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# ON EXTREMUM CONDITIONS IN PROBLEMS OF MATHEMATICAL PROGRAMMING

**Abstract**

*In the paper using the distance function in class of $\varphi-(\alpha,\beta,\nu,\delta)$ -Lipschitzian functions at the point the function of exact fine is constructed and the necessary conditions of high order are obtained for extremum problems by involving the limitations.*

In Clark paper (see [1]) for extremum problems involving the limitations for the first time the function of exact fine is constructed by using the distance function in classes of Lipschitzian functions. In [2] Lipschitzian functions at the point were determined and applied to the investigation of extremum problems with the limitation $(\alpha,\beta,\nu,\delta)$ -Lipschitzian function at the point. In [3] the $\varphi-(\alpha,\beta,\nu,\delta)$ -Lipschitzian functions at the point are determined and the extremum problems with limitations are considered. In the given paper by using the distance function a number of theorems on exact fine, and also the necessary conditions of high order involving the limitations are found.

1. About classes of $\varphi-(\alpha,\beta,\nu,\delta)$ -Lipschitzian functions. Let $X$ be a Banach space, $C\subset X$, $f:X\to R$, $\varphi:X\to R$, $\alpha>0$, $\nu>0$, $\beta\geq\alpha\nu$, $\delta>0$ and $B=\{x\in X:\|x\|\leq 1\}$.

The function $f$ is called $\varphi-(\alpha,\beta,\nu,\delta)$ -Lipschitzian with the constant $K$ at the point $x_0$, if $f$ satisfies the condition

$$|f(x_0+x+y)-f(x_0+x)-\varphi(x+y)+\varphi(x)|\leq K\|y\|^{\nu}\left(\|x\|^{\beta-\alpha\nu}+\|y\|^{\frac{\beta-\alpha\nu}{\alpha}}\right)$$

for $x,y\in\delta B$. If $\varphi=0$, then $f$ we'll call $(\alpha,\beta,\nu,\delta)$ Lipschitzian (see [2, 4]) with the constant $K$ at the point $x_0$.

Note that if $f$ satisfies $\varphi-(\alpha,\beta,\nu,\delta)$ -Lipschitzian condition at the point $x_0$ then $f(x)-\varphi(x-x_0)$ satisfies $(\alpha,\beta,\nu,\delta)$ -Lipschitzian condition at the point $x_0$. It is clear that if the functions $f_i, i=\overline{1,n}$, at the point $x_0$ are $\varphi_i-(\alpha,\beta,\nu,\delta)$ -Lipschitzian functions, then $\sum\limits_{i=1}^{n}f_i$ $\sum\limits_{i=1}^{n}\varphi_i-(\alpha,\beta,\nu,\delta)$ -Lipschitzian function at the point $x_0$.

If the derivative $f''(x_0)$ in terms of Freshe exists then by the theorem on Taylor formula the function $\omega(x)=0\left(\|x\|^2\right)$ holds, where $\frac{0(\lambda)}{\lambda}\to 0$ at $\lambda\downarrow 0$ such that

$$\left|f(x_0+x+y)-f(x_0+x)-\left(f'(x_0)(x+y)+\omega(x+y)\right)+f'(x_0)x+\omega(x)\right|=$$

$$=\left|\frac{1}{2}f''(x_0)(x+y,x+y)-\frac{1}{2}f''(x_0)(x,x)\right|\leq\left\|f''(x_0)\right\|\cdot\|y\|\left(\|x\|+\|y\|\right).$$

Therefore if we suppose $\varphi(x)=f'(x_0)x+\omega(x)$, then $f$ satisfies $\varphi-(1,2,1,\delta)$, $\delta>0$, the Lipschitzian condition at the point $x_0$.

If the derivative $f^{(3)}(x_0)$ in terms of Freshe exists then the function $\omega(x)=0\left(\|x\|^3\right)$ will be found such that $f$ satisfies $\varphi-(1,3,1,\delta)$, $\delta>0$, Lipschitzian condition at the point $x_0$, where $\varphi(x)=f'(x_0)x+\frac{1}{2}f''(x_0)(x,x)+\omega(x)$, $\frac{0(\lambda)}{\lambda}\to 0$ at $\lambda\downarrow 0$.

We'll denote the set of all continuous bilinear symmetric functionals from $X\times X$ to $R$ by $\overline{B}\left(X^2,R\right)$. The set $M\subset\overline{B}\left(X^2,R\right)$ is bounded if there exists a number $L>0$ such that $\left|x_2^*(x,y)\right|\le L\left\|x\right\|\cdot\left\|y\right\|$ at $x_2^*\in M$.

Let the function $f:X\to R$ be representable in the form

$$f\left(x_0+x\right)=f\left(x_0\right)+\sup_{x_1^*\in M_1}x_1^*\left(x\right)+\sup_{x_2^*\in M_2}x_2^*\left(x,x\right)+0\left(\left\|x\right\|^2\right),$$

where $M_1\subset X^*$ and $M_2\subset\overline{B}\left(X^2,R\right)$ are bounded sets, $\frac{0(\lambda)}{\lambda}\to 0$ at $\lambda\downarrow 0$. Assume $\varphi\left(x\right)=\sup\limits_{x_1^*\in M_1}x_1^*\left(x\right)+0\left(\left\|x\right\|^2\right)$. Then it easily can be checked that $f$ satisfies $\varphi-\left(1,2,1,\delta\right)$ -Lipschitzian condition at the point $x_0$, where $\delta>0$.

