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NECESSARY OPTIMALITY CONDITIONS OF QUASI-SINGULAR CONTROLS IN GOURSAT-DARBOUX SYSTEMS

Abstract

The present paper is devoted to investigation of optimality of quasi-singular in Goursat-Darboux systems, and some known results are generalized.

A number of papers was devoted to investigation of quasi-singular controls and a lot of various and important results were obtained [1-6]. It is known [1] that necessary optimality conditions of quasi-singular controls also allow to get additional information on nonsingular controls in terms of Pontrjagin maximum principle. On the other hand, the study of quasi-singular control is a simpler problem than the analogous problem for Pontrjagin's extremums. The present paper is devoted to investigation of optimality of quasi-singular controls in Goursat-Darboux systems and results from [3-4] are generalized.

1. Let on the class of piecewise continuous vector-functions $u(t,x)=(u_1(t,x),...,$ $u_r(t,x))'$, $(t,x)\in D=\{(t,x):t\in J_1[t_0,t_1],x\in J_2=[x_0,x_1]\}$ taking on values from the given convex set U of r -dimensional Euclidian space E^r (admissible controls), the following functional be minimized

$$S(u)=\Phi(z(t_1,x_1)) \tag{1}$$

which is defined by means of trajectories of the system

$$z_{tx}=f(z,z_t,z_x,u,t,x),\quad (t,x)\in D, \tag{2}$$

$$z(t,x_0)=\varphi_1(t),\ t\in J_1;\ z(t_0,x)=\varphi_2(x),\ x\in J_2,\ \varphi_1(t_0)=\varphi_2(x_0), \tag{3}$$

where the prime $(')$ denotes transposition and z_t,z_x,z_{tx} are partial derivatives of the n-dimensional vector function $z(t,x)$.

Suppose that $\varphi_1(t),\ t\in J_1$ and $\varphi_2(x),\ t\in J_2$ are continuously differentiable functions. Not specifying it, we assume that for functions $f:E^n\times E^n\times E^n\times$ $\times E^r\times I_1\times I_2\to E^n,\ \Phi(z),\ z\in E^n$ and investigated admissible controls those analytic properties are satisfied which we will need during consideration. Exact-formulation of these properties in concrete case is not complicated. We also assume that each admissible control $u(t,x),\ (t,x)\in D$ generates a unique solution (in terms of [7]) $z(t,x_0),\ (t,x)\in D$ of problem (2), (3) which is defined everywhere on D.

Admissible control $u^0(t,x)$, which is a solution of problem (1)-(3) is called optimal control and we call process $(u^0(t,x),z^0(t,x))$ an optimal process.

2. Denote by $\tilde C(\tilde D,\tilde U)$ a class of piecewise continuous vector functions $\tilde u:\tilde D\to\tilde U$. Let $(u^0(t,x),\ z^0(t,x))$ be a fixed optimal process. We define a special increment of control $u^0(t,x)$ in the following way:

$$\Delta u(t,x)=\varepsilon\delta u(t,x),(t,x)\in D, \tag{4}$$

where $\varepsilon\in(0,1],\ \delta u(t,x)=\tilde u(t,x)-u^0(t,x),\ \tilde u(t,x)\in\tilde C(D,U)$.

It is clear that $u^0(t,x)+\Delta u(t,x)$ is an admissible control. Denote by $\Delta z(t,x)$ an increment of $z^0(t,x)$ responding to increment (4) of the control.

Estimates established e.g., in [8] imply

$$\|\Delta z(t,x)\| \le K\cdot\varepsilon, \|\Delta z_t(t,x)\| \le$$
$$\le K\cdot\varepsilon, \|\Delta z(t,x)\| \le K\cdot\varepsilon, (t,x)\in D, K = const > 0. \tag{5}$$

Assuming $\Delta z(t,x) = \varepsilon\delta z(t,x) + o(\varepsilon)$, $\Delta z_t(t,x) = \varepsilon\delta z_t(t,x) + o(\varepsilon)$, $\Delta z_x(t,x) = \varepsilon\delta z_x + o(\varepsilon), (t,x)\in D$ and taking into account (4) using ordinary technique [1,p.80] it is easy to establish that the following inequality is satisfied along the optimal process $(u^0(t,x), z^0(t,x))$.

$$\begin{gathered}\Delta S(u^0;\delta u) = -\varepsilon\int_{t_0}^{t_1}\int_{x_0}^{x_1} H_u'(t,x)\delta u(t,x)dtdx+\\ +\frac{\varepsilon^2}{2}\left\{\delta z'(t_1,x_1)\Phi_{zz}\left(z^0(t_1,x_1)\delta z(t_1,x_1)\right.\right. -\\ -\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\delta p'(t,x)H_{pp}(t,x)\delta p(t,x) + 2\delta p'(t,x)H_{pu}(t,x)\delta u(t,x)+\right.\\ \left.\left.+\delta u'(t,x)H_{u\ u}(t,x)\delta u(t,x)\right]dtdx\right\} + o(\varepsilon^2) \ge 0\ ,\end{gathered} \tag{6}$$

for all $\tilde u(t,x)\in \tilde C(D,U)(\delta u(t,x) = \tilde u(t,x) - u^0(t,x)),\ \varepsilon\in(0,1]$.

