Mubariz G. HAJIBAYOV

ON THE CAPACITY OF THE SET OF TOTALLY NON-DIFFERENTIABILITY OF ANISOTROPIC RIESZ POTENTIALS

Abstract

The problem on total differentiability of Riesz potentials with anisotropic kernel is considered. Mizuta's theorem on total differentiability of classic Riesz potentials is generalized in certain sense (see [4]).

Introduction and preliminaries.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be positive numbers and $|\lambda| = \lambda_1 + \lambda_2 + ...\lambda_n$. For $x = (x_1, x_2, ..., x_n)$ define the following λ distance from zero (see [1]).

$$\|x\|_\lambda = \left(\sum_{i=1}^{n} |x_i|^{\frac{1}{\lambda_i}}\right)^{\frac{|\lambda|}{n}}. \tag{1}$$

Assuming $t^\lambda x = (t^{\lambda_1}x_1, t^{\lambda_2}x_2, ..., t^{\lambda_n}x_n)$ it is easy to see that for any $t > 0$

$$\left\|t^\lambda x\right\|_\lambda = t^{\frac{|\lambda|}{n}} \|x\|_\lambda .$$

Accepting the notion $\underline{\lambda} = \min\limits_{i=\overline{1,n}} \lambda_i$, $\bar{\lambda} = \max\limits_{i=\overline{1,n}} \lambda_i$ one can observe that the inequality of the triangle for the λ distance (1) will have the form:

$$\|x + y\|_\lambda \leq 2^{\frac{|\lambda|}{\underline{\lambda} n}} \left(\|x\|_\lambda + \|y\|_\lambda\right). \tag{2}$$

By $\sigma(x_0, r)$ we'll denote an open ball of radius r with center x_0:

$$\sigma(x_0, r) = \{x \in R^n : \|x - x_0\|_\lambda < r\}.$$

For $0 < \alpha < n$, the function

$$R_\alpha(y) = \|y\|_\lambda^{\alpha - n}$$

is called an anisotropic Riesz kernel, and the integral

$$R_\alpha f(x) = \int\limits_{R^n} \|x - y\|_\lambda^{\alpha - n} f(y) dy \tag{3}$$

is called an anisotropic Riesz potential.

Let $0 < \alpha < n$, $1 < p < \infty$.. The quantity $R_{\alpha,p}(A) = \inf\limits_f \|f\|^p_{L^p(R^n)}$ is called an anisotropic Riesz capacity of the set $A \subset R^n$, where the infimum being taken over all non negative functions $f \in L^p(R^n)$ such that $\int\limits_{R^n} R_\alpha(x - y) f(y) dy \geq 1$ for all $x \in A$. The properties of $R_{\alpha,p}$ are studied in [3], [6] and [7].

Let $\varepsilon_0 > 0$. For a non-decreasing and continuous function $h : [0, \varepsilon_0] \to [0, +\infty)$, $h(0) = 0$ and $A \subset R^n$ we define the quantity

$$H_{h,\varepsilon}(A) = \inf \left\{ \sum_{i=1}^{\infty} h(r_i) : A \subset \sigma(x^{(i)}, r_i), r_i < \varepsilon \right\},$$

where $0 < \varepsilon < \varepsilon_0$ and the infimum being taken over all coverings of the set A by balls $\sigma(x^{(i)}, r_i)$ with radius $r_i < \varepsilon$. It is obvious that there exists $\lim\limits_{\varepsilon \to 0} H_{h,\varepsilon}(A) = H_h(A) \leq \infty$. The quantity $H_h(A)$ is called Hausdorff h measure of the set A. If $h(r) = r^\beta$ where $\beta > 0$ we'll simply write $H_\beta(A)$.

