MATHEMATICS

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CONDITIONS OF CORRECT SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR ONE CLASS OF THIRD ORDER OPERATOR-DIFFERENTIAL EQUATION

Abstract

In the paper we found sufficient conditions imposed on the coefficients of one class of third order operator-differential equations which provide correct and unique solvability of some initial boundary value problem posed for such equations.

Let H be separable Hilbert space, and A normal invertible operator in H. Then A can be represented in the form $A = UC$, where U is a unitary operator, and C is a positive defined self-adjoint operator. Denote

$$H_\alpha = D\left(C^\alpha\right), \quad (x,y)_\alpha = \left(C_x^\alpha, C_y^\alpha\right), \ \alpha \geq 0.$$

Let $L_2\left(R_+; H\right)$ be Hilbert space of vector-functions $f\left(t\right)$ defined in $R_+ = (0,\infty)$ with values from H, measurable and

$$\|f\|_{L_2(R_+;H)} = \left(\int\limits_0^\infty \|f\left(t\right)\|^2 dt\right)^{1/2} < D\ .$$

Further, we define Hilbert space [1]

$$W_2^3\left(R_+; H\right) = \left\{u | u^{(3)} \in L_2\left(R_+; H\right), \quad A^3 u \in L_2\left(R_+; H\right)\right\}$$

and

$$\overset{\circ}{W}{}_2^3\left(R_+; H\right) = \left\{u | u \in W_2^3\left(R_+; H\right), \quad u\left(0\right) = 0\right\}$$

with norm

$$\|u\|_{W_2^3(R_+;H)} = \left(\left\|u^{(3)}\right\|^2_{L_2(R_+;H)} + \left\|A^3 u\right\|^2_{L_2(R_+;H)}\right)^{1/2}$$

For $R = (-\infty, \infty)$, spaces $L_2\left(R; H\right)$ and $W_2^3\left(R; H\right)$ are considered analogously. Note that here and later on derivatives are understood in the sense of the theory of distributions.

Consider the following operator-differential equation in the space H

$$\frac{d^3u}{dt^3} + A^3 u + A_1 \frac{d^2u}{dt^2} + A_2 \frac{du}{dt} + A_3 u = f\left(t\right) \tag{1}$$

with the boundary condition

$$u\left(0\right) = 0 \tag{2}$$

Definition 1. *If for $f(t) \in L_2(R_+; H)$ there exists the vector function $u(t) \in W_2^3(R_+; H)$, which satisfies equation (1) almost everywhere in R_+, then $u(t)$ is called a regular solution of equation (1).*

Definition 2. *If for any $f(t) \in L_2(R_+; H)$ there exists a regular solution of equation (1) which satisfies boundary condition (2)*

$$\lim_{t \to 0} \|u(t)\|_{3/2} = 0$$

and the inequality

$$\|u\|_{W_2^3(R_+;H)} \leq const \, \|f\|_{L_2(R_+;H)}$$

then problem (1)-(2) will be called regular solvable.

In the given paper we establish sufficient conditions, under which problem (1), (2) becomes regular solvable. When A is a positive-defined self-adjoint operator this problem in general from was considered in [2]. In the paper [3] operator-differential equation of even order was considered.

Denote

$$P_0 u = \tfrac{d^3 u}{dt^3} - A^3 u, \qquad u \in \mathring{W}_2^3(R_+; H),$$

$$P_1 u = A_1 \tfrac{d^3 u}{dt^3} + A_2 \tfrac{du}{dt} + A_3 u, \qquad u \in \mathring{W}_2^3(R_+; H).$$

Lemma 1. *Let the normal operator A be invertible and it have a spectrum on angular sector*

$$S_\varepsilon = \{\infty| \qquad |\arg \lambda| \leq \varepsilon\}, \quad 0 \leq \varepsilon < \frac{\pi}{6}.$$

Then operator P_0 isomorphically maps the space $\mathring{W}_2^3(R_+; H)$ on $L_2(R_+; H)$.