There $p = (z, z_t, z_x)'$, $\delta p(t,x) = (\delta z(t,x),\ \delta z_t(t,x),\ \delta z_x(t,x))'$, $\delta z(t,x)$ is a variation of state, which is a solution of the equation in variations:

$$\delta z_{tx} = f_z(t,x)\delta z + f_{z_t}(t,x)\delta z_t + f_{z_x}(t,x)z_x + f_u(t,x)\delta u(t,x), (t,x)\in D, \tag{7}$$

$$\delta z(t_0,x) = \delta z(t,x_0) \equiv 0. \tag{8}$$

$H(\Psi, z, z_t, z_x, u, t, x) = \Psi' f(z, z_t, z_x, u, t, x)$ is a Hamiltonian of the system, $H(t,x) = H(\Psi^0(t,x),\ z^0(t,x),\ z_t^0(t,x),\ z_x^0(t,x),\ u^0(t,x),\ t,x)$, where $\Psi^0(\xi,\eta) = -\lambda'(\xi,\eta;t_1,x_1)$ $\Phi_z(z^0(t_1,x_1))$ and $\lambda(\xi,\eta;t,x) - (n\times n)$ is Riemannian matrix of variational equation (7), (8) which is a solution of the following problem [2,9]:

$$\lambda_{t,x} = f_z(t,x)\lambda + f_{z_t}(t,x)\lambda + f_{z_x}(t,x)\lambda_x,\ (t,x)\in D, \tag{9}$$

$$\lambda_t(\xi,\eta;t,\eta) = f_{z_x}(t,\eta)\lambda(\xi,\eta;t,\eta),\ \lambda_x(\xi,\eta;\xi,x) = f_{z_t}(\xi,x)\lambda(\xi,\eta;\xi,x), \tag{10}$$

$$\lambda(\xi,\eta;\xi,\eta)\in E,\ (\xi,\eta)\in D\ (E \text{ is unit } n\times n \text{ matrix}). \tag{11}$$

Note that following [2] the solution $\delta z(t,x), (t,x)\in D$ of problem (7),(8) and its partial derivatives with respect to t, x in domain D can be represented in the following form using $\lambda(\xi,\eta;t,x)$:

$$\delta z(t,x) = \int_{t_0}^{t_1}\int_{x_0}^{x_1}\lambda(\xi,\eta;t,x)f_u(\xi,\eta)\delta u(\xi,\eta)d\xi d\eta, \tag{12}$$

$$\delta z_t(t,x) = \int_{x_0}^{x_1}\left[\lambda(t,\eta;t,x)f_u(t,\eta)\delta u(t,\eta) + \int_{t_0}^{t_1}\lambda_t(\xi,\eta;t,x)f_u(\xi,\eta)\delta u(\xi,\eta)d\xi\right]d\eta, \tag{13}$$

$$\delta z_x(t,x) = \int_{t_0}^{t_1}\left[\lambda(\xi,x;t,x)f_u(\xi,x)\delta u(\xi,x) + \int_{x_0}^{x_1}\lambda_t(\xi,\eta;t,x)f_u(\xi,\eta)\delta u(\xi,\eta)d\eta\right]d\xi. \tag{14}$$

Note that inequality (6) is implicit optimality criterion. On its basis we can obtain a number of simpler effective optimality conditions which we'll study in this paper. To this end at first we investigate $\Delta S(u^0;\delta u)$ for special variation of the following type (variation of control is performed only on the strip parallel to x axis)

$$\delta u(t,x)\equiv\delta_\mu u(t,x)=\begin{cases} \upsilon(x)-u^0(t,x),(t,x)\in D_\mu=[\theta,\theta+\mu)\times J_2\subset D, \\ 0, \qquad (t,x)\in D\backslash D_\mu=D_0\cup D_1, \end{cases} \tag{15}$$

where $t=\theta\in J_1$ is not a line of discontinuity of control $u^0(t,x)$, $\mu>0$ is a sufficiently small number such that

$$D_0=[t_0,\theta]\times J_2, D_1=[\theta+\mu,t_1]\times J_2;\upsilon(x)\in\tilde{C}(J_2,U).$$

Denote by $\delta_\mu z(z,t)$, $(t,x)\in D$ solution of problem (7), (8) corresponding to variation (15). Taking into account (15), from (12)-(14) we obtain: 1) equalities D_0 hold in domain $\delta_\mu z(t,x)=\delta_\mu z_t(t,x)=\delta_\mu z_k(t,x)\equiv 0$; 2) in domain D_μ vector-functions $\delta_\mu z(t,x),\delta_\mu z_x(t,x)$ have order μ and $\delta_\mu z_t(t,x)$ is finite with respect to μ; 3) in domain D_1 vector-functions $\delta_\mu z,\delta_\mu z_t,\delta_\mu z_x$ have the same order μ. Therefore according to (13), (15) inequality (6) takes the following form:

$$\begin{gathered}\Delta S(u^0;\delta_\mu u)=-\varepsilon\int\limits_\theta^{\theta+\mu}\int\limits_{x_0}^{x_1}H_u'(t,x)\left[\upsilon(x)-u^0(t,x)\right]dtdx- \\ -\frac{\varepsilon^2}{2}\left\{\mu M^{(1)}[u^0;\theta;\upsilon(x)]+o(\mu)\right\}+o(\varepsilon^2)\geq 0,\end{gathered} \tag{16}$$

$\forall\upsilon(x)\in\tilde{C}(J_2,U),\varepsilon(0,1),\mu>0$ are sufficiently small numbers.

