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THE SCATTERING PROBLEM FOR HYPERBOLIC SYSTEM OF n EQUATIONS OF THE FIRST ORDER ON A SEMI-AXIS WITH THE $n-1$ SAME VELOCITIES

Abstract

In the paper the scattering problem is studied for hyperbolic system of n equations of the second order on semi-axis with $n-1$ same velocities and the transformation operators are constructed.

Consider on the semi-axis $x \geq 0$ hyperbolic system $n \geq 3$ of the first order differential equations of the form

$$\xi_k \frac{\partial \psi_k(x,t)}{\partial t} - \frac{\partial \psi_k(x,t)}{\partial x} = \sum_{j=1}^{n} \left(\xi_k - \xi_j\right) C_{kj}(x,t)\, \psi_j(x,t), \quad (k = 1,2,...n), \tag{1}$$

where $C_{kj}(x,t)$ are measurable complex-valued functions ($C_{kk}(x,t) \equiv 0$) and satisfy the following conditions

$$|C_{kj}(x,t)| \leq C\,(1+|x|)^{-1-\varepsilon}\,(1+|t|)^{-1-\varepsilon}, \tag{2}$$

$k,j = 1,2,...,n$ (c and ε are positive constants) $\xi_1 = \xi_2 = ... = \xi_{n-1} > 0 > \xi_n$, $t \in (-\infty, \infty)$.

The direct and inverse scattering problems on whole axis and on semi-axis for hyperbolic system of two ($n = 2$) equations are explicitly studied in the L.P. Nijnik papers [1]. The inverse scattering problem for the Dirac system in characteristic variables is applied to integrating the Kortveg-de Friz two-dimensional modified equation [2-3].

The inverse scattering problem is studied in the L.P. Nijnik and V.G. Tarasov [4] papers for the hyperbolic system of $n \geq 3$ equations of the first order on whole axis with different velocities, in other statement in the L.Y. Sung and A.S. Fokas papers [5]. The inverse scattering problem on the semi-axis when there is $n-1$ or one incident wave is studied by N.Sh. Iskenderov [6-7] and for the system of four hyperbolic equations of the first order with two given incident waves is studied by N.Sh. Iskenderov and M.I. Ismayilov [8]. The direct and inverse scattering problem on whole axis for the system the three equations of form (1) for $\xi_1 = \xi_2 > \xi_3$ is studied by N.Sh. Iskenderov and M.I. Ismayilov [9], and on semi-axis by M.I. Ismayilov [10].

For simplicity of the statement we assume, that $\xi_1 = \xi_2 = ... = \xi_{n-1} = 1$, $\xi_n = -1$. Then system (1) is reduced to a system of the equations of form

$$\frac{\partial \psi_k(x,t)}{\partial t} - \frac{\partial \psi_k(x,t)}{\partial x} = u_{kn}(x,t)\, \psi_n(x,t), \qquad k = 1,2,...,n-1, \tag{3}$$

$$\frac{\partial \psi_n(x,t)}{\partial t} + \frac{\partial \psi_n(x,t)}{\partial x} = \sum_{j=1}^{n-1} u_{nj}(x,t)\, \psi_j(x,t).$$

Here $u_{kn}(x,t) = 2C_{kn}(x,t)$, $u_{nj}(x,t) = 2C_{nj}(x,t)$ $(k,j = 1,2,...,n-1)$.

The coefficients of the system of equation (3) are measurable complex-valued functions and satisfy the conditions

$$|u_{kj}(x,t)| \leq c\,(1+|x|)^{-1-\varepsilon}\,(1+|t|)^{-1-\varepsilon}\,. \qquad (4)$$

1. The scattering problem. Any bounded solution of system (3) with the coefficients satisfying the conditions (4) admits on the semi-axis $x \geq 0$ the asymptotic representation

$$\psi_i(x,t) = a_i(t+x) + o(1)\,, \quad i = 1,2,...,n-1, \qquad (5)$$

$$\psi_n(x,t) = b(t-x) + o(1)\,,$$

where the functions $a_1(s)\,,...,a_{n-1}(s) \in L_\infty(E)$ $(E = (-\infty,\infty))$ determine the profiles of incident waves, and the function $b(s) \in L_\infty(E)$ determines the profile of scattering wave. The scattering problem on semi-axis is in finding the solution of system (3) by given incident waves $a_1(s)\,,...,a_{n-1}(s)$ and the boundary conditions for $x = 0$.