Theorem 1. *Let f be a non-negative locally integrable function in R^n. For the convergence of the potential $R_\alpha f$ almost everywhere in R^n it is necessary and sufficient that one of the following equivalent conditions is fulfilled.*

1) There exists such $x_0 \in R^n$ that

$$\int\limits_{R^n \setminus \sigma(x_0, 1)} \|x_0 - y\|_\lambda^{\alpha - n} f(y) dy < \infty.$$

2) For any $x \in R^n$

$$\int\limits_{R^n \setminus \sigma(x, 1)} \|x - y\|_\lambda^{\alpha - n} f(y) dy < \infty.$$

3)

$$\int\limits_{R^n} (1 + \|y\|_\lambda)^{\alpha - n} f(y) dy < \infty. \tag{4}$$

Let $p = \dfrac{n}{\alpha} > 1$ and w be a positive, non-decreasing function on $(0, \infty)$, such that

$$\int\limits_1^\infty w^{-\frac{1}{p-1}}(t) t^{-1} dt < \infty$$

and there exists a positive constant $A > 0$ that for any $r > 0$

$$w(2r) < Aw(r).$$

Denote by Φ_w a class of all non-negative functions f for which

$$\int\limits_{R^n} f^p(y) w(f(y)) dt < \infty.$$

Theorem 2. *Let $f \in \Phi_w$ u and (4) be fulfilled. Then the potential $R_\alpha f$ is continuous on R^n.*

The proofs of above indicated theorems in isotropic case are in [4], in anisotropic case in [2].

A function $u : R^n \to R^1$ is said to be totally m times differentiable at the point x_0, if there exists a polynomial $P(x)$ of degree at most m such that

$$\lim_{\|x-x_0\|\to 0} \|x - x_0\|_\lambda^{-m} [u(x) - P(x)] = 0.$$

In [4] it is proved that if a distance is Euclidean, $(\lambda_1 = \lambda_2 = ... = \lambda_n = \frac{1}{2})$ and if the condition of theorem 2 is fulfilled, then for natural $m \le \alpha$ there will be found a set $E \subset R^n$ that $R_{\alpha-m,p}(E) = 0$, and $R_\alpha f$ is totally m times differentiable outside of E. In this case, we shall say that E is the set of total non-differentiability of the potential $R_\alpha f$.

The goal of the paper is to generalize this statement for anisotropic distance and anisotropic Riesz potentials.

Total differentiation of $R_\alpha f$.

In sequel, positive real constants will be denoted by C, M, M_1, M_2 and etc.

Lemma1. *Let $k = (k_1, k_2, ..., k_n)$ be a multi-index and $\mathcal{R}_k$ be a domain of determination for the function*

$$\frac{\partial^k}{\partial y^k} R_\alpha(y) = \frac{\partial^{|k|}}{\partial y_1^{k_1} \partial y_2^{k_2} ... \partial y_n^{k_n}} R_\alpha(y).$$

Then for any $y \in \mathcal{R}_k$

$$\frac{\partial^k}{\partial y^k} R_\alpha(y) = \sum_{\substack{\nu_1=\overline{1,k_1} \\ \nu_2=\overline{1,k_2} \\ \cdots\cdots\cdots \\ \nu_n=\overline{1,k_n}}} M_{\nu_1,\nu_2...\nu_n} \|y\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}(\nu_1+\nu_2+...+\nu_n)} |y_1|^{\frac{\nu_1}{\lambda_1}-k_1} |y_2|^{\frac{\nu_2}{\lambda_2}-k_2} ... |y_n|^{\frac{\nu_n}{\lambda_n}-k_n}, \tag{5}$$

where $|M_{\nu_1,\nu_2,...,\nu_n}|$ is independent of y and if $\frac{1}{\lambda_i}$ is a natural number for some i then *values ν_i where $\frac{\nu_i}{\lambda_i} - k_i < 0$ don't included in the sum.*

The lemma is proved by the mathematical induction method.