There

$$\begin{gathered}M^{(1)}\left[u^0;\theta;\upsilon(x)\right]\equiv\int\limits_{x_0}^{x_1}\left[y'(x)H_{z_tz_t}(\theta,x)y(x)+2y'(x)H_{z_tu}(\theta,x)(\upsilon(x)-\right. \\ \left.-u^0(\theta,x))+(\upsilon(x)-u^0(\theta,x))'H_{u\ u}(\theta,x)(\upsilon(x)-u^0(\theta,x))\right]dx,\end{gathered} \tag{17}$$

where $y(x)$ is a solution of ordinary linear problem:

$$\frac{dy}{dx}=f_{z_t}(\theta,x)y+f_u(\theta,x)(\upsilon(x)-u^0(\theta,x)),x\in J_2, \tag{18}$$

$$y(x_0)=0. \tag{19}$$

It is interesting to note that taking into account problems (18), (19) by virtue of the technique from [1,p.177] functional (17) can be represented in the form:

$$\begin{gathered}M^{(1)}\left[u^0;\theta;\upsilon(x)\right]\equiv\int\limits_{x_0}^{x_1}(\upsilon(x)-u^0(\theta,x))'H_{u\ u}(\theta,x)(\upsilon(x)-u^0(\theta,x))dx+ \\ +2\int\limits_{x_0}^{x_1}(\upsilon(x)-u^0(\theta,x))'\left[H_{z_tu}'(\theta,x)+f_u'(\theta,x)\Psi^{(1)}(\theta,x)\right]y(x)dx,\end{gathered} \tag{20}$$

where

$$\begin{gathered}\frac{\partial}{\partial x}\Psi^{(1)}(\theta,x)=-f_{z_t}'(\theta,x)\Psi^{(1)}(\theta,x)-\Psi^{(1)}(\theta,x)f_{z_t}(\theta,x)-H_{z_tz_t}(\theta,x), \\ x\in J_2,\Psi^{(1)}(\theta,x_1)=0.\end{gathered} \tag{21}$$

For further investigation we consider the functional $M^{(1)}[u^0;\theta;\upsilon(x)]$ on "needle-shaped" functions of the form:

$$\upsilon(x)=\upsilon^*(x)=\begin{cases}\upsilon\in U, x\in[\sigma,\sigma+\tilde{\varepsilon})\equiv J_{\tilde{\varepsilon}}\subset J_2,\\ u^0(\theta,x), x\in J_2\backslash J_{\tilde{\varepsilon}},\end{cases}$$

where (θ,σ) is a point of continuity $u^0(t,x), \tilde{\varepsilon}>0$ is a sufficiently small number.

Using the scheme from [10], applying Taylor's formula at the point σ and Leibnitz formula on differentiation of product, we obtain the following representation for $M^{(1)}[u^0;\theta;\upsilon^*(x)]$:

$$\begin{aligned}M^{(1)}[u^0;\theta;\upsilon^*(x)]=\int\limits_{\sigma}^{\sigma+\tilde{\varepsilon}}(\upsilon-u^0(\theta,x))'H'_{u\ u}(\theta,x)(\upsilon-u^0(\theta,x))dx+\\ +2\int\limits_{\sigma}^{\sigma+\tilde{\varepsilon}}(\upsilon-u^0(\theta,x))'\left[H_{z_tu}(\theta,x)+f'_u(\theta,x)\Psi^{(1)}(\theta,x)\right]y(x)dx=\\ =\int\limits_{\sigma}^{\sigma+\tilde{\varepsilon}}(\upsilon-u^0(\theta,x))'H_{u\ u}(\theta,x)(\upsilon-u^0(\theta,x)dx+\\ +2\sum_{m=1}^{k+1}\sum_{i=1}^{m}C_m^i\left.\frac{d^{m-i}q^{(1)'}(x,\upsilon-u^0(\theta,x))}{dx^{m-i}}\right|_{x=\sigma+oP_{i-1}^{(1)}(\sigma,\upsilon-u^0(\theta,x))}\frac{\tilde{\varepsilon}^{m+1}}{(m+1)!}+o(\tilde{\varepsilon}^{k+2}).\end{aligned}\tag{22}$$

Here $C_m^i=\dfrac{m!}{i!(m-i)!}$;

$$q^{(1)}(x,\upsilon-u^0(\theta,x))=\left[H_{z_tu}(\theta,x)+\Psi^{(1)}(\theta,x)f_u(\theta,x)\right](\upsilon-u^0(\theta,x)),x\in J_2,\tag{23}$$

$$\omega_j^{(1)}(x)=\frac{d}{dx}\omega_{j-1}^{(1)}(x)+\omega_{j-1}^{(1)}(x)\omega_0^{(1)}(x),\ \omega_0^{(1)}(x)\equiv f_{z_t}(\theta,x),$$

$$P_j^{(1)}(x;\upsilon-u^0(\theta,x))=\frac{d}{dx}P_{j-1}^{(1)}(x;\upsilon-u^0(\theta,x))+\omega_{j-1}^{(1)}(x)P_0^{(1)}(x;\upsilon-u^0(\theta,x)),$$

$$P_0^{(1)}(x;\upsilon-u^0(\theta,x))\equiv f_u(\theta,x)(\upsilon-u^0(\theta,x)),\ j=1,2,...,x\in J_2.\tag{24}$$