Proof. It is obvious that equation $P_0 u = 0$ has only trivial solution from the space $\mathring{W}_2^3(R_+; H)$. On the other hand, equation $P_0 u = f$, $f \in L_2(R_+; H)$, $u \in \mathring{W}_2^3(R_+; H)$ is always solvable. In fact, vector function

$$u_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(-\xi^3 E + A^3\right)^{-1} \left(\int_0^\infty f(s)\, e^{i(t-s)\xi} ds \right) d\xi$$

satisfies equation $P_0 u = f$ almost everywhere. Let us show that $u_1(t) \in W_2^3(R; H)$. According to Plancherel theorem it suffices to prove that

$$A^3 \hat{u}_1(\xi) \in L_2(R; H), \quad \xi^3 \hat{u}_1(3) \in L_2(R; H)$$

where $\hat{u}(\xi)$ is Fourier transformation of the vector function $u_1(t)$.

Obviously,

$$\left\|A^2 \hat{u}_1(\xi)\right\|_{L_2(R;H)} = \left\|A^2 \left(-i\xi^3 + A^3\right)^{-1} \hat{f}(\xi)\right\|_{L_2(R;H)} \leq$$
$$\leq \sup_{\xi \in R} \left\|A^3 \left(-i\xi^3 + A^3\right)^{-1}\right\| \|f\|_{L_2(R_+;H)}, \tag{3}$$

where $\hat{f}(\xi)$ is Fourier transformation of the vector function $f(t)$.

Since for any $\xi \in R$

$$\left\| A^3 \left(-i\xi^3 E + A^3 \right)^{-1} \right\| = \sup_{\mu \in \sigma(A)} \left| \mu^3 \left(-i\xi^3 + \mu^3 \right)^{-1} \right| =$$

$$= \sup_{\mu \in \sigma(A)} \left| \mu^3 \right| \left(\xi^6 + |\mu|^6 - 2\xi^3 \mu^3 \sin 3\varphi \right)^{-\frac{1}{2}} \Bigg| \leq$$

$$\leq \sup_{\mu \in \sigma(A)} \left| \mu^3 \right| \left(\xi^6 + |\mu|^6 - \left(\xi^6 + |\mu|^6 \sin^2 3\varepsilon \right) \right)^{-\frac{1}{2}} \Bigg| = \frac{1}{\cos 3\varepsilon},$$

we obtain that inequality (3) implies $A^3 u_1(t) \in L_2(R; H)$. One can similarly prove that $\dfrac{d^3 u_1}{dt^3} \in L_2(R; H)$, i.e., $u_1(t) \in W_2^3(R; H)$. Denote by $u_2(t)$ the contraction of vector function $u_1(t)$ on $[0, \infty]$. Then $u_2(t) \in W_2^3(R_+; H)$. By the theorem of traces $u_2(0) \in H_{5/2}$ [1].

Then general solution of equation $P_0 u = f$ is represented in the form $u(t) = u_2(t) + e^{-tA}\varphi_0$, where $\varphi_0 \in H_{5/2}$. Since $u(0) = 0$ we obtain that $\varphi_0 \in H_{5/2}$. Thus $u(t) \in W_2^3(R_+; H)$. On the other hand

$$\|P_0 u\|_{L_2}^2 = \left\| \frac{d^3 u}{dt^3} - A^3 u \right\|_{L_2}^2 \leq 2\, \|u\|_{W_2^3(R_+; H,)}^2$$

i.e. $P_0 : \mathring{W}_2^3(R; H) \to L_2(R_+; H)$ is a continuous operator. Then the statement of the lemma follows from the Banach theorem on the inverse operator.

Lemma 2. *For any* $u \in \mathring{W}_2^3(R_+; H)$*, the following inequalities*

$$\|P_0 u\|_{L_2(R_+; H)}^2 \geq \|u\|_{W_2^3(R; H,)}^2 + \cos 3\varepsilon \left\| u'(0) \right\|_{3/2}^2 - \sin 3\varepsilon \left\| A^3 u \right\|_{L_2} \left\| u^{(3)} \right\|_{L_2}$$

hold.