Lemma 2. *Let $k = (k_1, k_2, ..., k_n)$ be a multi-index and $\mathcal{R}_k$ be a domain of determination for the function $\frac{\partial^k}{\partial y^k} R_\alpha(y)$. Assume that*

$$\frac{1}{\lambda_i} \in N \ \ or \ \ \frac{1}{\lambda_i} > |k| \ \ for \ any \ i = \overline{1,n}.$$

Then 1) For $y \in \mathcal{R}_k$

$$\left| \frac{\partial^k}{\partial y^k} R_\alpha(y) \right| \le M \|y\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum_{j=1}^{n} k_j \lambda_j} \tag{6}$$

2) For $y \in \mathcal{R}_k$, $\|y\|_\lambda < 1$

$$\left| \frac{\partial^k}{\partial y^k} R_\alpha(y) \right| \le M \|y\|_\lambda^{\alpha-n-\frac{\bar{\lambda}\, n}{|\lambda|}|k|} \tag{7}$$

3) For $y \in \mathcal{R}_k$, $\|y\|_\lambda \geq 1$

$$\left|\frac{\partial^k}{\partial y^k} R_\alpha(y)\right| \leq M \|y\|_\lambda^{\alpha-n} \tag{8}$$

where M *is a positive constant depending only on* n, α *and* k.

Proof. We have from (5)

$$\left|\frac{\partial^k}{\partial y^k} R_\alpha(y)\right| \leq M \sum_{\substack{\nu_1=\overline{1,k_1} \\ \nu_2=\overline{1,k_2} \\ \cdots\cdots \\ \nu_n=\overline{1,k_n}}} \|y\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{j=1}^n \nu_j} \times$$

$$\times \left(\sum_{i=1}^n |y_i|^{\frac{1}{\lambda_i}}\right)^{\nu_1-k_1\lambda_1} \left(\sum_{i=1}^n |y_i|^{\frac{1}{\lambda_i}}\right)^{\nu_2-k_2\lambda_2} \cdots \left(\sum_{i=1}^n |y_i|^{\frac{1}{\lambda_i}}\right)^{\nu_n-k_n\lambda_n} =$$

$$= M \|y\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{j=1}^n k_j\lambda_j}.$$

It is clear that $\sum\limits_{j=1}^n k_j\lambda_j \leq \bar{\lambda}\,|k|$. Then (7) and (8) are easily obtained from (6). The lemma is proved.

Let $x_0 \in R^n$. Determine

$$Q_{\alpha,|k|}(x,y) = R_\alpha(x-y) - \sum_{|l|\leq|k|} \frac{(x-x_0)^l}{l!} \frac{\partial^l}{\partial x^l} R_\alpha(x_0-y)$$

where $l = (l_1, l_2, ..., l_n)$ is a multiindex, $|l| = l_1 + l_2 + ... + l_n$, $l! = l_1!l_2!...l_n!$, $(x-x_0)^l = \left(x_1 - x_1^0\right)^{l_1} (x_2 - x_2^0)^{l_2} ... (x_n - x_n^0)^{l_n}$, $x_0 = (x_1^0, x_2^0, ..., x_n^0)$.

Let $\mathcal{R}(x)$ be a domain determination of $Q_{\alpha,|k|}(x,\cdot)$ for the fixed x.

Lemma 3. *Let* $k = (k_1, k_2, ..., k_n)$ *be a multiindex. Assume that*

$$\frac{1}{\lambda_i} \in N \text{ or } \frac{1}{\lambda_i} > |k|.$$

for any $i = \overline{1,n}$. *Then following estimates are true.*

1) If $y \in \mathcal{R}(x)$, $\|y-x_0\|_\lambda \geq 2^{\frac{|\lambda|}{\lambda_n}+1} \|x-x_0\|_\lambda$ and $|l| = |k|+1$, *then*

$$\left|Q_{\alpha,|k|}(x,y)\right| \leq C \|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i} \|y-x_0\|_\lambda^{\alpha-n-\frac{n}{\lambda}\sum\limits_{i=1}^n \lambda_i l_i} \tag{9}$$

2) If $y \in \mathcal{R}(x)$, $\|y-x_0\|_\lambda > 1$, $|l| = |k|+1$ *and* x *is sufficiently close to* x_0,*then*

$$\left|Q_{\alpha,|k|}(x,y)\right| \leq C \|x-x_0\|_\lambda^{\frac{\lambda_n}{|\lambda|}|l|} \|y-x_0\|_\lambda^{\alpha-n}. \tag{10}$$