Consider the functional $M^{(1)}[u^0;\theta;\upsilon(x)]$ on a number of "needle shaped" functions of the form:

$$\upsilon(x)=\tilde{\upsilon}(x)=\begin{cases}\upsilon_j\in U, x\in[\sigma_i,\sigma_i+l_i\tilde{\varepsilon}), i=\overline{1,m},\\ u^0(\theta,x), x\in J_2\backslash\left(\bigcup\limits_{i=1}^{m}[\sigma_i,\sigma_i+l_i\tilde{\varepsilon})\right),\end{cases}\tag{25}$$

where $x_0\leq\sigma_1<\sigma_2<...<\sigma_m<x_1\equiv\sigma_{m+1}, l_i\in[0,+\infty), \tilde{\varepsilon}>0$ is a sufficiently small number such that $\sigma_i+l_i\tilde{\varepsilon}<\sigma_{i+1}, i=\overline{1,m}; (\theta,\sigma_i), i=\overline{1,m}$ are points of continuity of $u^0(t,x)$.

Obviously, $\upsilon(x)\in\tilde{C}(J_2,U)$. Denote by $\tilde{y}(x)$ solution of problem (18), (19) corresponding to function (25). Then taking into account (25) and problems (10),

(11) we have:

$$\tilde{y}(x)=\int\limits_{x_0}^{x}\lambda(\theta,\eta;\theta,x)f_u(\theta,\eta)\tilde{\upsilon}((\eta)-u^0(\theta,\eta))d\eta=$$

$$=\tilde{\varepsilon}\cdot sign\,(i-1)\sum_{j=1}^{i-1}l_j(\theta,\sigma_j;\theta,x)f_u(\theta,\sigma_j)\left(\upsilon_j-u^0(\theta,\sigma_j)\right)+$$

$$+l_i\lambda(\theta,\sigma_i;\theta,x)f_u(\theta,\sigma_i)\left(\upsilon_i-u^0(\theta,\sigma_i)\right)(x-\sigma_i)+o(\tilde{\varepsilon}), \tag{26}$$

$$x\in[\sigma_i,\sigma_i+l_i\cdot\tilde{\varepsilon}),i\in\{1,..,m\}.$$

Thus, taking into account (25) and (26), subject to (20) we obtain the following representation for $M^{(1)}[u^0;\theta;\tilde{\upsilon}(x)]$:

$$M^{(1)}[u^0;\theta;\tilde{\upsilon}(x)]=\sum_{i=1}^{m}\int\limits_{\sigma_i}^{\sigma_i+l_i\tilde{\varepsilon}}\left(\upsilon_i-u^0(\theta,x)\right)'H_{u\ u}(\theta,x)(\upsilon_i-u^0(\theta,x))dx+$$

$$+2\sum_{i=1}^{m}\int\limits_{\sigma_i}^{\sigma_i+l_i\tilde{\varepsilon}}(\upsilon_i-u^0(\theta,x))'[H'_{z_tu}(\theta,x)+f'_u(\theta,x)\Psi^{(1)}(\theta,x)]\times$$

$$\times\left(l_i\lambda(\theta,\sigma_i;\theta,x)f_u(\theta,\sigma_i)(\upsilon_i-u^0(\theta,\sigma_i))(x-\sigma_i)\right)+$$

$$+\tilde{\varepsilon}\cdot sign\,(i-1)\sum_{j=1}^{i-1}l_j\lambda(\theta,\sigma_j;\theta,x)f_u(\theta,\sigma_j)\left(\upsilon_j-u^0(\theta,\sigma_j)\right)+o(\tilde{\varepsilon}))dx=$$

$$=\sum_{i=1}^{m}\int\limits_{\sigma_i}^{\sigma_i+l_i\tilde{\varepsilon}}(\upsilon_i-u^0(\theta,x))'H_{u\ u}(\theta,x)(\upsilon_i-u^0(\theta,x))dx+ \tag{27}$$

$$+\varepsilon^2\sum_{i=1}^{m}l_i^2(\upsilon_i-u^0(\theta,\sigma_i))'\left[H'_{z_tu}(\theta,\sigma_i)+f'_u(\theta,\sigma_i)\Psi^{(1)}(\theta,\sigma_i)\right]\times$$

$$\times f_u(\theta,\sigma_i)(\upsilon_i-u^0(\theta,\sigma))+2\varepsilon^2\sum_{i=1}^{m}\sum_{j=1}^{i-1}sign(i-1)l_il_j(\upsilon_i-u^0(\theta,\sigma_i))'\times$$

$$\times\left[H'_{z_tu}(\theta,\sigma_i)+f'_u(\theta,\sigma_i)\Psi^{(1)}(\theta,\sigma_i)\right]\times$$

$$\times\lambda(\theta,\sigma_j;\theta,\sigma_i)f_u(\theta,\sigma_j)(\upsilon_j-u^0(\theta,\sigma_j))+o(\varepsilon^2).$$