**Lemma 1.** *Let $Q\subset X$ be an open set, and Gato derivative of the function $f$ satisfies Holder condition of order $\alpha$, $0<\alpha\le 1$, on the set $Q$ with the constant $L\ge 0$. Let $x_0\in Q$ be some point, then there exists $\delta>0$ such that $f$ satisfies $f'\left(x_0\right)x-\left(1,1+\alpha,1,\delta\right)$ Lipschitzian condition at the point $x_0$ with the constant $L$.*

**Proof.** Since $Q$ is open and $x_0\in Q$ then there exists such $\delta>0$ that $x_0+2\delta B\subset Q$. Then by the the mean value theorem (see [5], p.38) we have

$$f\left(x\right)-f\left(y\right)-f'\left(y\right)\left(x-y\right)=\int_0^1\left(f'\left(y+t\left(x-y\right)\right)-f'\left(y\right)\right)\left(x-y\right)dt$$

at $x,y\in x_0+2\delta B$. Therefore

$$\left|f\left(x\right)-f\left(y\right)-f'\left(y\right)\left(x-y\right)\right|\le\int_0^1\left\|f'\left(y+t\left(x-y\right)\right)-f'\left(y\right)\right\|\cdot\left\|x-y\right\|dt\le$$

$$\le\int_0^1 L\cdot\left\|x-y\right\|^{1+\alpha}\cdot t^{\alpha}dt=\frac{L}{1+\alpha}\left\|x-y\right\|^{1+\alpha}$$

for all $x,y\in x_0+2\delta B$. Then it is clear that

$$\left|f\left(x_0+x+y\right)-f\left(x_0+x\right)-f'\left(x_0+x\right)y\right|\le L\frac{\left\|y\right\|^{1+\alpha}}{1+\alpha}$$

at $x,y\in\delta B$. Hence we have

$$\left|f\left(x_0+x+y\right)-f\left(x_0+x\right)-f'\left(x_0\right)\left(x+y\right)+f'\left(x_0\right)x\right|-$$

$$-\left|\left(f'\left(x_0+x\right)-f'\left(x_0\right)\right)y\right|\le L\frac{\left\|y\right\|^{1+\alpha}}{1+\alpha}$$

Since

$$\left|\left(f'\left(x_0+x\right)-f'\left(x_0\right)\right)y\right|\le\left\|\left(f'\left(x_0+x\right)-f'\left(x_0\right)\right)\right\|\cdot\left\|y\right\|\le L\left\|x\right\|^{\alpha}\cdot\left\|y\right\|$$

then

$$\left|f\left(x_0+x+y\right)-f\left(x_0+x\right)-f'\left(x_0\right)\left(x+y\right)+f'\left(x_0\right)x\right|\le L\left\|y\right\|\left(\left\|x\right\|^{\alpha}+\left\|y\right\|^{\alpha}\right)$$

at $x, y \in \delta B$, i.e., $f$ satisfies $f'(x_0)x - (1, 1+\alpha, 1, \delta)$ Lipschitzian condition at the point $x_0$. The lemma is proved.

**2. Higher order necessary conditions.** Denoting

$$Q_\alpha = \left\{ 0(\cdot) : 0(\lambda) \in R_+ \text{ at } \lambda \in R_+ \text{ and } \frac{0(\lambda)}{\lambda^\alpha} \to 0 at \lambda \downarrow 0 \right\}$$

$d(x) = \inf\{\|x - y\| : y \in C\}$ and let $x_0 \in C$. Assume that

$$K_\alpha(x_0; C) = \left\{ x \in X : \overline{\lim_{\lambda \downarrow 0}} \frac{d(x_0 + \lambda x)}{\lambda^\alpha} = 0 \right\}, \ \Gamma_\alpha(x_0; C) =$$

$$= \left\{ x \in X : \underline{\lim_{\lambda \downarrow 0}} \frac{d(x_0 + \lambda x)}{\lambda^\alpha} = 0 \right\}$$

Note that (see [2]) $x \in K_\alpha(x_0; C)$ $(\alpha > 0)$ if and only if $\lambda_x > 0$ and $0(x, \lambda) : [0, \lambda_x] \to X$ can be found such that $x_0 + \lambda x + 0(x, \lambda) \in C$ for any $\lambda \in [0, \lambda_x]$ and $\frac{0(x,\lambda)}{\lambda^\alpha} \to 0$ at $\lambda \downarrow 0$. Also $x \in \Gamma_\alpha(x_0; C)$ $(\alpha > 0)$ if and only if there exist the sequences $\lambda_i \downarrow 0, \{\nu_i\} \subset X$ such that $\frac{1}{\lambda_i^{\alpha-1}} \|\nu_i - x\| \to 0$ and $x_0 + \lambda_i \nu_i \in C$.

Assume that

$K_{\alpha,\beta}(x_0; C, \varphi) = \{x \in X : \exists \lambda_x > 0, \ \exists 0_1(x, \lambda) : [0, \lambda_x] \to X, \exists 0_2(x, \lambda) : [0, \lambda_x] \to R_+$

where $\frac{0_1(x,\lambda)}{\lambda^\alpha} \to 0, \frac{0_2(x,\lambda)}{\lambda^\beta} \to 0$ at $\lambda \downarrow 0$, where $x_0 + \lambda x + 0_1(x, \lambda) \in C$ and $\varphi(\lambda x + 0_1(x, \lambda)) \le 0_2(x, \lambda)$ at $0 \le \lambda \le \lambda_x\}$,

$\Gamma_{\alpha,\beta}(x_0; C, \varphi) = \{x \in X : \exists 0(x, \lambda) \in R_+$ where $\frac{0(x,\lambda)}{\lambda^\beta} \to 0$, at $\lambda \downarrow 0$ and $\exists \lambda_i \downarrow 0, \ \exists \{\nu_i\} \subset X$
where $\frac{1}{\lambda_i^{\alpha-1}} \|\nu_i - x\| \to 0$ that $x_0 + \lambda_i \nu_i \in C, \ \varphi(\lambda_i \nu_i) \le 0(\lambda_i, x)\}$.

