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INTEGRAL GENERALIZATION OF THE SECOND ORDER MATRIX DIFFERENTIAL EQUATION

Abstract

The generalization of some results from the theory of matrix differential equations was realized in case of Volterra matrix integral equations with Stieltjes integral on a finite segment.

Consider the matrix integral equation

$$Y(x) = c_1 \cos\sqrt{\lambda} x I + \frac{c_2}{\sqrt{\lambda}} \sin\sqrt{\lambda} x I + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} dF(t) Y(t), \qquad (1)$$

where Y and F are square matrices of the order $n \times n$, I is a unique matrix of the same order, c_1, c_2 are some constants. Assume that $F(x)$ is continuous from the right (in the sense of continuity of each of its elements) matrix function of constrained variation (in the sense of constrained variation of each of its elements) on any finite segment from $[0, \infty)$. It is evident that if $F(x)$ is an absolutely continuous matrix function then integral equation (1) is equivalent to the boundary value problem generated on a semi-axis $[0, \infty)$ by the matrix differential equation

$$Y'' + \lambda Y = V(x) Y \qquad (2)$$

and the boundary condition

$$Y(0) = c_1 I, \qquad Y'(0) = c_2 I, \qquad (3)$$

where $V(x) = F'(x)$ ([1]).

Equation (1) is Volterra type integral equation with Stieltjes integral and it is a generalization of problem (2), (3) in the class of integral equations. Such "Integral generalization" of Sturm-Liouville type problems was investigated in [2]. In [3], [4] the generalization of some results from the theory of ordinary differential equations were executed in case of the Volterra integral equation with Stieltjes integral on the finite segment.

1. Uniqueness and existence of solutions

Theorem 1.1. *If $F(x)$ is continuous from the right matrix function of constrained variation on the segment $[0, b]$, then at the given c_1, c_2 equation (1) has a unique solution in the class of continuous matrix functions on $[0, b]$.*

Let $|\lambda| \leq N$. Since the expression (1) is even relative to $\sqrt{\lambda}$, then not losing generality we assume that $\operatorname{Im}\sqrt{\lambda} \geq 0$.

Assume

$$Y(x) = \exp\left(\operatorname{Im}\sqrt{\lambda} x\right) A(x). \qquad (1.1)$$

Then from (1) we'll obtain

$$A(x)=\left(c_1\cos\sqrt{\lambda}x+\frac{c_2}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right)e^{-\operatorname{Im}\sqrt{\lambda}x}I+\frac{1}{\sqrt{\lambda}}\int\limits_0^x\sin\sqrt{\lambda}(x-t)\times$$

$$\times e^{-\operatorname{Im}\sqrt{\lambda}(x-t)}dF(t)\,A(t)\,. \tag{1.2}$$

Let's pass to the proof of the theorem. Assuming that there exist two solutions $Y_1,\ Y_2$ we see that their difference which will be called $Z(x,\lambda)$ would satisfy the equation

$$Z(x,\lambda)=\int\limits_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}dF(t)\,Z(t,\lambda) \tag{1.3}$$

or subject to (1.1), (1.2) the equation

$$B(x)=\int\limits_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}e^{-\operatorname{Im}\sqrt{\lambda}(x-t)}dF(t)\,B(t)\,, \tag{1.4}$$

where $B(x)=e^{-\operatorname{Im}\sqrt{\lambda}x}Z(x,\lambda)$.

To perform estimations, introduce the norm (denote it $|\cdot|$) for the matrix as the sum of absolute values of all its elements.

Since $F(x)$ is a function of constrained variation we can choose $x_1>0$ such that

$$x_1\int\limits_0^{x_1}|dF(t)|<\frac{1}{2}. \tag{1.5}$$

Here $\int\limits_0^x|dF(t)|$ is a sum of overall variations of all elements the matrix $F(t)$ on the interval $(0,x)$ and vanishes as $x\to 0+$.

Assume that the maximum $|B(x)|$ in $[0,x_1]$ is attained at the point x_2. Then from (1.4) replacing x by x$_2$, using inequality

$$\left|\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right|e^{-\operatorname{Im}\sqrt{\lambda}x}\le x,\quad\text{when}\qquad x\ge 0,\quad \operatorname{Im}\sqrt{\lambda}\ge 0, \tag{1.5$'$}$$

we have

$$|B(x_2)|\le\frac{1}{2}|B(x_2)|\,.$$

From this it follows that $B(x_2)=0$, consequently $B(x)\equiv 0$ in $[0,x_1]$, i.e. $Z(x,\lambda)=0$ in $[0,x_1]$. Further let b' be the upper bound of those x_3 in $[0,b]$ for which $B(x)\equiv 0$ in $[0,x_3]$. Then we can replace (1.4) by the equality

$$B(x)=\int\limits_{b'}^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}e^{-\operatorname{Im}\sqrt{\lambda}(x-t)}dF(t)\,B(t) \tag{1.6}$$

for $b' < x < b$ and making the same arguments we obtain that $B(x)$ is equal to zero at the right neighbourhood of the point b', which results in contradiction. This proves the uniqueness of the solution of equation (1).