To write this formula briefly and convenient analogously to [11] we introduce the function

$$\Omega^{(1)}[u^0](\xi;\eta;\upsilon;x,\omega)=\begin{cases}(\omega-u^0(\xi,x))'\left[H'_{z_tu}(\xi,x)+f'_u(\xi,x)\Psi^{(1)}(\xi,x)\right]\times\\ \times\lambda(\xi,\eta;\xi,x)f_u(\xi,\eta)(\upsilon-u^0(\xi,\eta)),\eta\le x\\ \Omega^{(1)}[u^0](\xi;x,\omega;\eta,u),\eta>x.\end{cases} \tag{28}$$

In fact if we take into consideration (28) then formula (27) takes on the compact form:

$$M^{(1)}[u^0;\theta;\tilde{\upsilon}(x)] = \sum_{i=1}^{m}\int_{\sigma_i}^{\sigma_i+l_i\tilde{\varepsilon}} \left(\upsilon_i - u^0(\theta,x)\right)' H_{u\ u}(\theta,x)(\upsilon_i - u^0(\theta,x))dx+$$
$$+\varepsilon^2\sum_{i,j=1}^{m}\Omega^{(1)}\left[u^0\right](\theta;\sigma_i,\upsilon_i,\sigma_j,\upsilon_j)l_il_j + o(\varepsilon^2). \tag{29}$$

The case when investigation $\Delta S(u^0;\delta u)$ is carried out by special variation of the form:

$$\delta u(t,x)\equiv\delta_\alpha u(t,x) = \begin{cases} \upsilon(t)-u^0(t,x), & (t,x)\in D_\alpha = J_1\times[\sigma,\sigma+\alpha) \\ n0, & (t,x)\in D\backslash D_\alpha, \end{cases}$$

where $x=\sigma$ is not a line of discontinuity of the control $u^0(t,x)$, $\alpha>0$ is a sufficiently small number such that $D_\alpha\subset D$, $\upsilon(t)\in\tilde{C}(J_1,U)$ is a studied analogously.

Omitting the details of calculations we perform the finite results by analogy to (16)-(24), (27)-(29):

$$\Delta S(u^0;\delta_\alpha u) = -\varepsilon\int_{t_0}^{t_1}\int_{\sigma}^{\sigma+\alpha} H'_u(t,x)\left[\upsilon(x)-u^0(t,x)\right]dtdx-$$
$$-\frac{\varepsilon^2}{2}\left\{\alpha M^{(2)}\left[u^0;\sigma;u(t)\right]+o(\alpha)\right\}+o(\varepsilon^2)\geq 0. \tag{30}$$

Here

$$M^{(2)}[u^0;\sigma;\upsilon(t)]\equiv\int_{t_0}^{t_1}[v'(t)H_{z_xz_x}(t,\sigma)v(t)+2v'(t)H_{z_xu}(t,\sigma)(\upsilon(t)-$$
$$-u^0(t,\sigma))+(\upsilon(t)-u^0(t,\sigma))'H_{u\ u}(t,\sigma(\upsilon(t)-u^0(t,\sigma))]dt, \tag{31}$$

where $v[t]$ is a solution of the ordinary linear problem:

$$\frac{dv(t)}{dt} = f_{z_x}(t,\sigma)v(t)+f_u(t,\sigma)\left(\upsilon(x)-u^0(t,\sigma)\right), t\in J_1, v(t_0)=0;$$
$$M^{(2)}[u^0;\sigma;u^*(t)] = \int_{\theta}^{\theta+\tilde{\varepsilon}}(\upsilon-u^0(t,\sigma))'H_{u\ u}(t,\sigma)(\upsilon-u^0(t,\sigma)dt+$$
$$+2\int_{\theta}^{\theta+\tilde{\varepsilon}}(\upsilon-u^0(t,\sigma))'\left[H'_{z_xu}(t,\sigma)+f'_u(t,\sigma)\Psi^{(2)}(t,\sigma)\right]v(t)dt=$$
$$=\int_{\theta}^{\theta+\tilde{\varepsilon}}(\upsilon-u^0(t,\sigma))'H_{u\ u}(t,\sigma)(\upsilon-u^0(t,\sigma))dt+$$
$$+2\sum_{m=1}^{k+1}\sum_{i=1}^{m}C_m^i\left.\frac{d^{m-i}q^{(2)\prime}(t,\upsilon-u^0(t,\sigma))}{dt^{m-i}}\right|_{t=\theta+o}\times$$
$$\times p^{(2)}(\theta,\upsilon-u^0(\theta,\sigma))\frac{\tilde{\varepsilon}^{m+1}}{(m+1)!}+o(\tilde{\varepsilon}^{k+2}), \tag{32}$$

where $C_m^i=\frac{m!}{i!(m-i)!}$;

$$q^{(2)}(t;\upsilon-u^0(t,\sigma)) = [H_{z_xu}(t,\sigma)+\Psi^{(2)}(t,\sigma)f_u(t,\sigma)](\upsilon-u^0(t,\sigma)),$$
$$\omega_j^{(2)}(t) = \frac{d}{dt}\omega_{j-1}^{(2)}(t)+\omega_{j-1}^{(2)}(t)\omega_0^{(2)}(t), \omega_0^{(2)}(t) = f_{z_x}(t,\sigma), \tag{33}$$

