Araz R. ALIYEV

TO THE THEORY OF SOLVABILITY OF THE SECOND ORDER OPERATOR – DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

Abstract

In the present paper the sufficient conditions for the existence and uniqueness of regular solution of a class boundary value problems are found for the second order operator-differential equations with discontinuous coefficients.

Let H be a separate Hilbert space, A be a selfadjoint positive-defined operator in H. It is clear that the domain of definition of the operator $A^p (p > 0)$ is the Hilbert space H_p with respect to the scalar product $(x, y)_p = (A^p x, A^p y), x, y \in D(A^p)$. Denote by $L(X, Y)$ a set of linear bounded operators acting from the Hilbert space X to the other one Y, and by A_1, A_2 the linear, generally speaking, bounded operators.

Now we pass to the statement of the following boundary value problem:

we consider the following operator-differential equation in the Hilbert space

$$-u''(t) + \rho(t)A^2 u(t) + A_1 u'(t) + A_2 u(t) = f(t),\ t \in R_+ = [0; +\infty), \tag{1}$$

by fulfilling the boundary condition

$$u(0) = Su'(0), S \in L(H_{1/2}, H_{3/2}), \tag{2}$$

where $f(t) \in L_2(R_+; H),\ u(t) \in W_2^2(R_+; H)$ [1], and

$$\rho(t) = \begin{cases} \alpha, & if \quad 0 \le t \le 1, \\ \beta, & if \quad t > 1, \end{cases}$$

where α, β are positive, generally speaking, numbers nonequal to each other.

In the space $L_2(R_+; H)$ and $W_2^2(R_+; H)$ we determine the norms in the following form:

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2 \, dt \right)^{1/2},$$

$$\|u\|_{W_2^2(R_+;H)} = \left(\left\| u'' \right\|_{L_2(R_+;H)}^2 + \|u\|_{L_2(R_+;H_2)}^2 \right)^{1/2},$$

respectively.

Definition. *If the vector-function $u(t)$ from $W_2^2(R_+; H)$ satisfies equation (1) almost every where in R_+, and boundary condition (2) is satisfied in the sense of convergence of the space $H_{3/2}$, i.e.,*

$$\lim_{t \to 0} \left\| u(t) - Su'(t) \right\|_{H_{3/2}} = 0,$$

then we call $u(t)$ a regular solution of the boundary value problem (1), (2).

The corresponding problem is studied in the paper [2] in the case $\rho(t) \equiv 1,\ t \in R_+$, and in [3], [4] in the case $S = 0$.

Note that, in this paper the obtained existence and uniqueness conditions of regular solution are expressed only by the coefficients of operator-differential equations and by the operator being in boundary condition.

Having determined the space

$$W_2^2(R_+; H; S) = \{u(t) \left| u(t) \in W_2^2(R_+; H),\ u(0) = Su'(0) \right.\},$$

the following one holds

Theorem 1. *Let $B = A^{3/2} S A^{-1/2}$ and the operator*

$$T_{\alpha,\beta}(B) = E + \sqrt{\alpha} B + \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\beta}} (\sqrt{\alpha} B - E) e^{-2\sqrt{\alpha} A}$$

have a bounded inverse operator in H, where E is a unit operator. Then the operator L_0, acting from $W_2^2(R_+; H; S)$ to $L_2(R_+; H)$ in the form:

$$L_0 u(t) \equiv -u''(t) + \rho(t) A^2 u(t) = f(t),\ u(t) \in W_2^2(R_+; H; S),\ f(t) \in L_2(R_+; H).$$

realizes isomorphism from the space $W_2^2(R_+; H; S)$ to $L_2(R_+; H)$.

