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THE FORWARD JUMPING METHODS

Abstract

As is known there exists a wide range of numerical methods for solving the Cauchy problem for ordinary differential equations. Among them forward jumping methods are little-studied in sprite of the fact that the first paper was published at the begining of XX century. Because of some deficiencies of forward jumping methods many authors did not investigate these methods. Therefore in the given paper the advantages of forward jumping methods are considered.

Introduction. There are some classes of methods for numerical solving the Cauchy problem for ordinary differential equations. Two of them are popular and they are called one-step and multistep methods such as Runge-Kutta method and Adams method. Recently multistep methods are rapidly developed. Obviously, it is not accident that Krylov A.N. called the Adams method the advanced form of Runge-Kutta method (see [1, p.293]). Generalization of Adams methods in the form of k-step method with constant coefficients was completely theoretically investigated in Dahlquist's paper (see [2]). In this paper the upper found is found for accuracy of stable k -step method with constant coefficients which is written in the form: $p \leq 2[k/2]+2$. There p is the accuracy of k -step method, which is usually called a degree of k-step method and k is the order of k-step method with constant coefficients. The famous result belongs to Dahlquist G. (see [3, p.25]).

Note that stability of k-step method with constant coefficients was first investigated in [4] and it was called a boundedness of variance. The criterion obtained in [4] for the boundedness of variance was used in [5] as definition of stability for k-step method with constant coefficients.

The first numerical method that Clairaut applied to calculations of disturbance of Halley comet (see [6, p.133]) was indirect numerical method.

Now the similar methods are called analytical numerical methods (see, e.g.[7])

Methods obtained from the k-step method with constant coefficients' can be divided into three groups: extrapolation method, interpolation methods and forward jumping methods.

The described above investigations refer to extrapolation and interpolation methods. However, the first work devoted to investigations of forward jumping method published in 1910 was the Kowell method. In this paper Kowell method was applied to precomputation of return of Halley comet (see [6]). We have to note that these methods were not widely used because of some deficiencies in comparison with interpolation methods, by degree of running ahead methods constructed by Laplace, Steklov and obeying to Dahlquist laws. The advantages of forward jumping methods were theoretically proved in [9] and for their use forward jumping methods predictor corrector was constructed in [10]. The forward jumping method in more general form was investigated in [11].

Here we find relations between some coefficients of forward jumping method.

Consider the following Cauchy problem:

$$y' = f(x, y), \ y(x_0) = y_0. \tag{1}$$

Suppose that problem (1) has a unique solution $y(x)$ defined on the segment $[x_0, X]$ and $y(x)$ has continuous derivatives up to some order $p+1$ inclusively.

Let us divide segment $[x_0, X]$ into N equal parts by $x_i = x_0 + ih (i = 1, 2, ..., N)$ points of division. Denote by y_m an approximate value of solution of problem (1) at the point x_m. Then k-step method with constant coefficients can be written in the form

$$\sum_{i=0}^{k}\alpha_i y_{n+1} = h\sum_{i=0}^{k}\beta_i f_{n+i} \quad (n = 0, 1, ..., N-k), \tag{2}$$

here $\alpha_i, \beta_i (i = 0, 1, 2, ..., k)$ are some real numbers, k is an integer valued quantity called the order of method (2). If for $\alpha_k \neq 0$ values $y_0, y_1, ..., y_{k-1}$ are known, then using method (2) we can calculate $y_k, y_{k+1}, ..., y_N$.

The forward jumping methods which are obtained from (2) can be written in the following form:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+1} = h\sum_{i=0}^{k}\beta_i f_{n+i}. \tag{3}$$

If we suppose $m = 0$ in (3), then we obtain method (2). Consequently, we can suppose that $m > 0$ and $\alpha_{k-m} \neq 0$. In this case by means of method (3) we can calculate y_{k-m} if $y_0, y_1, ..., y_{k-m-1}$ and $y_{k-m+1}, y_{k-m+2}, ..., y_k$ are known. Usually values $y_0, y_1, ..., y_{k-m-1}$ are known because they are initial values. But values $y_{k-m+1}, y_{k-m+2}, ..., y_k$ are unknown and it is required to calculate them. Therefore application of methods (3) is more complicated than methods of type (2). As was mentioned above in this case one can use predictor corrector method (see [10]). To recommend the forward jumping method for numerical solving problem (1) we have to prove their advantages over the known methods. It was proved in [9] that if method (3) has order p and is stable for $\alpha_{k-m} \neq 0$, then $p \leq k+m+1$. Thus, we obtain that for maximal values of forward jumping methods it holds $p_{\max} \geq 2[k/2]+2$ for $m \geq 3k$ or $m \geq 3k+1$ depending on evenness of k or m. Hence it follows that it is appropriate to investigate methods of type (3).