$$p_j^{(2)}(t;\upsilon-u^0(t,\sigma))=\frac{d}{dt}p_{j-1}^{(2)}(t;\upsilon-u^0(t,\sigma))+\omega_{j-1}^{(2)}(t)p_0^{(2)}(t;\upsilon-u^0(t,\sigma)),$$
$$p_0^{(2)}(t;\upsilon-u^0(t,\sigma))\equiv f_u(t,\sigma)(\upsilon-u^0(t,\sigma)),j=1,2,..., \tag{34}$$

$$M^{(2)}[u^0;\theta;\tilde{\upsilon}(t)]=\sum_{i=1}^{m}\int_{\theta_i}^{\theta_i+l_i\tilde{\varepsilon}}(\upsilon_i-u^0(t,\sigma))'H_{u\,u}(t,\sigma)(\upsilon_i-u^0(t,\sigma))dt+$$
$$+\varepsilon^2\sum_{i,j=1}^{m}\Omega^{(2)}[u^0](\sigma;\theta_i,\upsilon_i;\theta_j;\upsilon_j)l_il_j+o(\varepsilon^2), \tag{35}$$

where

$$\Omega^{(2)}[u^0](\eta;\xi,\upsilon;t,\omega)=\begin{cases}(\omega-u^0(t,\eta))'\left[H'_{z_xu}(t,\eta)+f'_u(t,\eta)\Psi^{(2)}(t,\eta)\right]\times\\ \times\lambda(\xi,\eta;t,\eta)f_u(\xi,\eta)(\upsilon-u^0(\xi,\eta)),\xi\le t,\\ \Omega^{(2)}[u^0](\eta,t,\omega,\xi,\upsilon),\xi>t,\end{cases} \tag{36}$$

$$\frac{\partial}{\partial t}\Psi^{(2)}(t,\eta)=-f'_{z_x}(t,\eta)\Psi^{(2)}(t,\eta)-\Psi^{(2)}(t,\eta)f_{z_x}(t,x)-$$
$$-H_{z_xz_x}(t,\eta),t\in J_1,\Psi^{(2)}(t_1,\eta)=0. \tag{37}$$

3. From (6) one can easily obtain a simple necessary optimality condition (differential maximum principle for problem (1)-(3)).

$$\max_{\upsilon\in U}H'_u(t,x)(\upsilon-u^0(t,x))=0,\ \text{a.e.}\ (t,x)\in D. \tag{38}$$

Definition 1. *Control $u^0(t,x)$ satisfying condition (38) is called quasi-singular along the straight line $t=\theta$ $[x=\sigma]$ with the set $U_\theta\subset U[U_\sigma\subset U]$ if there exists $\alpha>0$ such that*

$$\int_{x_0}^{x_1}H'_u(t,x)\left(\upsilon-u^0(t,x)\right)dx=0,\forall\upsilon\in U_\theta,\forall\upsilon\in[\theta,\theta+\alpha),$$

$$\left[\int_{t_0}^{t_1}H'_u(t,x)\left(\upsilon-u^0(t,x)\right)dt=0,\forall\upsilon\in U_\sigma,\forall x\in[\sigma,\sigma+\alpha)\right],$$

where

$$U_\theta\backslash\{u^0(t,x)\}\ne\varnothing,(t,x)\in[\theta,\theta+\alpha)\times$$
$$\times J_2\ [U_\alpha\backslash\{u^0(t,x)\}\ne\varnothing,(t,x)\in J_1\times[\sigma,\sigma+\alpha)].$$

It is easy to show that any singular control in terms of definition from [3,4] is singular in terms of definition 1, but singular control n terms of definition 1 may not be singular control in term of definition from [3,4]. Allowing for definition 1 from (16) (30) we get the proof of the following theorem.

Theorem 1. *Let optimal control $u^0(t,x)$ be quasi-singular along the straight line $t=\theta[t_0,t_1)[x=\sigma\in[x_0,x_1)]$ on the set $U_\theta[U_\sigma]$. Then inequalities*

$$M^{(1)}[u^0;\theta;\upsilon(x)]\le 0,\forall\upsilon(x)\subset\tilde{C}(J_2,U_\theta), \tag{39}$$

$$\left[M^{(2)}[u^0;\sigma;\upsilon(t)]\le 0,\forall\upsilon(t)\in\tilde{C}(J_1,U_\sigma)\right], \tag{40}$$

hold where $t=\theta$ $[x=\sigma]$ *is not a line of discontinuity of control* $u^0(t,x),(t,x)\in D$ *functions* $M^{(1)}[.],M^{(2)}[.]$ *are defined by (17) and (31), respectively.*

Note that optimality conditions (39), (40) in this concrete case are not convenient because each of them requires complicated calculations. Therefore problem of obtaining simpler, more constructively verifiable necessary optimality conditions from these conditions appears.

From optimality conditions (39), [40] the simplest optimality condition is directly obtained

$$(\upsilon-u^0(\tau,x)'H_{u\ u}(\tau,x)(\upsilon-u^0(\tau,x))\le 0,\upsilon\in U_\theta,\tau\in[\theta,\theta+\alpha),x\in J_2\backslash\tilde{J}_2, \quad (41)$$

$$[(\upsilon-u^0(\tau,x))'H'_{u\ u}(\tau,x)(\upsilon-u^0(\tau,x))\le 0,\upsilon\in U_\sigma,\tau\in[\sigma,\sigma+\alpha),t\in J_1\backslash\tilde{J}_1] \quad (42)$$

where $\tilde{J}_1\subset J_1,\tilde{J}_2\subset J_2,\tilde{J}_1,\tilde{J}_2$ are some finite sets, i.e., the following assertion is valid.