Proof. At first we consider the homogeneous equation $L_0 u(t) = 0$, where $u(t) \in W_2^2(R_+; H; S)$. It is obvious that this equation has only a zero solution from the space $W_2^2(R_+; H; S)$. Really, the solution of the equation $L_0 u(t) = 0$ form $W_2^2(R_+; H; S)$ has the following form

$$u_0(t) = \begin{cases} u_{01}(t) = e^{-\sqrt{\alpha} t A} \varphi_0 + e^{-\sqrt{\alpha}(1-t) A} \varphi_1,\ 0 \le t < 1, \\ \\ u_{02}(t) = e^{-\sqrt{\beta}(t-1) A} \varphi_2,\ 1 < t < +\infty, \end{cases}$$

where the vectors $\varphi_j \in H_{3/2}, j = 0, 1, 2$. From the condition $u_0(t) \in W_2^2(R_+; H; S)$ for determination of $\varphi_j,\ j = 0, 1, 2$ we obtain the following relation

$$\begin{cases} u_0(0) = u_{01}(0) = S u_0'(0) = S u_{01}'(0), \\ \\ u_0(1) = u_{01}(1) = u_{02}(1), \\ \\ u_0'(1) = u_{01}'(1) = u_{02}'(1), \end{cases} \tag{3}$$

from which it is found that all $\varphi_j = 0,\ j = 0, 1, 2$, i.e. $u_0(t) = 0$. Really, from (3) we have the following system to determine the vectors $\varphi_j,\ \ j = 0, 1, 2$:

$$\begin{cases} \varphi_0 + e^{-\sqrt{\alpha} A} \varphi_1 = S(-\sqrt{\alpha} A \varphi_0 + \sqrt{\alpha} A e^{-\sqrt{\alpha} A} \varphi_1), \\ \\ e^{-\sqrt{\alpha} A} \varphi_0 + \varphi_1 = \varphi_2, \\ \\ -\sqrt{\alpha} A e^{-\sqrt{\alpha} A} \varphi_0 + \sqrt{\alpha} A \varphi_1 = -\sqrt{\beta} A \varphi_2. \end{cases}$$

From this system we obtain that

$$\varphi_2 = \sqrt{\frac{\alpha}{\beta}} e^{-\sqrt{\alpha}A}\varphi_0 - \sqrt{\frac{\alpha}{\beta}}\varphi_1,$$

$$\varphi_1 = \frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}} e^{-\sqrt{\alpha}A}\varphi_0,$$

$$(E+\sqrt{\alpha}SA + \frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}(\sqrt{\alpha}SA - E)e^{-2\sqrt{\alpha}A})\varphi_0 = 0.$$

From the last equation we have:

$$T_{\alpha,\beta}(B)A^{3/2}\varphi_0 = 0.$$

Since by the condition of the theorem, the operator $T_{\alpha,\beta}(B)$ has a bounded inverse operator in H, then $A^{3/2}\varphi_0 = 0$. Consequently, $\varphi_0 = 0$, so $\varphi_1 = 0$ and $\varphi_2 = 0$.

Now we show that the equation $L_0 u(t) = f(t)$ at any $f(t) \in L_2(R_+; H)$ has the solution $u(t) \in W_2^2(R_+; H; S)$. Really, in the space $W_2^2(R; H)$ [1] ($R = (-\infty; +\infty)$) we consider the equation

$$L_\alpha u(t) \equiv -\frac{d^2 u(t)}{dt^2} + \alpha A^2 u(t) = F(t), \tag{4}$$

where

$$F(t) = \begin{cases} f(t), \ if \ t \in [0;1), \\ 0, \ \ if \ \ t \in R\backslash[0;1). \end{cases}$$

It is obvious that the solution of equation (4) from the space $W_2^2(R; H)$ is represented in the form of

$$\tilde{u}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\lambda^2 E + \alpha A^2)^{-1} \left(\int_0^1 f(s) e^{-i\lambda s} ds \right) e^{it\lambda} d\lambda.$$