**Theorem 1.** *Let $f : X \to R$, $\varphi : X \to R$, $\alpha > 0$, $\nu > 0$, $\beta \ge \alpha\nu$ and $x_0$ be the minimum point of the function $f$ on the set $C$. And also assume that there exist finite positive homogeneous functions $\varphi_1 : X \to R$ of degree $\beta - \alpha\nu$, $0(\cdot) \in \Omega_1$ numbers $\delta > 0$ and $K$ such that*

$$|f(x_0 + x + y) - f(x_0 + x) - \varphi(x + y) + \varphi(x)| \le$$

$$\le K\|y\|^\nu \left( \varphi_1(x) + \|y\|^{\frac{\beta - \alpha\nu}{\alpha}} \right) + 0\left( \|x\|^\beta \right)$$

*for* $x \in K_{\alpha,\beta}(x_0; C, \varphi)\,(x \in \Gamma_{\alpha,\beta}(x_0; C, \varphi))$, $\|x\| \le \delta$, $y \in X$, $\|y\| \le \|x\|$, $x_0 + x + y \in C$.

*Then*

$$f_\varphi^{\{\beta\}-}(x_0; x) = \underline{\lim_{\lambda \downarrow 0}} \frac{1}{\lambda^\beta} (f(x_0 + \lambda x) - \varphi(\lambda x) - f(x_0)) \ge 0 \text{ at } x \in K_{\alpha,\beta}(x_0; C, \varphi),$$

$$\left( f_\varphi^{\{\beta\}+}(x_0; x) = \overline{\lim_{\lambda \downarrow 0}} \frac{1}{\lambda^\beta} (f(x_0 + \lambda x) - \varphi(\lambda x) - f(x_0)) \ge 0 \text{ at } x \in \Gamma_{\alpha,\beta}(x_0; C, \varphi) \right).$$

**Proof.** If $x \in K_{\alpha,\beta}(x_0; C, \varphi)$ then by the definition $\lambda_x > 0, \ 0_1(x, \lambda) : [0, \lambda_x] \to X, \ 0_2(x, \lambda) : [0, \lambda_x] \to R_+$ will be found, where $\frac{0_1(x,\lambda)}{\lambda^\alpha} \to 0, \ \frac{0_2(x,\lambda)}{\lambda^\beta} \to 0$ at $\lambda \downarrow 0$

that $x_0 + \lambda x + 0_1(x,\lambda) \in C$ and $\varphi(\lambda x + 0_1(x,\lambda)) \le 0_2(x,\lambda)$ at $0 \le \lambda \le \lambda_x$. Therefore

$$f_{\varphi}^{\{\beta\}-}(x_0;x) =$$

$$= \lim_{\lambda\downarrow 0}\left\{\frac{f(x_0+\lambda x+0_1(x,\lambda)) - f(x_0) + f(x_0+\lambda x) - f(x_0+\lambda x+0_1(x,\lambda))}{\lambda^\beta} + \right.$$

$$\left. + \frac{-\varphi(\lambda x+0_1(x,\lambda)) + 0_2(x,\lambda) + \varphi(\lambda x+0_1(x,\lambda)) - 0_2(x,\lambda) - \varphi(\lambda x)}{\lambda^\beta}\right\} \ge$$

$$\ge \underline{\lim}_{\lambda\downarrow 0}\frac{f(x_0+\lambda x+0_1(x,\lambda)) - f(x_0) - \varphi(\lambda x+0_1(x,\lambda)) + 0_2(x,\lambda)}{\lambda^\beta} +$$

$$+ \underline{\lim}_{\lambda\downarrow 0}\frac{-(f(x_0+\lambda x+0_1(x,\lambda)) - f(x_0+\lambda x) - \varphi(\lambda x+0_1(x,\lambda)) + \varphi(\lambda x)) - 0_2(x,\lambda)}{\lambda^\beta} \ge$$

$$\ge \underline{\lim}_{\lambda\downarrow 0}\frac{1}{\lambda^\beta}\left(f(x_0+\lambda x+0_1(x,\lambda)) - f(x_0) - \varphi(\lambda x+0_1(x,\lambda)) + 0_2(x,\lambda)\right) -$$

$$-\overline{\lim_{\lambda\downarrow 0}}\frac{K}{\lambda^\beta}\left[\|0_1(x,\lambda)\|^\nu\left(\lambda^{\beta-\alpha\nu}\varphi_1(x) + \|0_1(x,\lambda)\|^{\frac{\beta-\alpha\nu}{\alpha}}\right) + 0_2(x,\lambda) + 0\left(\|\lambda x\|^\beta\right)\right] \ge 0.$$

The second case is analogously proved. The theorem is proved.

**Remark 1.** From the proof of theorem 1 we have that if there exist $\lambda_x > 0$, $0(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{0(x,\lambda)}{\lambda^\beta} \to 0$ at $\lambda \downarrow 0$, that

$$|f(x_0+\lambda x+0_1(\lambda)) - f(x_0+\lambda x) - \varphi(\lambda x+0_1(\lambda)) + \varphi(\lambda x)| \le 0(x,\lambda),$$

for any $x \in K_{\alpha,\beta}(x_0;C,\varphi)$,
$0_1(\cdot) \in \overline{\Omega}_\alpha = \left\{0(\cdot) : 0(\lambda) \in X \text{ at } \lambda \in R_+ \text{ and } \frac{0(\lambda)}{\lambda^\alpha} \to 0 \text{ at } \lambda \downarrow 0\right\}$ and $0 \le \lambda \le \lambda_x$, then theorem 1 is also true.