Theorem 2. *Let optimal control* $u^0(t,x)$ *be quasi-singular along the straight line* $t=\theta$ $[x=\sigma]$ *on the set* $U_\theta[U_\sigma]$. *Then conditions (41), (42) are necessarily satisfied.*

Definition 2. *Control* $u^0(t,x)$, *which is quasi-singular along the straight line* $t=\theta$ $[x=\sigma]$ *on the set* $U_\theta[U_\sigma]$ *and satisfying condition (41) ([42]) is called strong quasi-singular along the straight line* $t=\theta$ $[x=\sigma]$ *at the point* $(\theta,\sigma)[(\theta,\sigma)]$ *on the set* $U_{c\theta}(\sigma)\subset U_\theta[U_{c\sigma}(\theta)\subset U_\sigma]$, *if there exists* $\tilde{\alpha}>0$ *such t*hat

$$(\upsilon-u^0(\theta,x)'H_{u\ u}(\theta,x)(\upsilon-u^0(\theta,x))=0,\forall\upsilon\in U_{c\theta}(\sigma),\forall x\in[\sigma,\sigma+\tilde{\alpha})\in J_2$$

$$[(\upsilon-u^0(\tau,\sigma))'H_{u\ u}(\theta,x)(\upsilon-u^0(\theta,x))=0,\forall\upsilon\in U_{c\sigma}(\theta),\forall t\in[\theta,\theta+\tilde{\alpha})\subset J_1]$$

where

$$U_{c\theta}(\sigma)\backslash\{u^0(\sigma,x)\}\ne\varnothing,x\in[\sigma,\sigma+\tilde{\alpha})\ [U_{c\sigma}(\theta)\backslash\{u^0(t,\sigma)\}\ne\varnothing],t\in[\theta,\theta+\tilde{\alpha}).$$

From theorem 1, taking into account this definition and formulae (29), (35) we obtain the following assertion.

Theorem 3. *Let control* $u^0(t,x)$ *be strong quasi-singular at the points* $U_{c\sigma}(\sigma_i)$, *on the sets* $(\theta,\sigma_i)(x_0\le\sigma_1<\sigma_2<...<\sigma_m<x_1),i=\overline{1,m}$, *on the sets* $U_{c\theta}(\sigma_i)$, $[(\theta_i,\sigma)(t_0\le\theta_1<\theta_2<...<\theta_m<t_1)\ i=\overline{1,m}$.

Then fulfilment of the inequality

$$\sum_{i,j=1}^{m}\Omega^{(1)}[u^0](\theta;\sigma_i,\upsilon_i;\sigma_j;\upsilon_j)l_il_j\le 0,\forall\upsilon_j\in U_{c\theta}(\sigma_i),\forall l_i\ge 0,i=\overline{1,m},$$
$$\left[\sum_{i,j=1}^{m}\Omega^{(2)}[u^0](\sigma;\theta_i,\upsilon_i;\theta_j;\upsilon_j)l_il_j\le 0,\forall\upsilon_i\in U_{c\theta}(\sigma_i),\forall l_i\ge 0,i=\overline{1,m}\right],$$

where $(\theta,\sigma_i),i=\overline{1,m}[(\theta_i,\sigma)i-\overline{1,m}]$*are continuity points of control* $u^0(t,x)$, $\Omega^{(1)}[u^0](.),\Omega^{(2)}[u^0](.)$ *are defined by (28) and (36) reprectively is necessary for optimality of* $u^0(t,x)$.

Note that according to definition 1,2 this theorem generalizes the analogous theorem from [3,4].

Let us introduce the following definition.

Definition 3. *Let control* $u^0(t,x)$ *be strong quasi-singular along the straight line* $t=\theta$ $[x=\sigma]$ *at the point* $(\theta,\sigma)[(\theta,\sigma)]$ *on the set* $U_{c\theta}(\sigma)[U_{c\sigma}(\theta)]$. *We call it* *strongly quasi-singular of order* $k \in \{1, 2, ...\}$ *along the straight line* $t = \theta$ $[x = \sigma]$ *at the point* $(\theta, \sigma)[(\theta, \sigma)]$ *on the set* $U^{(k)}_{c\theta}(\sigma) \subset U_{c\theta}(\sigma)[U^{(k)}_{c\sigma}(\theta) \subset U_{c\sigma}(\theta)]$ *if equalities:*

$$Q^{(1)}_m(\theta, \sigma; \upsilon - u^0(\theta, \sigma)) = 0, \forall \upsilon \in U^{(k)}_{c\theta}(\sigma), m = \overline{1, k}, \tag{43}$$

$$\left[Q^{(2)}_m(\theta, \sigma; \upsilon - u^0(\theta, \sigma)) = 0, \forall \upsilon \in U^{(k)}_{c\theta}(\theta), m = \overline{1, k}\right], \tag{44}$$

where

$$U^{(k)}_{c\theta}(\sigma) \backslash \{u^0(\theta, \sigma)\} \neq \varnothing [U^{(k)}_{c\sigma}(\theta) \backslash \{u^0(\theta, \sigma)\} \neq \varnothing];$$