Proof. Let $u \in \mathring{W}_2^3(R; H)$. Then

$$\|P_0 u\|_{L_2}^2 = \left\| u^{(3)} + A^3 u \right\|_{L_2}^2 = \|u\|_{W_2^3}^2 + 2\,\mathrm{Re} \left(u^{(3)}, A^{(3)} u \right)_{L_2} . \qquad (4)$$

Since $u \in \mathring{W}_2^3(R; H)$ $(u(0) = 0)$ using the integration by parts we have

$$\left(u^{(3)}, A^3 u \right)_{L_2} = \left(C^{3/2} u'(0), C^{3/2} U^3 u'(0) \right) - \left(A^{*3} u, u^3 \right) =$$

$$= \left(C^{3/2} u'(0), U^3 C^{3/2} u'(0) \right) - \left(A^3 u, u^{(3)} \right) + \left(A^{*3} u, u^{(3)} \right) .$$

Hence

$$2\,\mathrm{Re} \left(u^{(3)}, A^3 u \right)_{L_2} \geq \cos 3\varepsilon \left\| u'(0) \right\|_{3/2}^2 - \left| \left(A^3 - A^{*3} \right) u, u^3 \right| \geq$$

$$\geq \cos 3\varepsilon \left\| u'(0) \right\|_{3/2}^2 - \left\| \left(A^3 - A^{*3} \right) A^{-3} \right\| \ \left\| A^3 u \right\|_{L_2} \geq$$

$$\geq \cos 3\varepsilon \left\| u'(0) \right\|_{3/2} - \sin 3\varepsilon \left\| A^3 u \right\|_{L_2} \left\| u^{(3)} \right\|_{L_2} .$$

Taking into account this inequality in (4) we obtain the statement of the lemma.

Lemma 3. *For any $u \in \mathring{W}_2^3(R_+;H)$, the following inequalities hold.*

$$1)\ \left\|A^3u\right\|_{L_2(R_+;H)} \le \frac{1}{\cos 3\varepsilon}\left\|P_0u\right\|_{L_2(R_+;H)} \tag{5}$$

$$2)\ \left\|u^{(3)}\right\|_{L_2(R_+;H)} \le \frac{1}{\cos 3\varepsilon}\left\|P_0u\right\|_{L_2(R_+;H)} \tag{6}$$

$$3)\ \left\|A^2\frac{du}{dt}\right\|_{L_2(R_+;H)} \le \frac{2^{1/3}}{\cos 3\varepsilon}\left\|P_0u\right\|_{L_2(R_+;H)} \tag{7}$$

$$4)\ \left\|A\frac{d^2u}{dt^2}\right\|_{L_2(R_+;H)} \le \frac{2^{3/2}}{\cos 3\varepsilon}\left\|P_0u\right\|_{L_2(R_+;H)} \tag{8}$$

Proof. From lemma 2 using Cauchy's inequality we obtain

$$\left\|P_0u\right\|_{L_2}^2 \ge \left\|u^{(3)}\right\|_{L_2}^2 + \left\|A^3u\right\|_{L_2}^2 +$$

$$+\cos 3\varepsilon\left\|u'(0)\right\|_{3/2}^2 - \left(\left\|u^{(3)}\right\|^2 + \sin^2 3\varepsilon\left\|A^3u\right\|^2\right) =$$

$$= \cos^2 3\varepsilon\left\|A^3u\right\|_{L_2}^2 + \cos 3\varepsilon\left\|u'(0)\right\|_{3/2}^2 \ .$$

This implies the validity of inequality (5). Inequality (6) is proved analogously. Integrating by parts we obtain

$$\begin{gathered}\left\|A^2u'\right\|_{L_2}^2 = \left\|C^2u'\right\|_{L_2}^2 = -\left(C^3u, Cu^4\right)_{L_2} \le \left\|C^3u\right\|_{L_2}\left\|Cu^4\right\|_{L_2} = \\ = \left\|A^3u\right\|_{L_2} + \left\|Au^4\right\|_{L_2}.\end{gathered} \tag{9}$$