**Corollary 1.** *If $x_0$ is the minimum point of the function $f$ on the set $C$, and if there exist $\alpha > 0$, $\beta > 0$, $\nu > 0$, $\delta > 0$, $K > 0$, where $\beta \ge \alpha\nu$, the functions $0(\cdot) \in \Omega_1$, and $\varphi : X \to R$ such that*

$$|f(x_0+x+y) - f(x_0+x) - \varphi(x+y) + \varphi(x)| \le$$

$$\le K\|y\|^\nu\left(\|x\|^{\beta-\alpha\nu} + \|y\|^{\frac{\beta-\alpha\nu}{\alpha}}\right) + 0\left(\|x\|^\beta\right)$$

*for $x \in K_{\alpha,\beta}(x_0;C,\varphi)$ $(x \in \Gamma_{\alpha,\beta}(x_0;C,\varphi))$, $\|x\| \le \delta$, $y \in X$, $\|y\| \le \|x\|$, $x_0 + x + y \in C$ then*

$$f_{\varphi}^{\{\beta\}-}(x_0;x) = \underline{\lim}_{\lambda\downarrow 0}\frac{f(x_0+\lambda x) - \varphi(\lambda x) - f(x_0)}{\lambda^\beta} \ge 0 \text{ at } K_{\alpha,\beta}(x_0;C,\varphi),$$

$$\left(f_{\varphi}^{\{\beta\}+}(x_0;x) = \overline{\lim_{\lambda\downarrow 0}}\frac{f(x_0+\lambda x) - \varphi(\lambda x) - f(x_0)}{\lambda^\beta} \ge 0 \text{ at } \Gamma_{\alpha,\beta}(x_0;C,\varphi)\right).$$

Let $X = R^n$, $g_i : R^n \to R$, $i = \overline{1,m}$. Consider the minimization of the function $f$ on the set $C = \left\{x \in R^n : g_i(x) \le 0,\ i = \overline{1,k},\ \ g_i(x) = 0,\ i = \overline{k+1,m}\right\}$.

Assume that $I(x_0) = \left\{i \in \overline{1,k},\ g_i(x_0) = 0\right\}$ and let $f, g_1, ..., g_m$ be continuously differentiable in some neighbourhood of the point $x_0 \in R^n$, and the gradients

$g_i'(x_0)$, $i=\overline{k+1,m}$, are be linearly independent. Then it is known that (see [6], p.191)

$$K_1(x_0;C)=\left\{h\in R^n:\left\langle g_i'(x_0),h\right\rangle<0,\ i\in I(x_0),\ \left\langle g_i'(x_0),h\right\rangle=0,\ i=\overline{k+1,m}\right\},$$

and conjugate cone is represented in the following form:

$$K_1^*(x_0;C)=\left\{x^*:x^*=-\sum_{i=I(x_0)}\lambda_i g_i'(x_0)-\sum_{i=k+1}^{m}\lambda_i g_i'(x_0),\quad \lambda_i\geq 0,\ i\in I(x_0)\right\}.$$

By the condition there exist $\delta>0$ and $K>0$ such that

$$|f(x_0+x+y)-f(x_0+x)|\leq K\|y\|\quad\text{at}\quad x,y\in\delta B.$$

Hence it follows that $f$ satisfies the $(\alpha,\alpha,1,\delta)$ -Lipschitzian condition at the point $x_0$. Therefore if $x_0$ is the minimum point of the function $f$ on the set $C$ then by corollary 1 we have that $\langle f'(x_0),x\rangle\geq 0$ at $x\in K_1(x_0;C)$. Then we'll obtain that there exist the numbers $\lambda_1\geq 0,...,\lambda_k\geq 0$, $\lambda_{k+1},...,\lambda_m$, where $\lambda_i g_i(x_0)=0$ at $i=\overline{1,k}$, such that $f'(x_0)+\sum\limits_{i=1}^{m}\lambda_i g_i'(x_0)=0$..

Assume $L(x,\lambda)=f(x)+\sum\limits_{i=1}^{m}\lambda_i g_i(x)$, where $\lambda=(\lambda_1,...,\lambda_m)$ and let the functions $f,\ g_i,...,g_m$ be twice differentiable at the point $x_0$. It is easily verified that there exist $\delta>0$ and $\omega(x)=0\left(\|x\|^2\right)$ such that the function $q(x)=L(x,\lambda)$ satisfies $\omega(x)-$ $(1,2,1,\delta)$ Lipschitzian condition at the point $x_0$. Denote $\varphi(x)=-\sum\limits_{i=1}^{m}\lambda_i g_i(x_0+x)+$ $\omega(x)$. Then it is clear that the function $f$ satisfies $\varphi-(1,2,1,\delta)$ -Lipschitzian condition at the point $x_0$. Applying corollary 1 we have that

$$\overline{\lim_{\lambda\downarrow 0}}\frac{1}{\lambda^2}\left(f(x_0+\lambda x)+\sum_{i=1}^{m}\lambda_i g_i(x_0+\lambda x)-\omega(\lambda x)-f(x_0)-\sum_{i=1}^{m}\lambda_i g_i(x_0)\right)\geq 0$$

at $x\in K_{1,2}(x_0;C,\varphi)$. Hence by Taylor theorem it follows the validity of the following

**Proposition 1.** Let functions $f,g_1,...,g_m$ be twice differentiable at the point $x_0\in R^n$ and continuously differentiable in some neighbourhood of the point $x_0$, and the gradients $g_i'(x_0)$, $i=\overline{k+1,m}$ be linearly independent and $K_1(x_0;C)$ be not empty. Then if $x_0$ is a minimum point of the function $f$ on the set $C$, then

$$\langle L_{xx}(x_0;\lambda)x,x\rangle\geq 0\ \text{at all x}\in K_{1,2}(x_0;C,\varphi),$$

where $\varphi(x)=-\sum\limits_{i=1}^{m}\lambda_i g_i(x_0+x)$ and at any $\lambda_1\geq 0,...,\lambda_k\geq 0$, $\lambda_{k+1},...,\lambda_m$, satisfying the conditions $f'(x_0)+\sum\limits_{i=1}^{m}\lambda_i g_i'(x_0)=0$, $\lambda_i g_i(x_0)=0$ at $i=\overline{1,k}$..