As it is known by constructing algorithm and especially two-sided methods it is necessary to determine the signs of the coefficients $\beta_j (j = k-m, k-m+1, ..., k)$ and also relations between these coefficients. Therefore let us consider some properties of coefficients $\beta_j (j = k-m, k-m+1, ..., k)$.

On properties of some coefficients.

The signs of the coefficients and relations between them for stable forward jumping method obtained from (3), can be determined by means of the following theorem.

Theorem. *Let method (3) be stable and have order* $p = k-m+1$ *and* $\alpha_{k-m} \neq 0$. *Then* $\beta_{k-v} = (-1)^{m-v} l_{m-v} (l_{m-v} > 0; v = 0, 1, 2, ..., m), \left|\beta_{k-s}\right| > \left|\beta_{k-s+1}\right| \quad (s = 1, 2, ..., m)$.

Note that the following relation holds between k *and* m

$$k \geq 3m+1 \text{ or } k \geq 3.$$

Proof. Allowing for the conditions of theorem we can rewrite (3) in the following form:

$$\rho(E)y(x_n) - hv(E)y'(x_n) = ch^{p+1}y_n^{(p+1)}, h \to 0, \tag{4}$$

where E is a displacement operator, i.e., $Ey(x) = y(x+h)$ and

$$\rho(\lambda) \equiv \sum_{i=0}^{k-m}\alpha_i\lambda^i, v(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i.$$

Because of independence of the coefficients in method (3) in the function $y(x)$. Consider the special case $y(x) = \exp(x)$ and suppose $\tau = \exp(h)$. Then $h \to 0$ implies $\tau \to 1$. Moreover relation (4) can be rewritten in the form:

$$\rho(E)\exp(x_n) - hv(E)\exp(x_n) = C(\ln\tau)^{p+1}\exp(x_n), \tau \to 1. \tag{5}$$

Using denotations $z = (\tau+1)/(\tau-1), \tau = (z+1)/(z-1)$ relation (5) can be written in the form:

$$R(z)(\ln(z+1)/(z-1))^{-1} - S(z) = C(2/z)^{p-k+m}, z \to \infty \tag{6}$$

where

$$R(z) = ((z-1)/2)^{k-m}\rho(\tau) = \sum_{i=0}^{k-m} a_i z^i \quad (a_{k-m} = 0),$$
$$S(z) = ((z-1)/2)^{k-1}v(\tau) = \sum_{i=-\infty}^{k-m} b_i z^i.$$

If method (3) is stable we impose some restrictions on the coefficients of the polynomial $\rho(\lambda)$.

If we extend these conditions to the coefficients of the polynomial $R(z)$, then we have (see [2]);

1. $R(z)$ has no roots with positive real part.
2. $R(z)$ has no multiple roots on imaginary axis and $a_{k-m-1} \neq 0$.

In [2] it was proved that if method (3) is stable then the nonzero coefficients a_j have the same sign. Therefore we can assume that $a_j \geq 0$ $(i = 0,1,2,...,k-1)$.

Further to prove the theorem we use mathematical induction method. To this end suppose $m = 1$. In this case sign of β_k is defined in [12]. For $m = 1$ we choose the coefficient b_k such that relation (6) has the following form:

$$R(z)\left(\ln\frac{z+1}{z-1}\right)^{-1} - S(z) = \sum_{i=3}^{\infty} C_i z^{-i}, z \to \infty. \tag{7}$$

Taking into account that method (3) has the maximum degree, we can write $C_2 = 0$. Consequently,

$$d\beta_k = -\sum_i a_{2i+1}\mu_{2i+1}, \tag{8}$$

where $d > 0$. If we take into account that $\mu_{2i+1} > 0$ (see [2]), then it follows from (8) that $\beta_k < 0$ or $\beta_k = -l_1 (l_1 > 0)$.