The identity ($\lambda > 0$)

$$\left\|\lambda C^2u' + Cu'' + \tfrac{1}{\lambda}u^{(3)}\right\|_{L_2}^2 = \lambda^2\left\|C^2u'\right\|_{L_2}^2 + \tfrac{1}{\lambda^2}\left\|u^{(3)}\right\|_{L_2}^2 - \left\|Cu''\right\|_{L_2}^2 -$$

$$-\left\|\sqrt{\lambda}C^{3/2}u'(0) + \tfrac{1}{\sqrt{\lambda}}C^{1/2}u''(0)\right\|^2$$

implies that

$$\lambda^2\left\|C^2u'\right\|_{L_2}^2 + \frac{1}{\lambda^2}\left\|u^{(3)}\right\|_{L_2}^2 \ge \left\|Cu''\right\|_{L_2}^2 \ .$$

If we assume here $\lambda = \left\|C^2u'\right\|_{L_2}^{-\frac{1}{2}}\left\|u^{(3)}\right\|_{L_2}^{\frac{1}{2}}$ we obtain

$$\left\|Cu''\right\|_{L_2}^2 \le 2\left\|C^2u'_{L_2}\right\|\ \left\|u^{(3)}\right\|_{L_2}$$

or

$$\left\|Au''\right\|_{L_2}^2 \le 2\left\|A^2u'\right\|_{L_2}\ \left\|u^{(3)}\right\|_{L_2}. \tag{10}$$

Taking into account inequality (10) in (9) subject to inequalities (5) and (6) we obtain

$$\left\|A^2u'\right\| \le 2^{1/3}\left\|A^3u\right\|_{L_2}^{2/3}\ \left\|u^{(3)}\right\|_{L_2}^{\frac{1}{3}} \le 2^{1/3}\frac{1}{\cos 3\varepsilon}. \tag{11}$$

Taking into account inequality (9) in (10) subject to (5) and (6) we have

$$\left\|Au''\right\|_{L_2} \le 2^{2/3}\left\|A^3u\right\|_{L_2}^{1/3}\left\|u^{(3)}\right\|_{L_2}^{2/3} \le \frac{2^{2/3}}{\cos 3\varepsilon}. \qquad (12)$$

The lemma is proved.

Theorem. *Let A be normal operator and its spectrum be contained in the angular sector $S_\varepsilon = \{\lambda|\ |\arg\lambda| \le \varepsilon\}$ $0 \le \varepsilon < \frac{\pi}{6}$, operators $B_1 = A_1A^{-1}$, $B_2 = A_2A^{-2}$, $B_3 = A_3A^{-3}$ be bounded in H, moreover, the inequality*

$$2^{2/3}\left\|B_1\right\| + 2^{1/3}\left\|B_2\right\| + \left\|B_3\right\| < \cos 3\varepsilon$$

hold. Then problem (1), (2) is regular solvable.

Proof. We write equation $Pu = f$ in the form $(P_0 + P_0)\,u = f$.

After substitution $P_0u = v$ we obtain equation $\left(E + P_1P_0^{-1}\right)v = f$ in the space $L_2\left(R_+; H\right)$. By lemma 3

$$\left\|P_1P_0^{-1}v\right\|_{L_2} = \left\|P_1u\right\|_{L_2} \le \left\|B_1\right\|\ \left\|Au''\right\|_{L_2} + \left\|B_2\right\|\ \left\|A^2u'\right\|_{L_2} +$$

$$+\left\|B_2\right\|\ \left\|A^3u\right\|_{L_2} \le \left(2^{2/3}\left\|B_1\right\| + 2^{1/3}\left\|B_2\right\| + \left\|B_3\right\|\right)/\cos 3\varepsilon < 1$$

which means that the operator $E+P_1P_0^{-1}$ is invertible and $u = P_0^{-1}\left(E + P_1P_0^{-1}\right)^{-1}f$.

The theorem is proved.

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