Note that if under the condition of proposition 1 $x\in K_{1,2}(x_0;C,\varphi)$ then we can easily check that $\langle\lambda_i g_i'(x_0),x\rangle=0$ for $i=\overline{1,m}$.

**Remark 2.** Note that from proposition 1 it follows the necessaty conditions of the second order for classical problem on conditional extremum and for problem of mathematical programming (see theorems 1.1.7 and theorems 4.2.7 [7]).

**3. Construction of function of exact fine.** Assume that $C_\delta = \{x \in X : d(x) \leq \delta\}$. Denote some projection of the point $x$ on the set $C$ by $\mathrm{pr}_C x$.

**Theorem 2.** *Let $X$ be Hilbert space, $x_0$ be the minimum of the function $f$ on a closed set $C$, where either $C$ is convex or $X$ is finite dimensional; for every $z \in \partial C$ there exists the function $\varphi_z : X \to R$ such that $f - \varphi_z$ satisfies the $(\alpha, \beta, \nu, \delta)$ - Lipschitzian condition at the point $z$ with the constant $K$ and for any $y \in C_\delta \backslash C$ the inequality $\varphi_z(\mathrm{pr}_C y) - \varphi_z(y) \leq 0$ is fulfilled, where $z = \mathrm{pr}_C y$. Then for every $\lambda \geq K$ the function $g_\lambda(x) = f(x) + \lambda\left(d^{\frac{\beta}{\alpha}}(x) + d^{\beta-\alpha\nu+\nu}(x)\right)$ attains minimum at $C_\delta$ at the point $x_0$ and if $\lambda > K$ then any point minimizing $g_\lambda(x)$ on the set $C_\delta$ belongs to $C$.*

**Proof.** Assume the vice-verse. Let there exist the point $y \in C_\delta$ such that $g_\lambda(y) < f(x_0)$, where $\lambda \geq k$. Denote some projection of the point $y$ by $c$ on the set $C$. Assume $\psi = f - \varphi_c$. Since $\psi(y) = \psi(c + y - c)$, $\psi(c) = \psi(c + y - c + c - y)$ and $\psi$ at the point $c$ satisfies the $(\alpha, \beta, v, \delta)$-Lipschitzian condition, then we obtain:

$$f(c) \leq f(y) + \varphi_c(c) - \varphi_c(y) + K\|c-y\|^v\left(\|c-y\|^{\beta-\alpha v} + \|c-y\|^{\frac{\beta-\alpha v}{\alpha}}\right) \leq$$

$$\leq f(y) + \lambda\left(\|c-y\|^{\beta+v-\alpha v} + \|c-y\|^{\frac{\beta}{\alpha}}\right) =$$

$$= f(y) + \lambda\left(d^{\beta+v-\alpha v}(y) + d^{\frac{\beta}{\alpha}}(y)\right) < f(x_0) \ .$$

This contradicts the assumption that $f$ reaches the minimum at the point $x_0$ on the set $C$. If $\lambda > K$ and $y \in C_\delta$ also minimizes the functions $g_\lambda(x)$ on the set $C_\delta$, then from the first part of the theorem we obtain

$$f(y) + \lambda\left(d^{\frac{\beta}{\alpha}}(y) + d^{\beta+v-\alpha v}(y)\right) =$$

$$= f(x_0) \leq f(y) + \frac{\lambda + K}{2}\left(d^{\frac{\beta}{\alpha}}(y) + d^{\beta+v-\alpha v}(y)\right) \ .$$

Hence we'll obtain that $d(y) = 0$, i.e., $y \in C$. The theorem is proved.

**Corollary 2.** *If the condition of theorem 2 and $\psi \equiv 0$ is satisfied then*

$$f_\psi^{\{s\}-}(x_0; x) \geq 0 \quad at \quad x \in \mathbf{K}_{\frac{s\alpha}{\beta}} \cap \mathbf{K}_{\frac{s}{\beta+v-\alpha v}},$$

$$f_\psi^{\{s\}+}(x_0; x) \geq 0 \quad at \quad x \in \mathbf{\Gamma}_{\frac{s\alpha}{\beta}} \cap \mathbf{K}_{\frac{s}{\beta+v-\alpha v}} \quad \left(x \in \mathbf{K}_{\frac{s\alpha}{\beta}} \cap \mathbf{\Gamma}_{\frac{s}{\beta+v-\alpha v}}\right) \ .$$

**Proof.** It is evident that

$$0 \leq \lim_{t\downarrow 0}\frac{g_\lambda(x_0+tx) - g_\lambda(x_0)}{t^s} \leq \lim_{t\downarrow 0}\frac{f(x_0+tx) - f(x_0)}{t^s} +$$

$$+\lambda\overline{\lim_{t\downarrow 0}}\frac{d^{\frac{\beta}{\alpha}}(x_0+tx)}{t^s} + \lambda\overline{\lim_{t\downarrow 0}}\frac{d^{\beta+v-\alpha v}(x_0+tx)}{t^s} =$$

$$= f_\psi^{\{s\}-}(x_0; x) + \lambda\overline{\lim_{t\downarrow 0}}\left(\frac{d(x_0+tx)}{t^{\frac{s\alpha}{\beta}}}\right)^{\frac{\beta}{\alpha}} + \lambda\overline{\lim_{t\downarrow 0}}\left(\frac{d(x_0+tx)}{t^{\frac{s}{\beta+v-\alpha v}}}\right)^{\beta+v-\alpha v} .$$

Hence it follows that $f_\psi^{\{s\}-}(x_0; x) \geq 0$ at $x \in \mathbf{K}_{\frac{s\alpha}{\beta}} \cap \mathbf{K}_{\frac{s}{\beta+v-\alpha v}}$. The second relation is analogously checked.