Really, by the Plancherel known theorem

$$\|\tilde{u}\|^2_{W_2^2(R;H)} = \left\| \frac{d^2\tilde{u}}{dt^2} \right\|^2_{L_2(R;H)} + \|\alpha A^2 \tilde{u}\|^2_{L_2(R;H)} = \left\| \lambda^2 \hat{\tilde{u}}(\lambda) \right\|^2_{L_2(R;H)} +$$

$$+ \left\| \alpha A^2 \hat{\tilde{u}}(\lambda) \right\|^2_{L_2(R;H)} \le \left\| (\lambda^2 E + \alpha A^2)^{-1} \lambda^2 \right\|^2_{H\to H} \cdot \left\| \hat{f}(\lambda) \right\|^2_{L_2(R;H)} +$$

$$+ \left\| \alpha A^2 (\lambda^2 E + \alpha A^2)^{-1} \right\|^2_{H\to H} \cdot \left\| \hat{f}(\lambda) \right\|^2_{L_2(R;H)} \le const \left\| \hat{f}(\lambda) \right\|^2_{L_2(R;H)} =$$

$$= const \, \|f(t)\|^2_{L_2([0;1);H)} \, .$$

Here $\overset{\wedge}{\tilde{u}}(\lambda)$, $\hat{f}(\lambda)$ are Fourier transformations of the functions $\tilde{u}(t)$, $f(t)$, respectively.

Further, we define the contraction of solution $\tilde{u}(t)$ on $[0;1)$ and denote it by $u_\alpha(t)$.

Analogously, we can consider the equation

$$L_\beta u(t) \equiv -\frac{d^2u(t)}{dt^2} + \beta A^2 u(t) = \Phi(t), \tag{5}$$

where

$$\Phi(t) = \begin{cases} f(t), & if \ \ t \in (1;+\infty), \\ 0, & if \ \ t \in R\backslash(1;+\infty) \end{cases}$$

and define the solution $u_\beta(t)$ of equation (5) from the space $W_2^2((1;+\infty);H)$.

Thus, the solution of the equation $L_0u(t) = f(t)$ from the space $W_2^2(R_+;H;S)$ is represented in the following form:

$$u(t) = \begin{cases} u_1(t) = u_\alpha(t) + e^{-\sqrt{\alpha}tA}\Psi_0 + e^{-\sqrt{\alpha}(1-t)A}\Psi_1, \ 0 \le t < 1, \\ u_2(t) = u_\beta(t) + e_2^{-\sqrt{\beta}(t-1)A}\Psi_2, \ 1 < t < +\infty, \end{cases}$$

where the vectors $\Psi_j \in H_{3/2}$, $j = 0,1,2$ are uniquely determined from the following conditions:

$$\begin{cases} u(0) = u_1(0) = Su'(0) = Su_1'(0), \\ u(1) = u_1(1) = u_2(1), \\ u'(1) = u_1'(1) = u_2'(1). \end{cases}$$

The boundedness of the operator L_0 follows from the inequality

$$\|L_0u\|^2_{L_2(R_+;H)} \le 2 \cdot \max(1;\alpha^2;\beta^2) \, \|u\|^2_{W_2^2(R_+;H)} \, .$$

Thus, the operator L_0 is bounded and in one-to-one manner acts from the space $W_2^2(R_+;H;S)$ to $L_2(R_+;H)$. Then by the Banach theorem on the inverse operator, the operator L_0 realizes isomorphism between the spaces $W_2^2(R_+;H;S)$ and $L_2(R_+;H)$.

The theorem is proved.

Further, we obtain the estimation of norms for operators of intermediate derivatives in the space $W_2^2(R_+;H;S)$.

The following one is valid

Theorem 2. *Let* $\mathrm{Re}\, B \ge 0$. *Then for all* $u(t) \in W_2^2(R_+;H;S)$ *the following inequalities hold:*

$$\left\|Au'\right\|_{L_2(R_+;H)} \le \frac{1}{2\min^{1/2}(\alpha;\beta)} \|L_0u\|_{L_2(R_+;H)} \, ,$$

$$\left\|A^2u\right\|_{L_2(R_+;H)} \le \frac{1}{\min(\alpha;\beta)} \|L_0u\|_{L_2(R_+;H).}$$