Consider the $n-1$ problem; the k-th solution is in finding the solution of system (3) satisfying the boundary condition

$$\psi_n^k(0,t) = \psi_k^k(0,t)\,, \quad k = 1,2,...,n-1, \qquad (6)$$

by the given incident waves $a(s) = (a_1(s)\,,...,a_{n-1}(s))$ determining for $x \to \infty$ the asymptotics of solutions $\psi_1^k(x,t)\,,...,\psi_{n-1}^k(x,t)$ of form (5).

The joint consideration of these $n-1$ problems we'll call the scattering problem for system (3) on the semi-axis.

Theorem 1. *Let the coefficients of system (3) satisfy conditions (4). Then, there exists a unique solution of the first and second scattering problem on the semi-axis for the system of equations (3) with arbitrary given incident waves $a_1(s)\,, a_2(s)\,,...,$ $a_{n-1}(s) \in L_\infty(E)$.*

Proof. The scattering problem for k-th problem is equivalent to the following system of integral equations

$$\psi_i^k(x,t) = a_i(t+x) + \int_x^{+\infty} u_{in}(s,x+t-s)\,\psi_n^k(s,x+t-s)\,ds, \quad i = 1,2,...,n-1$$

$$\psi_n^k(x,t) = b_k(t-x) - \int_x^{+\infty} \sum_{j=1}^{n} u_{nj}(s,t-x+s)\,\psi_j^k(s,t-x+s)\,ds, \qquad (7)$$

where

$$b_k(t) = a_k(t) + \int_0^{+\infty} \sum_{j=1}^{n} u_{nj}(s,t+s)\,\psi_j^k(s,t+s) - u_{kn}(s,t-s)\,\psi_n^k(s,t-s)\,ds. \quad (8)$$

The existence and uniqueness o solutions of system (7) follows from its Volterra property by the variable x by virtue of conditions (2). The theorem is proved.

By virtue of conditions (2) from (7) we obtain the asymptotic representation for $\psi_n^k(x,t)$ as $x \to +\infty$ of form (5)

$$\psi_n^k(x,t) = b_k(t-x) + o(1)\,, \quad b_k(s) \in L_\infty(E)\,, \quad k = 1,2,...,n-1. \qquad (9)$$

On the basis of theorem 1 according to (9) to each vector-function $a(s) = (a_1(s), ..., a_{n-1}(s)) \in L_\infty(E)$ corresponds the $n-1$ solutions of system (3) the solutions of $n-1$ problems with boundary conditions (6), respectively. These $n-1$ solutions determine $n-1$ functions $b(s) = (b_1(s), ..., b_{n-1}(s)) \in L_\infty(E)$ according to (9). Thus, in the space $L_\infty(E, C_{n-1})$ the operator S transforming $a(s)$ to $b(s)$ is determined:

$$b(s) = Sa(s), \quad S = \|S_{ij}\|_{i,j=1}^{n-1}. \tag{10}$$

This operator we'll call the scattering operator for system (3) on a semi-axis.

2. The transformation operator. Each solution of system (3) on the semi-axis we can express by the following vector-functions

$$g^1(t) = \{\psi_1(0,t), \psi_2(0,t), ..., \psi_n(0,t)\}.$$

$$g^2(t) = \{a_1(t), ..., a_{k-1}(t), \psi_k(0,t), ..., \psi_n(0,t)\} \quad (2 \le k \le n),$$

$$g^{n+1}(t) = \{a_1(t), ..., a_{n-1}(t), b(t)\},$$

$$g^{n+k}(t) = \left\{\psi_1(0,t), ..., \psi_{k-1}(0,t), a_k(t), ..., a_{n-1}(t), b(t)\right\} \ (2 \le k \le n-1),$$