We represent the function $S(z)$ in the form:

$$S(z) = S_0(z) + S_1(z),$$

where

$$S_0(z) = \sum_{i=0}^{k-1} b_i z^i, \quad S_1(z) = \frac{\beta_k}{2^{k-1}}(z+1)^{k-1}\frac{z+1}{z-1}.$$

Taking into account $S(z)$ in (7) and choosing $S_0(z)$ as a principal part of the obtained function in the left-hand side of relation (7), we can write

$$S_0(z) = R(z)\left(\frac{z}{2} - \sum_{i=0}^{[(k-1/2)]} \mu_{2i+1} z^{-(2i+1)}\right) + \sum_{v=1}^{k-1} \mu'_v z^{-v} + \frac{\beta_k}{2^{k-1}}(z+1)^{k-1}$$
$$\times \left(\sum_{v=1}^{k-1} \hat{\mu}_v z^{-v} - 1 - 2z^{-1} - ... - 2z^{-(k-1)}\right).$$

Hence,

$$S_0(1) = R(1)\left(\frac{1}{2} - \sum_{i=0}^{[(k-1/2)]} \mu_{2i+1}\right) + \sum_{v=1}^{k-1}\mu'_v - \beta_k\left(1 + 2(k-1) - \sum_{v=1}^{k-1}\hat{\mu}_v\right), \quad (9)$$

where $\mu'_1 > 0$ and $\hat{\mu}_1 + \hat{\mu}_2 + ... + \hat{\mu}_{k-1} \le 2(k-2)$.

Consider the polynomial of the form:

$$v_1(\lambda) \equiv \sum_{i=0}^{k-1}\beta_i\lambda^i.$$

Then

$$S_0(z) = \left(\frac{1}{2}(z-1)\right)^{k-1} v_1(\tau).$$

$\beta_{k-1} \ne 0$, because method (3) has the maximal degree. But

$$\beta_{k-1} = \lim_{z\to\infty} z^{-(k-1)}v_1(z) = \lim_{\tau\to\infty}\left(\frac{\tau-1}{\tau+1}\right)^{k-1}\left(\frac{1}{2}(\tau-1)\right)^{-(k-1)} S(\tau) = S(1).$$

From expression (9) we have $S_0(1) > 0$. Consequently, $\beta_{k-1} > 0$. Thus expression (9) can be rewritten in the form:

$$\beta_{k-1} + \beta_k = R(1)\left(\frac{1}{2} - \sum_{i=0}^{[(k-1/2)]} \mu_{2i+1}\right) - \beta_k\left(2(k-1) - \sum_{v=0}^{k-1}\hat{\mu}_v\right). \quad (10)$$

Taking into account $\beta_k < 0$ in (10) we obtain that

$$\beta_{k-1} + \beta_k > 0.$$

Hence $\left|\beta_{k-1}\right| > |\beta_k|$.

Thus the theorem is proved for $m = 1$.

We suppose now that the result of the theorem is valid for $v \le m-1$ and let us prove it for $v = m$. In this case if we take into account that method (3) has the maximum degree, then in relation (7) $C_1 = C_2 = ... = C_{2m} = 0$, which implies:

$$d\beta_{k-m+1} + d_1^{(2)}\beta_{k-m+2} + ... + d_1^{(m)}\beta_k = -\sum_{i=0}^{[(k-m-1)/2]} a_{2i}\mu_{2+1},$$
$$d\beta_{k-m+1} + d_2^{(2)}\beta_{k-m+2} + ... + d_2^{(m)}\beta_k = -\sum_{i=0}^{[(k-m-1)/2]} a_{2i+1}\mu_{2+3},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d\beta_{k-m+1} + d_{2m-1}^{(2)}\beta_{k-m+2} + ... + d_{2m-1}^{(m)}\beta_k = -\sum_{i=0}^{[(k-m-1)/2]} a_{2i}\mu_{2+4m-3},$$

$$d\beta_{k-m+1} + d_{2m}^{(2)}\beta_{k-m+2} + ... + d_{2m}^{(m)}\beta_k = -\sum_{i=0}^{[(k-m-1)/2]} a_{2i+1}\mu_{2i+4m-1}. \quad (11)$$

In this case $S(z)$ can be represented in the form:

$$S(z) = S_0(z) + S_1(z) + ... + S_m(z),$$

where

$$S_j(z) = \beta_{k-m+j}(z+1)^{k-m}(z+1)^j/2^{k-m}(z-1)^j \quad (j = 1, 2, ..., m).$$