**Remark 3.** Let $C_\delta = C \cup B(x_0;\delta)$. Assume that $f - \varphi_z$ satisfies the $(\alpha,\beta,v,\delta)$-Lipschitzian condition at each point $z = \mathrm{pr}_C y$ at $y \in B(x_0;\delta)\backslash C$ then we can show that the assertion of theorem 2 is also true.

**Remark 4.** Note that the condition of theorem 1 and 2 at $\varphi(x) \equiv f(x_0 + x)$ and $\varphi_z \equiv f$ is trivially satisfied. The proof of these theorems at the noted conditions is also trivial.

**Corollary 3.** *Let $X$ be Hilbert space, $x_0$ be minimum of the function $f$ on the closed set $C$, where $C$ is convex or $X$ finite-dimensional, and there exists the function $\varphi : X \to R$ such that $f - \varphi$ satisfies the $(\alpha,\beta,v,\delta)$-Lipschitzian condition at each point $z \in \partial C$ with the constant $K$, $\varphi\left(\mathrm{pr}_C y\right) - \varphi(y) \le 0$ at $y \in C_\delta$. Then for any $\lambda \ge K$ the function $g_\lambda(x) = f(x) + \lambda\left(d^{\frac{\beta}{\alpha}}(x) + d^{\beta-\alpha v+v}(x)\right)$ attains the minimum on $C_\delta$ at the point $x_0$ and if $\lambda > K$, then any point which minimizing $g_\lambda(x)$ on the set $C_\delta$ belongs to $C$.*

**Corollary 4.** *Let $X$ be Hilbert space, where $x_0$ is the minimum of the function $f$ on the close set $C$, where either $C$ is convex $X$ is finite-dimensional, for every $z \in \partial C$ there exists the function $\varphi_z : X \to R$ such that $f$ satisfies the $\varphi_z - (\alpha,\beta,v,\delta)$ - Lipschitzian condition at the point $z$ with the constant $K$ and for any $y \in C_\delta \backslash C$ the inequality $\varphi_z\left(y - \mathrm{pr}_C y\right) \ge \varphi_z(0)$ is fulfilled, where $z = \mathrm{pr}_C y$. Then for any $\lambda \ge K$ the function $g_\lambda(x) = f(x) + \lambda\left(d^{\frac{\beta}{\alpha}}(x) + d^{\beta-\alpha v+v}(x)\right)$ attains minimum on $C_\delta$ at the point $x_0$ and if $\lambda > K$, then any point minimizing $g_\lambda(x)$ on the set $C_\delta$ belongs to $C$.*

Note that corollary 4 is the another formulation of theorem 2.

It is clear that if $f$ satisfies $\varphi_z - (\alpha,\beta,v,\delta)$ -Lipschitzian condition at the point $z$ with the constant $K$, then $f(x) - \varphi_z(x - z)$ satisfies $(\alpha,\beta,v,\delta)$ -Lipschitzian condition at the point $z$ with the constant $K$. Therefore the correctness of corollary 4 follows from theorem 2.

From lemma 1 and theorem 2 it follows the following

**Corollary 5.** *Let $X$ be Hilbert space, $x_0$ be the minimum of the function $f$ on closed convex set $C$, int $C \ne \varnothing$, $\delta_0 > \delta > 0$ and Gato derivative of the function $f$ satisfy Holder condition of order $\alpha\,(0 < \alpha \le 1)$ on the set int $C_{\delta_0}$ with the constant $K$, for any $y \in C_\delta \backslash C$ the inequality $f'(z)(z - y) \le 0$ is fulfilled where $z = \mathrm{pr}_C y$. Then for any $\lambda \ge K$ the function $g_\lambda(x) = f(x) + 2\lambda d^{1+\alpha}(x)$ attains minimum on $C_\delta$ at the point $x_0$, and if $\lambda > K$ then any point minimizing $g_\lambda(x) = f(x) + 2\lambda d^{1+\alpha}(x)$ on the set $C_\delta$ belongs to $C$.*

Note that if $f$ is a continuous function at each point $z \in \partial C$ and for any $y \in C_\delta \backslash C$ the inequality $\varphi_z\left(\mathrm{pr}_{\bar{C}} y\right) - \varphi_z(y) \le 0$ is fulfilled, where $z = \mathrm{pr}_{\bar{C}} y$ then the first part of theorem 2 is true without closure condition $C$.