Now prove the existence of the solution of matrix integral equation. At first assume that $F(x)$ is a step function with the finite number of jumps a_k where $0 < a_1 < a_2 < ... < a_n < b$. In this case equation (1) or (1.2) can be solved recurrently. Write the equation (1.2) replacing the Stieljes integral by the sum

$$A(x) = \left[c_1 \cos\sqrt{\lambda}x + \frac{c_2}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right] e^{-\operatorname{Im}\sqrt{\lambda}x} I +$$

$$+\sum_{a_k<x} \frac{\sin\sqrt{\lambda}(x-a_k)}{\sqrt{\lambda}} e^{-\operatorname{Im}\sqrt{\lambda}(x-a_k)} \left[F(a_k) - F(a_k - 0)\right] A(a_k). \tag{1.7}$$

In particular

$$A(a_{s+1}) = \left[c_1 \cos\sqrt{\lambda}a_{s+1} + \frac{c_2}{\sqrt{\lambda}}\sin\sqrt{\lambda}a_{s+1}\right] e^{-\operatorname{Im}\sqrt{\lambda}a_{s+1}} I +$$

$$+\sum_{k\le s} \frac{\sin\sqrt{\lambda}(a_{s+1}-a_k)}{\sqrt{\lambda}} e^{-\operatorname{Im}\sqrt{\lambda}(a_{s+1}-a_k)} \left[F(a_k) - F(a_k - 0)\right] A(a_k) \tag{1.8}$$

and

$$A(0) = c_1 I, \quad A(a_1) = \left[c_1 \cos\sqrt{\lambda}a_1 + \frac{c_2}{\sqrt{\lambda}}\sin\sqrt{\lambda}a_1\right] e^{-\operatorname{Im}\sqrt{\lambda}a_1} I. \tag{1.9}$$

Passing to the norm in (1.8) and using (1.5') we'll obtain the estimation of the form

$$|A(a_{s+1})| \le n|c_1| + n|c_2|b + b\sum_{1}^{s} |F(a_k) - F(a_k - 0)| \, |A(a_k)|. \tag{1.10}$$

Denote

$$\omega(x) = \sum_{a_k\le x} |F(a_k) - F(a_k - 0)|, \quad c_3 = (|c_1| + |c_2|b)\, n.$$

It is proved by the mathematical induction that

$$|A(a_s)| \le c_3 \exp\left[b\omega(a_{s-1})\right], \qquad s = 2, 3, ... \tag{1.11}$$

Introducing in the right hand side of equality (1.7) this estimation in weakened form $|A(a_s)| \le c_3 \exp(b\omega(b))$, we obtain

$$|A(x)| \le c_3 \exp(2b\omega(b)). \tag{1.12}$$

Subject to (1.12) in (1.1)

$$|Y(x,\lambda)| \le c_3 \exp\left(\operatorname{Im}\sqrt{\lambda}b + 2b\omega(b)\right). \tag{1.12'}$$

We'll need the obtained estimation in general case to which we'll go on by means of passage to limit.

Now assume that $F(x)$ is a matrix-function of constrained variation and continuous from the right. In this case we approximate $F(x)$ by means of sequence of step matrix-functions $F_m(x)$, $m=1,2,...$, choosen so that the number of their jumps is at most m and they coincide with $F(x)$ at the points obtained by dividing the interval $(0,b)$ into m equal parts and between these points they remain constant, i.e.

$$F_m(x)=F\left(b\cdot\frac{j}{m}\right),\qquad b\cdot\frac{j}{m}\le x<b\cdot\frac{j+1}{m},\quad j=\overline{0,m-1}$$

and we construct the corresponding solutions $Y_m(x,\lambda)$ in the form

$$Y_m(x,\lambda)=\left(c_1\cos\sqrt{\lambda}x+\frac{c_2}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right)I+$$

$$\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}dF_m(t)\,Y_m(t,\lambda).\tag{1.13}$$

By virtue of inequality (1.12$'$) $Y_m(x,\lambda)$ $(m=1,2,...)$ satisfy the estimation

$$|Y_m(x,\lambda)|\le c_3\exp\left(Jm\sqrt{\lambda}b+2b\omega_m(b)\right).$$

Since

$$\omega_m(b)=\int_0^b|dF_m(x)|\le\int_0^b|dF(x)|=\omega(b),\tag{1.15}$$

then

$$|Y_m(x,\lambda)|\le c_3\exp\left(2b\omega(b)+Jm\sqrt{\lambda}b\right).\tag{1.16}$$

Thus a family of functions $Y_m(x,\lambda)$, $m=1,2,...$ is uniformly bounded.