Taking into account $S(z)$ in (7) and choosing $S_0(z), S_1(z), ..., S_m(z)$ according to the mentioned sequence as a principal part in the function

$$R(z)\left(\frac{z}{2} - \sum_{i=3}^{\infty}\mu_{2i+1}z^{-(2i+1)}\right),$$

and then assuming that $z = 1$ we have:

$$S_0(1) = R(1)\left(\frac{1}{2} - \sum_{i=0}^{[k/2]}\mu_{2i+1}\right) - d_1\beta_{k-m+1} - ... - d_m\beta_k + \sum_{v=1}^{k}\left(\mu_v^{(1)} + \mu_v^{(2)}\right), \quad (12)$$

where $\mu_v^{(1)} > 0$ and $\mu_v^{(2)} > 0$ since they are some finite sum of quantities $a_s\mu_s > 0$.

It is clear that $d_1\beta_{k-m+1} + ... + d_m\beta_k$ will coincide with one of equations in the left-hand side of system (11) since method (3) is stable and has the maximum degree. Consequently, $d_1\beta_{k-m+1} + ... + d_m\beta_k < 0$.

If we take into account the last fact in (12), we obtain that

$$\beta_{k-m} = S_0(1) > 0.$$

Eliminating $\beta_{k-m+2}, ..., \beta_k$ in system (11) and taking into account in (12) we come to the case $m = 1$ according to which

$$\beta_{k-m} + \beta_{k-m+1} > 0.$$

If we take into account that $\beta_{k-m+1} < 0$ then we have

$$\left|\beta_{k-m}\right| > \left|\beta_{k-m+1}\right|.$$

Now we cite some concrete examples as an illustration

$$y_{n+1} = y_n + h(5f_n + 8f_{n+1} - f_{n+2})/12, \quad p = 3,$$

$$y_{n+1} = y_n + h(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3})/24, \quad p = 4,$$

$$y_{n+1} = y_n + h(251f_n + 646f_{n+1} - 264f_{n+2} + 106f_{n+3} - f_{n+4})/720, p = 5.$$

References

[1]. Krylov A.N. *Lectures on approximate calculations.* M.: Gostekh-izdat, 1954, 400 p. (Russian)

[2]. Dahlquist G. *Convergence and stability in the numerical integration of ordinary differential equations.* Math.Scand., 1956, No4, pp.33-53.

[3]. Shtetter Kh. *Analysis of discretization technique for ordinary differential equations.* M.: "Mir", 1978, 461 p. (Russian)

[4]. Shura-Bura M.R. *Error estimates of numerical integration of ordinary differential equations.* Prikl. Matem.i Mekh. 1952, No5, pp.575-588. (Russian)

[5]. Mukhin I.S. *On accumulation of errors at numerial integration of differential equations.* Prikl. Matem. i Mekh., 1952, issue 6, pp.752-756. (Russian)

[6]. Subbotin M.F. *Course of celestial mechanics.* M.: ONTI, 1937, v.2, 404 p. (Russian)

[7]. Dzyadik V.K. *Approximate method for solutions of ordinary differential equations by algebraic polynomials.* Izv. SSSR, ser. mat. 1974, No4, pp.937-967.

[8]. Tokmalayeva S.S. *Ordinary formulae for numerical integration of first order ordinary differential equations.* Vychislitel'naya Matematika, 1959, No5, pp.3-57.

[9]. Ibragimov V.R. *Relation between the order and degree of stable diference.* Running ahead formula. Priblizh. methody oper. uravneniy, Baku, 1984, pp.55-63. (Russian)

[10]. Ibragimov V.R. *Convergence of method of forecasting-correction.* Godishnik na vyshite uchebni zavedeniya. Priblezhno matematika Sophia NRB, 1984, No4, pp.187-197. (Russian)

[11]. Ibragimov V.R. *On one relation between the order and degree for stable running ahead formula.* Vhichis. Matem. I Matem. Fizika, 1990, No7, pp.1045-1056. (Russian)

[12]. Ibragimov V.R. *On one construction technique of A-stable methods.* Priblizh. metody resh. diff. I integr. Uravnenij. Baku, 1983, pp.55-62. (Russian)

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