Assume

$$f^{[2]}(x_0; x_1, x_2) = \overline{\lim_{\substack{y \to x_0 \\ \lambda_1 \downarrow 0,\ \lambda_2 \downarrow 0}}} \frac{1}{\lambda_1 \lambda_2}\left(f(y + \lambda_1 x_1 + \lambda_2 x_2) - f(y + \lambda_1 x_1) - \right.$$

$$\left. - f(y + \lambda_2 x_2) + f(y)\right),$$

$$\partial_2 f(x_0) = \left\{x^* \in \bar{B}\left(X^2, R\right) : f^{[2]}(x_0; x_1, x_2) \ge x^*(x_1, x_2),\, x_1, x_2 \in X\right\}.$$

We call the function $f$ 2-Lipschitzian with the constant $L$ in neighbourhood $x_0$ if $f$ for some $\varepsilon > 0$ satisfies the condition

$$|f(x + x_1 + x_2) - f(x + x_1) - f(x + x_2) + f(x)| \le L\,\|x_1\| \cdot \|x_2\|,$$

$$x_1, x_2 \in \varepsilon B, \quad x \in x_0 + \varepsilon B \ .$$

Corollary 6 follows from theorem 2, corollary 1.4.1. and proposition 1.1.7. [2].

**Corollary 6.** *If the condition of theorem 2 is fulfilled at* $\alpha = 1$, $\beta = 2$, $v > 0$, $C$ *is convex,* $\lambda \geq K$ *and* $f$ *2-Lipschitzian function in the neighbourhood* $x_0$ *then*

$$\max\left\{x^*(x, x) : x^* \in \partial_2 f(x_0) + 2\lambda \partial_2 d^2(x_0)\right\} \geq 0 \ .$$

**Proposition 2.** *Let* $X$ *and* $Y$ *be Hilbert space,* $\Lambda : X \to Y$ *be linearly continuous surjective operator from* $X$ *to* $Y$, $x_1^*, ..., x_s^*$ *be elements of conjugate space* $X^*$ *and let* $C = \left\{x : \langle x_i^*, x\rangle \leq 0, \quad i = \overline{1, s}, \ \Lambda x = 0\right\}$ *for every* $z \in \partial C$ *there exists function* $\varphi_z : X \to R$ *such that* $f - \varphi_z$ *satisfies the* $(\alpha, \beta, v, \delta)$ *−Lipschitzian condition at the point* $z$ *with the constant* $K$ *and for any* $y \in C_\delta \backslash C$ *the inequality* $\varphi_z\left(\mathrm{pr}_C y\right) - \varphi_z(y) \leq 0$ *is fulfilled, where* $z = \mathrm{pr}_C y$. *Then if the set of solutions of the problem* $f(x) \to \min$, $x \in C$ *is non empty then there exists such* $\lambda_0 > 0$ *that at* $\lambda \geq \lambda_0$ *the set of solutions of the problem*

$$f(x) \to \min, \ x \in C$$

*and the problem*

$$f(x) + \lambda\left(\psi(x)^{\frac{\beta}{\alpha}} + \psi(x)^{\beta+v-av}\right) \to \min, \quad x \in C_\delta$$

*coincide, where* $\psi(x) = \sum\limits_{i=1}^{s} \langle x_i^*, x\rangle_+ + \|\Lambda x\|, \quad (g(x))_+ = \max\{0, g(x)\}$.

**Proof.** By Hoffmann lemma (see [8], p.279) there exists such constant $M$ independent of $x$ that $d(x) \leq M\left\{\sum\limits_{i=1}^{s} \langle x_i^*, x\rangle_+ + \|\Lambda x\|\right\}$. Besides it is clear that $C = \{x \in X : d(x) = 0\} = \{x \in X : \psi(x) = 0\}$. Therefore if the set of solutions of the problem $f(x) \to \min$, $x \in C$ is non empty, then by theorem 2 there exists such $l_0 > 0$, that at $\lambda \geq l_0$ the set of solutions of the problem $f(x) \to \min$, $x \in C$ and problem $f(x) + \lambda\left(d^{\beta+v-\alpha v}(y) + d^{\frac{\beta}{\alpha}}(y)\right)$ coincide. If we assume $\lambda_0 = l_0 \max\left\{M^{\beta+v-\alpha v}, M^{\frac{\beta}{\alpha}}\right\}$ then from here follows the correctness of assertion of proposition 2.

**Theorem 3.** *Let* $x_0$ *be minimum of the function* $f$ *on the set* $C$ *for each* $z \in C$, *there exists the function* $\varphi_z : X \to R$ *such that* $f - \varphi_z$ *satisfies the* $(\alpha, \beta, v, \delta_0)$ *Lipschitzian condition at the point* $z$ *with the constant* $K$ *and for any* $y \in C_\delta \backslash C$ *and* $\varepsilon > 0$ *there exists the point* $c \in C$ *such that* $\|c - y\| \leq d(y) + \varepsilon$ *and* $\varphi_c(c) - \varphi_c(y) \leq 0$, *where* $\delta_0 > \delta > 0$. *Then for any* $\lambda \geq K$ *the function* $g_\lambda(x) = f(x) + \lambda\left(d^{\frac{\beta}{\alpha}}(x) + d^{\beta-\alpha v+v}(x)\right)$ *attains minimum on* $C_\delta$ *at the point* $x_0$ *and if* $\lambda > K$ *and* $C$ *is closed, then any point minimizing* $g_\lambda(x)$ *on the set* $C_\delta$ *belongs to* $C$.