$$\begin{gathered} Q^{(1)}_m(\theta, \sigma; \upsilon - u^0(\theta, \sigma)) = \\ = \sum_{i=1}^{m} C^i_m \frac{d^{m-i}}{dx^{m-i}} q^{(1)\prime}(x, \upsilon - u^0(\theta, x))\bigg|_{x=\sigma+o} \times \\ \times p^{(1)}_{i-1}(\sigma, \upsilon - u^0(\theta, \sigma)), m = 1, 2, \cdots, \end{gathered} \tag{45}$$

$$\begin{gathered} Q^{(2)}_m(\theta, \sigma; \upsilon - u^0(\theta, \sigma)) = \\ = \sum_{i=1}^{m} C^i_m \frac{d^{m-i}}{dt^{m-i}} q^{(2)\prime}(t, \upsilon - u^0(t, \sigma))\bigg|_{t=\theta+o} p^{(2)}_{i=1}(\theta, \upsilon - u^0(\theta, \sigma)), m = 1, 2, ..., \end{gathered} \tag{46}$$

are satisfied.

If equalities (43)[(44)] are impossible for any natural k then we assume that the order of singularity equals to zero.

Thus taking into account definition 3 and formulae (27), (32) on the basis of theorem 1 we obtain the following statement.

Theorem 4. *Let the control* $u^0(t, x)$ *being strong quasi-singular along the straight line* $t = \theta$ $[x = \sigma]$ *at the point* $(\theta, \sigma)[(\theta, \sigma)]$ *on the set* $U_{c\theta}(\sigma)[U_{c\sigma}(\theta)]$ *be also strongly quasi-singular of the order* k *along the straight line* $t = \theta$ $[x = \sigma]$ *at the point on the set* $U^{(k)}_{c\theta}(\sigma) \subset U_{c\theta}(\sigma)[U^{(k)}_{c\sigma}(\theta)]$. *Then fulfillment of inequalities:*

$$\begin{gathered} Q^{(1)}_1(\theta, x; \upsilon - u^0(\theta, x)) \leq 0, \forall x \in [\sigma, \sigma + \tilde{\alpha}], \forall \upsilon \in U_{c\theta}(\sigma), \\ Q^{(1)}_{k+1}(\theta, \sigma; \upsilon - u^0(\theta, x)) \leq 0, \forall \upsilon \in U^{(k)}_{c\theta}(\sigma), \text{ for } k\{1, 2, ...\} \\ [Q^{(2)}_1(\upsilon, \sigma; \upsilon - u^0(t, \sigma)) \leq 0, \forall t \in [\theta, \theta + \tilde{\alpha}], \forall \upsilon \in U_{c\sigma}(\theta), \\ Q^{(2)}_{k+1}(\theta, \sigma; \upsilon - u^0(\theta, x)) = 0, \forall \upsilon \in U^{(k)}_{c\theta}(\theta), \text{ for } k\{1, 2, ...\}], \end{gathered}$$

where $Q^{(1)}_m(.), [Q^{(2)}_m(.)], m = 1, 2, ...,$ *are defined by (45) [(46)], is necessary for optimality of* $u^0(t, x)$. *Note that theorem 4 is a generalization of the analogous theorem from [4].*

References

[1]. Gabasov R., Kirillova F.M. *Singular optimal controls.* M.: "Nauka", 1973. (Russian)

[2]. Melikov T.K. *Investigation of singular processes in some optimal systems.* Ph.D. thesis, Baku, 1976. (Russian)

[3]. Mansimov K.B. *On optimality of quasi-singular controls in Goursat-Darboux Goursat-systems.* Diff.uravnenija, 1986, v.22, No11, pp.1952-1960. (Russian)

[4]. Melikov T.K. *Necessary optimality conditions of quasi-sinqular controls in Goursat-Darboux systems.* Deposited in Az NIINTI, 1992, No1846. Az.92, 20 p. (Russian)

[5]. Melikov T.K. *Singular controls in systems with contagion.* Baku, "Elm", 2002, 188 p. (Russian)

[6]. Mansimov K.B., Mardanov M.J. *Singular controls in systems with distributed parameters.* Baku, 2003, 43 p. (Preprint Inst.of Cybern. NAS of Azerb.)

[7]. Ashepkov L.T., Vasil'ev O.V. *On optimality of singular controls in Goursat-Darboux systems.* Zhurnal vychis.mathem.phis., 1975, v.15, No5, pp.1157-1167. (Russian)

[8]. Egorov A.I. *On optimal control by processes in some systems with distributed parameters.* Avtomatika i telemekhanika, 1964, v.25, No5, pp.613-623. (Russian)

[9]. Srochko V.V. *Optimality conditions for one class of systems with distributed parameters.* Sib.math.zhurnal, 1976, v.17, No6, pp.1108-1115. (Russian)

[10]. Melikov T.K. *On necessary optimality conditions of high order.* Zhurnal vychis.math. i matem.phis, 1995, v.35, No7, pp.1134-1138. (Russian)

[11]. Melikov T.K. *On necessary optimality conditions of the second order.* In book "Mathematical optimization and control methods in complex systems", Kalinin, 1983, pp.100-106. (Russian)

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