From the equality (1.13) after simple calculations we have

$$|Y_m(x_2,\lambda)-Y_m(x_1,\lambda)|\le|x_2-x_1|\times$$

$$\times\left\{e^{\mathrm{Im}\sqrt{\lambda}b}\left[|c_1|\sqrt{\lambda}n+n|c_2|+2\max_{0\le x\le b}|Y_m(x,\lambda)|\int_0^b|dF_m(t)|\right]\right\}.$$

According to (1.15), (1.16) in the obtained inequality the coefficient on $|x_2-x_1|$ is bounded uniformly with respect to m such that all $Y_m(x,\lambda)$ are uniformly continuous.

Now applying the principle of Arzeli compactness we conclude that there is infinite sequence of values m such that $Y_m(x,\lambda)$ converge uniformly to limiting function $Y(x,\lambda)$. Since the estimation which obtained for $Y_m(x,\lambda)$ doesn't depend on m, then

$$|Y(x,\lambda)|\le c_3\exp\left\{2b\omega(b)+\mathrm{Im}\sqrt{\lambda}b\right\}.$$

Now pass to the limit as $m\to\infty$ in equality (1.13) as $m\to\infty$. Since $Y(t,\lambda)-Y_m(t,\lambda)\to 0$ uniformly, and $F_m(t)$ are matrix-functions of uniformly bounded variation, then

$$\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}dF_m(t)\,Y_m(t,\lambda)\to\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}dF(t)\,Y(t,\lambda).$$

Thus, $Y(x,\lambda)$ satisfies equation (1) and theorem 1.1 is completely proved.

Remark. Let now $F_m(x)$ be absolutely continuous matrix-functions of uniformly bounded variation tending to $F(x)$, i.e.

$$\int_a^b |dF_m(x)| < const, \quad F_m(x) \to F(x).$$

Then solution $Y_m(x,\lambda)$ of the integral equation

$$Y_m(x) = \left(c_1 \cos\sqrt{\lambda}x + \frac{c_2}{\sqrt{\lambda}} \sin\sqrt{\lambda}x\right) I + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} dF_m(t)\, Y_m(t)$$

is the solution of the problem

$$-Y_m'' + V_m(x)\, Y_m = \lambda Y_m,$$

$$Y_m(0,\lambda) = c_1 I, \quad Y_m'(0,\lambda) = c_2 I \ ,$$

where $V_m(x) = F_m'(x)$ ($V_m(x)$ is a matrix of the order $(n \times n)$).

It is obvious that a family of functions $Y_m(x,\lambda)$ $n = 1, 2, ...$ is uniformly bounded and equipotentially continuous. Then $Y_m(x,\lambda)$ converge uniformly to the limiting function Y(x,λ) which satisfies equation (1).

Using the result of theorem 1.1 we proved theorem 1.2.

Theorem 1.2. *Solution of equation (1) has a right derivative as* $0 \le x \le b$ *given by the equality*

$$Y'(x,\lambda) = \left(-c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x\right) I +$$

$$+\int_0^x \cos\sqrt{\lambda}(x-t)\, dF(t)\, Y(t,\lambda), \qquad Y'(0,\lambda) = c_2 I. \tag{1.16}$$

It is a two-sided derivative at the points, where $F(x)$ is continuous or $Y(x,\lambda) = 0$.

Corollary. *The right derivative* $Y'(x,\lambda)$ *is uniformly bounded when* $0 \le x \le b$, $|\lambda| \le N$, *and it is a matrix-function of bounded variation.*

Note that the special solutions of Sturm-Lioville equation were also studied in [5], [6].