**Proof.** Assume the contradiction. Let there exist the point $y \in C_\delta$ and $\varepsilon > 0$ such that $g_\lambda(y) < f(x_0) - \lambda\varepsilon$, where $\lambda \geq K$. Let's take $c \in C$ satisfying the inequalities $\|y - c\| \leq \delta_0$, $\|y - c\|^{\frac{\beta}{\alpha}} + \|y - c\|^{\beta-\alpha v+v} \leq d^{\frac{\beta}{\alpha}}(y) + d^{\beta-\alpha v+v}(y) + \varepsilon$ and $\varphi_c(c) - \varphi_c(y) \leq 0$. Assume $\psi = f - \varphi_c$ Since $\psi(y) = \psi(c + y - c)$, $\psi(c + y - c + c - y)$ and $\psi$ at the point $x_0$ satisfies $(\alpha, \beta, v, \delta_0)$ −Lipschitzian condition, then we obtain

$$f(c) \leq f(y) + \varphi_c(c) - \varphi_c(y) + K\|c - y\|^v\left(\|c - y\|^{\beta-\alpha v} + \|c - y\|^{\frac{\beta-\alpha v}{\alpha}}\right) \leq$$

$$\leq f(y) + \lambda\left(\|c - y\|^{\beta+v-\alpha v} + \|c - y\|^{\frac{\beta}{\alpha}}\right) \leq$$

$$\leq f(y) + \lambda \left( d^{\beta+v-\alpha v}(y) + d^{\frac{\beta}{\alpha}}(y) \right) + \lambda\varepsilon < f(x_0) \ .$$

This contradicts the assumption that $f$ attains the minimum at the point $x_0$ on the set $C$. If $\lambda > K$ and $y \in C_\delta$ also minimize the function $g_\lambda(x)$ on the set $C_\delta$ then from the first part of theorem we obtain

$$f(y) + \lambda \left( d^{\beta+v-\alpha v}(y) + d^{\frac{\beta}{\alpha}}(y) \right) = f(x_0) \leq f(y) + \frac{\lambda + K}{2} \left( d^{\frac{\beta}{\alpha}}(y) + d^{\beta+v-\alpha v}(y) \right)$$

Hence it follows that $d(y) = 0$, i.e., $y \in C$. The theorem is proved.

Consider the minimization of the function $f$ on the set $\left\{ x \in C \colon g_i(x) \leq 0, i = \overline{1,m} \right\}$ where $g_i : X \to R$. Assume

$$F(y) = \max \left\{ r_0 (f(y) - f(x_0)) + \sum_{i=1}^{m} r_i g_i(y) : r_i \geq 0, \quad i = \overline{0,m}, \ \sum_{i=1}^{m} r_i = 1 \right\} .$$

**Theorem 4.** *Let $X$ be Hilbert Space and $x_0$ minimize the function $f$ on the set $\left\{ x \in C : g_i(x) \leq 0, \quad i = \overline{1,m} \right\}$, where $g_i : X \to R$, $C \subset X$ is closed at each point $x \in \partial C$, $f$ and $g_i$, $i = \overline{1,m}$ satisfy the $(\alpha, \beta, v, \delta)$ -Lipschitzian condition with the constant $K$. Besides either $C$ is convex or $X$ is finite-dimensional. Then for any $\lambda \geq K$ the function $g_\lambda(x) = F(x) + \lambda \left( d^{\beta+v-\alpha v}(x) + d^{\frac{\beta}{a}}(x) \right)$ attains minimum on $C_\delta$ at the point $x_0$ and if $\lambda > K$ then any point minimizing $g_\lambda(x)$ on the set $C_\delta$ belongs to $C$.*

**Proof.** Let $r_i \geq 0$, $\quad i = \overline{0,m}$ and $\sum\limits_{i=1}^{m} r_i = 1$. Then, if $\bar{x} \in \partial C$, then we have that

$$\Bigg| r_0 (f(\bar{x} + x + y) - f(x_0)) + \sum_{i=1}^{m} r_i g_i(\bar{x} + x + y) -$$
$$- r_0 (f(\bar{x} + x) - f(x_0)) - \sum_{i=1}^{m} r_i g_i(\bar{x} + x) \Bigg| \leq$$
$$\leq r_0 \left| f(\bar{x} + x + y) - f(\bar{x} + x) \right| + \sum_{i=1}^{m} r_i \left| g_i(\bar{x} + x + y) - g_i(\bar{x} + x) \right| \leq$$
$$\leq r_0 K \|y\|^v \left( \|x\|^{\beta-\alpha v} + \|y\|^{\frac{\beta-\alpha v}{\alpha}} \right) + \sum_{i=1}^{m} r_i K \|y\|^v \left( \|x\|^{\beta-\alpha v} + \|y\|^{\frac{\beta-\alpha v}{\alpha}} \right) =$$
$$= K \|y\|^v \left( \|x\|^{\beta-\alpha v} + \|y\|^{\frac{\beta-\alpha v}{\alpha}} \right)$$

at $x, y \in \delta B$. Therefore from lemma 1.2.2 [4] it follows that

$$\left| F(\bar{x} + x + y) - F(\bar{x} + x) \right| \leq K \|y\|^v \left( \|x\|^{\beta-\alpha v} + \|y\|^{\frac{\beta-\alpha v}{\alpha}} \right)$$

at $x, y \in \delta B$. Show that $F$ is non-negative on the set $C$. Assume the opposite. Let there exist $\bar{y} \in C$ such that $F(\bar{y}) < 0$. Then $f(\bar{y}) - f(x_0) < 0$ and $g_i(\bar{y}) < 0$. And this contradicts the assumption that $x_0$ delivers the minimum of the function $f$ on the set $\left\{ x \in C : g_i(x) \leq 0, \quad i = \overline{i,m} \right\}$. Obtain that $F(y) \geq 0$ at $y \in C$. Besides $F(x_0) = 0$, i.e., $x_0$ minimize the function $F$ on the set $C$. Then from corollary 3 it follows that for any $\lambda \geq K$ the function $g_\lambda(x) = F(x) + \lambda \left( d^{\frac{\beta}{a}}(x) + d^{\beta+v-\alpha v}(x) \right)$ attains minimum on $C_\delta$ at the point $x_0$ and if $\lambda > K$, then any point minimizing $g_\lambda(x)$ on the set $C_\delta$ belongs to $C$. The theorem is proved.

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