$$g^{2n}(t) = \{\psi_1(0,t), ..., \psi_n(0,t), b(t)\}.$$

Lemma 1. *Let the coefficients* $u_{kn}(x,t)$ $(k = 1, 2, ..., n-1)$, $u_{nj}(x,t)$ $(j = 1, ..., n-1)$ $(u_{kn}(x,t) \equiv 0,\ u_{nj}(x,t) = 0,\ x < 0)$ *of system (3) satisfy condition (4). Then for each* $g^j(t) \in L_\infty(E, C_n)$ $(j = 1, 2, ..., 2n)$ *there exists a unique bounded solution of system (4) admitting the following integral representation*

$$\psi_i(x,t) = g_i^1(t + \xi_i x) + \int_{t-x}^{t+x} \sum_{j=1}^{n} A_{ij}^1(x,t,s)\, g_j^1(s)\, ds, \tag{11_1}$$

$$\psi_i(x,t) = g_i^2(t + \xi_i x) + \int_{-\infty}^{t+x} \sum_{j=1}^{n} A_{ij}^2(x,t,s)\, g_j^2(s)\, ds, \tag{11_2}$$

$$\psi_i(x,t) = g_i^k(t + \xi_i x) + \int_{-\infty}^{t+x} \sum_{j=1}^{n-1} A_{ij}^k(x,t,s)\, g_j^k(s)\, ds, \ 3 \le k \le n-1 \tag{11_k}$$

$$\psi_i(x,t) = g_i^n(t + \xi_i x) + \int_{-\infty}^{t+x} \sum_{j=1}^{n-1} A_{ij}^n(x,t,s)\, g_j^n(s)\, ds + \int_{-\infty}^{t-x} A_{in}^n(x,t,s)\, g_n^n(s)\, ds \tag{11_n}$$

$$\psi_i(x,t) = g_i^{n+1}(t + \xi_i x) +$$
$$+ \int_{t+x}^{+\infty} \sum_{j=1}^{n-1} A_{ij}^{n+1}(x,t,s)\, g_j^{n+1}(s)\, ds + \int_{-\infty}^{t-x} A_{in}^{n+1}(x,t,s)\, g_n^{n+1}(s)\, ds, \tag{11_{n+1}}$$

$$\psi_i(x,t) = g_i^{n+k}(t + \xi_i x) +$$
$$+ \int_{t+x}^{+\infty} \sum_{j=1}^{n-1} A_{ij}^{n+k}(x,t,s)\, g_j^{n+k}(s)\, ds + \int_{-\infty}^{t-x} A_{in}^{n+k}(x,t,s)\, g_n^{n+k}(s)\, ds, \tag{11_{n+k}}$$

$$\psi_i(x,t) = g_i^{2n}(t+\xi_i x)+$$
$$+\int\limits_{t+x}^{+\infty}\sum_{j=1}^{n-1} A_{ij}^{2n}(x,t,s)\, g_j^{2n}(s)\,ds + \int\limits_{t-x}^{+\infty} A_{in}^{2n}(x,t,s)\, g_n^{2n}(s)\,ds, \qquad (11_{2n})$$

The kernels of these transformations of fixed x are summable with the square by t,s, i.e., they are Hilbert-Shmidt kernels which are uniquely determined by the coefficients $u_{kn}(x,t), u_{nj}(x,t)\,(k,j=1,2,...,n-1)$. For arbitrary $g_i(t)\in L_\infty(E,C_n)$ the bounded solution, of system (3) are determined by the formulae $(11_1 - 11_{2n})$.

Let's prove the lemma for example for the representations (11_{n+1}). The problem of finding the boundary solutions of system (3) for the given $a_i(s)\,(1,2,...,n-1)$, $b(s)\in L_\infty(E)$ asymptotics $\psi_i(x,t)\,(i=1,2,...,n)$ as $x\to+\infty$ is equivalent to the following system of integral equations:

$$\psi_i(x,t) = a_i(t+x) + \int\limits_x^{+\infty} u_{nj}(s,x+t-s)\,\psi_n(s,x+t-s)\,ds,\ \ i=1,2,...,n-1,$$

$$\psi_n(x,t) = b(t-x) - \int\limits_x^{+\infty}\sum_{j=1}^{n} u_{nj}(s,t-x+s)\,\psi_j(s,t-x+s)\,ds, \qquad (12)$$