Proof. Subject to the condition of the theorem, after integration by parts we obtain

$$(L_0u, A^2u)_{L_2(R_+;H)} = (-u'' + \rho(t)A^2u, A^2u)_{L_2(R_+;H)} = -(u'', A^2u)_{L_2(R_+;H)}+$$

$$+\left\|\rho^{1/2}(t)A^2u\right\|^2_{L_2(R_+;H)} = -(u', A^2u)_H\Big|_0^{+\infty} + \left\|Au'\right\|^2_{L_2(R_+;H)} + \tag{6}$$

$$+\left\|\rho^{1/2}(t)A^2u\right\|^2_{L_2(R_+;H)} \geq \left\|Au'\right\|^2_{L_2(R_+;H)} + \left\|\rho^{1/2}(t)A^2u\right\|^2_{L_2(R_+;H)}.$$

Further, applying the known Bunyakovskii-Schwartz – Young inequalities to the left hand side of inequality (6), we have

$$\left|(L_0u, A^2u)_{L_2(R_+;H)}\right| \leq \|L_0u\|_{L_2(R_+;H)} \times \left\|A^2u\right\|_{L_2(R_+;H)} \leq$$

$$\leq \left\|\max\left(\frac{1}{\sqrt{\alpha}};\frac{1}{\sqrt{\beta}}\right)L_0u\right\|_{L_2(R_+;H)} \times \left\|\rho^{1/2}(t)A^2u\right\|_{L_2(R_+;H)} \leq \tag{7}$$

$$\leq \frac{\mathcal{E}}{2}\max\left(\frac{1}{\alpha};\frac{1}{\beta}\right)\|L_0u\|^2_{L_2(R_+;H)} + \frac{1}{2\mathcal{E}}\left\|\rho^{1/2}(t)A^2u\right\|^2_{L_2(R_+;H)}, \quad \mathcal{E} >0.$$

Choosing $\mathcal{E} = \frac{1}{2}$ in inequality (7) and allowing for (6), we find that

$$\left\|Au'\right\|^2_{L_2(R_+;H)} \leq \frac{1}{4}\max\left(\frac{1}{\alpha};\frac{1}{\beta}\right)\|L_0u\|^2_{L_2(R_+;H)} =$$

$$= \frac{1}{4\min(\alpha;\beta)}\|L_0u\|^2_{L_2(R_+;H)},$$

i.e.,

$$\left\|Au'\right\|_{L_2(R_+;H)} \leq \frac{1}{2\min^{1/2}(\alpha;\beta)}\|L_0u\|_{L_2(R_+;H)}.$$

And from inequality (6) subject to (7) we obtain:

$$\left\|Au'\right\|^2_{L_2(R_+;H)} + \left\|\rho^{1/2}(t)A^2u\right\|^2_{L_2(R_+;H)} \leq \left|(L_0u, A^2u)_{L_2(R_+;H)}\right| \leq$$

$$\leq \|L_0u\|_{L_2(R_+;H)} \times \left\|A^2u\right\|_{L_2(R_+;H)}.$$

Hence, we find that

$$\|L_0u\|_{L_2(R_+;H)} \times \left\|A^2u\right\|_{L_2(R_+;H)} \geq \left\|\rho^{1/2}(t)A^2u\right\|^2_{L_2(R_+;H)} \geq$$

$$\geq \min(\alpha;\beta)\left\|A^2u\right\|^2_{L_2(R_+;H)}.$$

Thereby we finally have

$$\left\|A^2u\right\|_{L_2(R_+;H)} \leq \frac{1}{\min(\alpha;\beta)}\|L_0u\|_{L_2(R_+;H)}.$$

The theorem is proved.

Now we return to boundary value problem (1), (2) and prove the theorem on regular solvability for it.

Theorem 3. *Let* $\operatorname{Re} B \geq 0$, *the operator* $T_{\alpha,\beta}(B)$ *have a bounded inverse operator in* H, *and the operators* $A_1 A^{-1}$, $A_2 A^{-2}$ *be bounded in* H, *where the inequality*

$$\frac{1}{2\min^{1/2}(\alpha;\beta)}\left\|A_1 A^{-1}\right\| + \frac{1}{\min(\alpha;\beta)}\left\|A_2 A^{-2}\right\| < 1. \tag{8}$$

is satisfied.

