gamma} u \cdot u_t dxdt = \frac{1}{2} \int_{Q_T} |x|^{\gamma} \left(u^2\right)_t dxdt = \frac{1}{2} \int_{\Omega} |x|^{\gamma} u^2(T,x)\, dx. \tag{2.6}$$

From (2.1)-(2.6) we obtain

$$\frac{1}{2} \int_{\Omega} |x|^{\gamma} u^2(T,x)\, dx + \int_{Q_T} |x|^{\gamma} \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j \right) dxdt =$$

$$= \frac{(1+\lambda)\gamma + [(2n-3)\lambda + (n-2)]\lambda + (n-2)(n-1)\lambda}{2} \int_{Q_T} |x|^{\gamma-2} u^2 dxdt -$$

$$- \int_{Q_T} |x|^{\gamma} u \mathcal{L} u dxdt. \tag{2.7}$$

From (2.7) we obtain

$$\int\limits_{Q_T} |x|^{\gamma} \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j \right) dxdt \leq$$

$$\leq \frac{(1+\lambda)\,\gamma + [(2n-3)\,\lambda + (n-2)]\,\lambda + (n-2)\,(n-1)\,\lambda}{2} \times$$

$$\times \int\limits_{Q_T} |x|^{\gamma-2}\, u^2 dxdt - \int\limits_{Q_T} |x|^{\gamma}\, u\mathcal{L}u dxdt. \tag{2.8}$$

Investigate the sign of square trinomial

$$(1+\lambda)\,\gamma^2 + ((2n-3)\,\lambda + n - 2)\,\gamma + (n-2)\,(n-1)\,\lambda \tag{2.9}$$

relatively to γ.

Solving the equation we have

$$\gamma_1 = 2-n, \quad \gamma_2 = \frac{\lambda\,(1-n)}{1+\lambda} \tag{2.10}$$

It is easy to see, that at $\lambda > n-2$ $\quad \frac{\lambda(1-n)}{1+\lambda} < 2-n$ and at $-1 < \lambda < n-2$ $\quad \frac{\lambda(1-n)}{1+\lambda} > 2-n$

At $\lambda = n-2$ $\quad \frac{\lambda(1-n)}{1+\lambda} = 2-n$

Thus

1) at $\lambda > n-2$, $\gamma \in \left(\frac{\lambda(1-n)}{1+\lambda};\ 2-n\right)$, and at $\lambda < n-2$ $\quad \gamma \in \left(2-n; \frac{\lambda(1-n)}{1+\lambda}\right)$ the inequality

$$(1+\lambda)\,\gamma^2 + ((2n-3)\,\lambda + n - 2)\,\gamma + (n-2)\,(n-1)\,\gamma \leq 0.$$

is fulfilled.

2) at $\lambda > n-2$, $\gamma \overline{\in} \left[\frac{\lambda(1-n)}{1+\lambda}; 2-n\right]$ or $\lambda < n-2$, $\gamma \overline{\in} \left[2-n; \frac{\lambda(1-n)}{1+\lambda}\right]$ the inequality

$$(1+\lambda)\,\gamma^2 + ((2n-3)\,\lambda + n - 2)\,\gamma + (n-2)\,(n-1)\,\gamma > 0$$

is fulfilled.

For the first case from (2.8) we obtain

$$\int\limits_{Q_T} |x|^{\gamma} \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j \right) dxdt \leq -\int\limits_{Q_T} |x|^{\gamma}\, u\mathcal{L}u dxdt. \tag{2.11}$$

Note that for any function $u\,(x,t) \in \overset{\circ}{A}{}^{\infty}_{0}\,(Q_T)$ at $\gamma \neq 2-n$ it holds the inequality

$$\int\limits_{Q_T} |x|^{\gamma-2}\, u^2 dxdt \leq \frac{4}{(n+\gamma-2)^2} \int\limits_{Q_T} |x|^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} u_i u_j dxdt. \tag{2.12}$$