If the solution of system (12) can be represented in the form (11_{n+1}) for any $a_i(s)\,(1,2,...,n-1)$, $b(s)\in L_\infty(E)$, then substituting (11_{n+1}) in (12) we obtain the system of equations for the kernels

$$\frac{1}{2}u_{in}\left(\frac{x+t-\tau}{2},\frac{x+t+\tau}{2}\right) + \int\limits_x^{\frac{x+t-\tau}{2}} u_{in}(s,x+t-s)\,A_{nn}^{n+1}(s,x+t-s,\tau)\,ds -$$
$$-A_{in}^{n+1}(x,t,\tau)=0, \quad -\infty<\tau\le t-x,$$
$$\int\limits_x^{+\infty} u_{in}(s,x+t-s)\,A_{nj}^{n+1}(s,x+t-s,\tau)\,ds = A_{ij}^{n+1}(x,t,\tau),$$
$$t+x\le\tau<+\infty,$$
$$\frac{1}{2}u_{nj}\left(\frac{\tau+x-t}{2},\frac{t-x+\tau}{2}\right) + \int\limits_x^{\frac{\tau+x-t}{2}}\sum_{k=1}^{n-1} u_{nk}(s,t-x+s)\,A_{jk}^{n+1}(s,t-x+s,\tau)\,ds -$$
$$-A_{nj}^{n+1}(x,t,\tau)=0, t+x\le\tau<+\infty,$$
$$\int_x^{+\infty}\sum_{j=1}^{n-1} u_{nj}(s,t-x+s)\,A_{jn}^{n+1}(s,t-x+s,\tau)\,ds +$$
$$+A_{nn}^{n+1}(x,t,\tau), \quad -\infty<\tau\le t-x. \qquad (13)$$

Thus, for proof of the representation (11_{n+1}) it is sufficient to prove, that the system of equations (13) has a unique solution. It follows from Volterra property of these systems.

Note, that the kernels of transformations (11_{n+1}) are connected with the coefficients by the equalities

$$A_{in}^{n+1}(x,t,t-x)=\frac{1}{2}u_{in}(x,t),\ \ i=1,2,...,n-1,$$

$$A_{nj}^{n+1}(x,t,t+x)=-\frac{1}{2}u_{nj}(x,t),\ \ j=1,2,...,n-1.$$

The equalities (14) immediately follow (14) from the system (13) for $\tau=t-x$.

3. Factorization of elements of scattering operator on a semi-axis. Using the representation (11_{n+1}), boundary conditions (6) and determination (10) of scattering operator we obtain

$$S_{11}=\left(I+A_{nn+}^{n+1}-A_{1n+}^{n+1}\right)^{-1}\left(I+A_{11-}^{n+1}-A_{n1-}^{n+1}\right),$$

$$S_{1i}=\left(I+A_{nn+}^{n+1}-A_{1n+}^{n+1}\right)\left(A_{1i-}^{n+1}-A_{ni-}^{n+1}\right)\quad(i=1,...,n-1),$$

$$S_{22}=\left(I+A_{nn+}^{n+1}-A_{2n+}^{n+1}\right)\left(I+A_{22-}^{n+1}-A_{n2-}^{n+1},\right)$$

$$S_{2i}=\left(I+A_{nn+}^{n+1}-A_{2n+}^{n+1}\right)^{-1}\left(A_{2i-}^{n+1}-A_{ni-}^{n+1}\right)\quad(i=1,3,...,n-1),$$

....

$$S_{n-1,n-1}=\left(I+A_{nn+}^{n+1}-A_{n-1,n+}^{n+1}\right)^{-1}\left(I+A_{n-1,n-1-}^{n+1}-A_{n,n-1-}^{n+1}\right),$$

$$S_{n-1,i}=\left(I+A_{nn+}^{n+1}-A_{n-1,n+}^{n+1}\right)^{-1}\left(A_{n-1,i-}^{n+1}-A_{ni-}^{n+1}\right)\quad(i=1,2,...,n-2),$$

where

$$A_{+}f(t)=\int_{-\infty}^{t}A(t,s)f(s)\,ds,$$

$$A_{-}f(t)=\int_{t}^{+\infty}A(t,s)f(s)\,ds$$

are Volterra operators of corresponding polarity.

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