2. The identity for Wronskian

Theorem 2.1. *Let* $F(x)$ *be a continuous matrix-function of bounded variation continuous from the right in the interval* $0 \le x \le b$, $Y(x,\lambda)$ *be unique solution of equation (1), and* $Z(x,\lambda)$ *be a solution of the equation*

$$Z(x) = \left(c_3\cos\sqrt{\lambda}x + c_4\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right) I + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} Z(t)\, dF(t), \tag{2.1}$$

where $c_3,\ c_4$ *are some constants. Then when* $0 \le x < b$

$$W(Z,Y) = Z'(x,\lambda)\,Y(x,\lambda) - Z(x,\lambda)\,Y'(x,\lambda) = (c_1c_4 - c_2c_3)\,I. \tag{2.2}$$

The extremity of interval $x = b$ needs the special consideration. If $F(x)$ is continuous when $x = b$ then there exists a derivative from the left

$$Y'(b,\lambda) = \left(-c_1\sqrt{\lambda}\sin\sqrt{\lambda}b + c_2\cos\sqrt{\lambda}b\right)I + \int_0^b \cos\sqrt{\lambda}\,(b-t)\,dF(t)\,Y(t,\lambda) \tag{2.3}$$

(and analogously for $Z'(b,\lambda)$) which coincides with $\lim\limits_{x\to b} Y'(x,\lambda)$ where $Y'(x,\lambda)$ is a right derivative and therefore (2.2) is true when $x = b$ established by passage to the limit as $x \to b$. In the presence of the jump $F(x)$ at the point $x = b$ the equality (2.2) is valid for $Y'(b,\lambda)$ understood in the sense of equality (2.3), since according to (2.3) the jump of the funciton $Y'(x,\lambda)$ when $x = b$ will be equal to $[F(b) - F(b-0)]\,Y(b,\lambda)$.

Therefore the result of influence of this jump and the alike jump for matrix-function $Z'(x,\lambda)$ on Wronskian in equality (2.2) is jump which is equal to

$$\{Z(b,\lambda)\,[F(b) - F(b-0)]\}\,Y(b,\lambda) - Z(b,\lambda)\,\{[F(b) - F(b-0)]\,Y(b,\lambda)\} = 0,$$

so Wronskian is continuous at the point $x = b$ even if $F(x)$ has there jump, i.e. (2.2) is true at the point $x = b$ if $Y'(b,\lambda)$ understood in the sense of equality (2.3).

We construct more general result from which the assertion of Theorem 2.1 will follow

Theorem 2.2. *If* $F(x),\ F^{+}(x)$ *are matrix-functions of bounded variation continuous from the right in the interval* $0 \le x \le b,\ Y(x,\lambda)$ *is a solution of equation (1) and* $Z(x,\lambda)$ *is a solution of the equation*

$$Z(x) = \left(c_3\cos\sqrt{\lambda}x + \frac{c_4}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\right)I + \int_0^x \frac{\sin\sqrt{\lambda}\,(x-t)}{\sqrt{\lambda}}\,Z(t)\,dF^{+}(t), \tag{2.4}$$

then

$$\left[Z'Y - ZY'\right]_0^x = \int_0^x Z(t,\lambda)\left[dF^{+}(t) - dF(t)\right]Y(t,\lambda) \tag{2.5}$$

holds.

Proof. Taking into account that in the products $Z'Y,\ ZY'$ one of cofactors is continuous and another one is of bounded variation, the left- hand side (2.5) can be written in the form

$$\left[Z'Y - ZY'\right]_0^x = \int_0^x \left[Z'(dY) + (dZ')\,Y - Z\,(dY') - (dZ)\,Y'\right]. \tag{2.6}$$

Allowing for (1.19) and (2.4) we'll obtain

$$\int_0^x Z\,(dY') = -\lambda\int_0^x Z(t,\lambda)\,Y(t,\lambda)\,dt + \int_0^x Z(t,\lambda)\,dF(t)\,Y(t,\lambda). \tag{2.7}$$

Analogously

$$\int_0^x (dZ')\, Y = -\lambda \int_0^x Z(t,\lambda)\, Y(t,\lambda)\, dt + \int_0^x Z(t,\lambda)\, dF^+(t)\, Y(t,\lambda)\,. \tag{2.8}$$

Further from (1) integrating by parts we'll obtain

$$Y(x,\lambda) = c_1 I + \int_0^x \left(\left(-c_1\sqrt{\lambda}\sin\sqrt{\lambda}t + c_2\cos\sqrt{\lambda}t \right) I + \right.$$

$$\left. + \int_0^t \cos\sqrt{\lambda}\,(t-s)\, dF(s)\, Y(s,\lambda) \right) dt,$$

and this by virtue of (1.16) coincides with

$$Y(x,\lambda) = Y(0) + \int_0^x Y'(t,\lambda)\, dt. \tag{2.9}$$

We get the analogous equality for $Z(x,\lambda)$. Taking into account (2.9) and analogous for $Z(x,\lambda)$

$$\int_0^x Z'(dY) = \int_0^x Z'Y'dt = \int_0^x (dZ)\, Y'. \tag{2.10}$$

Putting in (2.6) equality (2.7)-(2.10) we'll obtain the required equality.

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