Using the last estimation in (2.8) we obtain

$$\int\limits_{Q_T} |x|^\gamma \left(u_x^2 + \lambda \sum_{i,j=1}^n \frac{x_i x_j}{|x|^2} u_i u_j \right) dxdt \le$$

$$\le \frac{2\left((1+\lambda)\gamma^2 + ((2n-3)\lambda + n-2)\gamma + (n-2)(n-1)\lambda\right)}{(n+\gamma-2)^2} \times$$

$$\times \int\limits_{Q_T} |x|^\gamma \sum_{i,j=1}^n \frac{x_i x_j}{|x|^2} u_i u_j dxdt - \int\limits_{Q_T} |x|^\gamma u\mathcal{L}udxdt. \tag{2.13}$$

Solving the inequality

$$\frac{2\left((1+\lambda)\gamma^2 + ((2n-3)\lambda + n-2)\gamma + (n-2)(n-1)\lambda\right)}{(n+\gamma-2)^2} < 1+\lambda$$

we find that it is fulfilled at

$$\gamma \in \left(-\frac{\lambda n - (n-2)}{1+\lambda}; \ \frac{\lambda(1-n)}{1+\lambda} \right), \quad \text{if} \quad \lambda > n-2,$$

and at

$$\gamma \in \left(-\frac{\lambda(1-n)}{1+\lambda}; \ -\frac{\lambda n - (n-2)}{1+\lambda} \right), \quad \text{if} \quad -1 < \lambda < n-2.$$

Let γ be chosen by the abovementioned method. Then there exists such $\varepsilon_1 > 0$ that

$$\frac{2\left((1+\lambda)\gamma^2 + ((2n-3)\lambda + n-2)\gamma + (n-2)(n-1)\lambda\right)}{(n+\gamma-2)^2} \le 1+\lambda-\varepsilon_1. \tag{2.14}$$

We can assume that $0 < \varepsilon_1 < 1$. Allowing for (2.14) in (2.13) we obtain

$$\varepsilon_1 \int\limits_{Q_T} |x|^\gamma u_x^2 dxdt \le -\int\limits_{Q_T} |x|^\gamma u\mathcal{L}udxdt. \tag{2.15}$$

Since

$$\lambda \sum_{i,j=1}^n \frac{x_i x_j}{|x|^2} u_i u_j = \lambda \left(\sum_{i=1}^n \frac{x_i}{|x|} u_i \right)^2 \ge 0, \quad \text{at} \quad \lambda \ge 0$$

and

$$\lambda \sum_{i,j=1}^n \frac{x_i x_j}{|x|^2} u_i u_j \ge \lambda u_x^2, \quad \text{at} \quad -1 < \lambda < 0.$$

then subject to (2.10) we conclude that estimation (2.15) holds at any $\gamma \in \left(-\frac{\lambda n-(n-2)}{1+\lambda}; \ 2-n\right)$ if $\lambda > n-2$, and at $\gamma \in \left(2-n; -\frac{\lambda n-(n-2)}{1+\lambda}\right)$ if $\lambda < n-2$.

Further, applying the Friedrichs inequality we obtain

$$\int\limits_{Q_T} |x|^\gamma \left(u^2 + u_x^2\right) dxdt \le \varepsilon \int\limits_{Q_T} |x|^\gamma u\mathcal{L}udxdt, \tag{2.16}$$

where $\varepsilon > 0$.

Theorem. *Let $Q_T = \Omega \times (0,T)$, $0 \in \Omega$, $\partial\Omega \in C$ and numerical parameter γ satisfy condition (0.2). Then for any function $u(x,t) \in \overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$ the inequality*

$$\|u\|_{W^{1,0}_{2,\gamma}(Q_T)} \le C(\lambda, n, \gamma, diam\Omega) \|\mathcal{L}u\|_{L_{2,\gamma}(Q_T)} \qquad (2.17)$$

is fulfilled.

3. Unique solvability of the first boundary value problem

Theorem. *Let γ satisfy condition (0.2). Then problem (0.3)-(0.4) is uniquely solvable in the space $\overset{\circ}{W}{}^{1,0}_{2,\gamma}(Q_T)$.*

The theorem is proved by the scheme described in [5].

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