Then boundary value problem (1), (2) at any $f(t)$ *from the space* $L_2(R_+;H)$ *has a unique regular solution.*

Proof. We rewrite problem (1), (2) in the form of operator equation

$$L_0 u(t) + L_1 u(t) = f(t),$$

where

$$f(t) \in L_2(R_+;H), u(t) \in W_2^2(R_+;H;S),$$

but

$$L_1 u(t) \equiv A_1 u^{'}(t) + A_2 u(t),\ u(t) \in W_2^2(R_+;H;S).$$

By theorem 1 , the operator L_0 has a bounded inverse operator L_0^{-1} acting from space $L_2(R_+;H)$ to $W_2^2(R_+;H;S)$. Then after substitution $L_0 u(t) = v(t)$ we obtain the following equation in the space $L_2(R_+;H)$:

$$(E + L_1 L_0^{-1}) v(t) = f(t).$$

By virtue of the fact that

$$\left\|L_1 L_0^{-1} v\right\|_{L_2(R_+;H)} = \left\|L_1 u\right\|_{L_2(R_+;H)}$$

$$\leq \left\|A_1 A^{-1}\right\| \left\|A u^{'}\right\|_{L_2(R_+;H)} + \left\|A_2 A^{-2}\right\| \left\|A^2 u\right\|_{L_2(R_+;H)},$$

applying theorem 2 to the last inequality we have

$$\left\|L_1 L_0^{-1} v\right\|_{L_2(R_+;H)} \leq$$

$$\leq \left(\frac{1}{2\min^{1/2}(\alpha;\beta)}\left\|A_1 A^{-1}\right\| + \frac{1}{\min(\alpha;\beta)}\left\|A_2 A^{-2}\right\|\right)\|v\|_{L_2(R_+;H)} < \|v\|_{L_2(R_+;H)}.$$

Thus, the norm of the operator $L_1 L_0^{-1}$ is less than unit.

Therefore, the operator $E + L_1 L_0^{-1}$ is reversible in the space $L_2(R_+;H)$ and we can find $u(t)$, i.e.,

$$u(t) = L_0^{-1}(E + L_1 L_0^{-1})^{-1} f(t).$$

The theorem is proved.

Corollary. *Let the operators A_1A^{-1}, A_2A^{-2} be bounded in H and inequality be (8) satisfied. Then equation (1) by fulfilling the boundary condition $u(0) = 0$, has a unique solution from the space $W_2^2(R_+; H)$ [3], [4].*

Remark. We can analogously study problem (1), (2) when $\rho(t)$ represents any positive function having the finite number of discontinuity points.

Note that we can analogously investigate equation (1) by fulfilling the boundary condition

$$u'(0) = Qu(0),\ Q \in L(H_{3/2}, H_{1/2})$$

and obtain the corresponding results.

References

[1]. Lions Zh.L., Madjenes E. *Nonhomogenous boundary value problems and their applications.* M.: "Mir", 1971, 371 p. (Russian)

[2]. Gasymov M.G., Mirzoev S.S. *On solvability of boundary value problems for the second order elliptic type operator-differential equations.* Differential'nye Uravnenija, 1992, v.28, No4, pp.651-661. (Russian)

[3]. Mirzoev S.S., Aliyev A.R. *On one boundary value problem for second order operator-differential equations with discontinuous coefficients.* Trudy IMM AN Azerb., Baku, 1997, v.VI(XIV), pp.117-121. (Russian)

[4]. Aliyev A.R. *On well-posed solvability of boundary value problems for operator differential equations with discontinuous coefficients.* Authors abstract for Ph.D. Thesis, BSU, Baku, 1998, 13 p. (Russian)

Araz R. Aliyev.
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 394 720 (off.)

Received January 27, 2003; Revised November 10, 2003.
Translated by Mirzoyeva K.S.