Proof. Expanding the function $R_\alpha(x-y)$ in Taylor series in the vicinity of the point $x = x_0$, we'll see that $Q_{\alpha,|k|}$ equals remainder term.

$$Q_{\alpha,|k|}(x,y) = \sum_{|l|=|k|+1} \frac{(x-x_0)^l}{l!} \frac{\partial^l}{\partial x^l} R_\alpha\left(x_0 + \theta(x-x_0) - y\right),$$

where $0<\theta<1$. It is clear that $\left|x_i-x_i^0\right|^{l_i}\le\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\lambda_i l_i}$. Then $\left|(x-x_0)^l\right|\le\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n\lambda_i l_i}$. Hence

$$\left|Q_{\alpha,|k|}(x,y)\right|\le M\,\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n\lambda_i l_i}\,\|x_0+\theta(x-x_0)-y\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n\lambda_i l_i}.$$

By (2) we get

$$\|x+\theta(x-x_0)-y\|_\lambda\ge 2^{-\frac{|\lambda|}{\lambda n}}\,\|y-x\|_\lambda-\|\theta(x-x_0)\|_\lambda\ge 2^{-\frac{|\lambda|}{\lambda n}-1}\,\|y-x\|_\lambda.$$

Then

$$\left|Q_{\alpha,|k|}(x,y)\right|\le C\,\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n\lambda_i l_i}\,\|y-x_0\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n\lambda_i l_i}$$

and so (9) is proved.

It is clear that $\sum\limits_{i=1}^n\lambda_i l_i\le\bar\lambda\left(|k|+1\right)$ and if $\|x-x_0\|<1$, then $\left|(x-x_0)^l\right|\le\|x-x_0\|_\lambda^{\frac{\lambda_n}{\lambda}|l|}$. Then from (9) we get (10). The lemma is proved.

We cite also the following covering lemma whose proof is similar to the proof of lemma called Vitali lemma in isotropic case (for example, see [8], or [5, ch.I, lemma 1.6]).

Lemma 4. *Let* $\mathbf{B}$ *be any class of closed balls* $\bar\sigma$ *and* $\sup\{radius\,\bar\sigma:\bar\sigma\in\mathbf{B}\}<\infty$. *Then a class* $\{\bar\sigma\left(x^{(j)},r_j\right)\}_{j=1}^\infty$ *composed of pairwise dispoint balls can be chosen from class* $\mathbf{B}$, *that for any* $\bar\sigma\in\mathbf{B}$ *there will be found a number* $j\in N$ *that*

$$\bar\sigma\cap\bar\sigma(x^{(j)},r_j)\ne\emptyset\text{ and }\bar\sigma\subset\bar\sigma\left(x^{(j)},5\cdot 2^{\frac{|\lambda|}{\lambda n}}r_j\right).$$

Lemma 5. *Let* $\varepsilon_0>0$ *and* $h:(0,\varepsilon_0)\to(0,\infty)$ *be a non-decreasing continuous function* $\lim\limits_{r\to0}h(r)=0$, $h(2r)\le Ch(r)$, $\lim\limits_{r\to0}h(r)r^{-n}=\infty$. *Assume that* f *is a non-negative function in* $L^1(R^n)$ *and*

$$E=\left\{\xi\in R^n:\limsup_{r\downarrow0}h^{-1}(r)\int\limits_{\sigma(\xi,r)}f(y)dy>0\right\}.$$

Then $H_h(E)=0$.

Proof. For $\delta>0$ consider a set

$$E(\delta)=\left\{\xi\in R^n:\limsup_{r\downarrow0}h^{-1}(r)\int\limits_{\sigma(\xi,r)}f(y)dy>\delta\right\}.$$

It sufficies to show $H_h(E(\delta))=0$.

Let $\varepsilon>0$. Then for any $\xi\in E(\delta)$ there will be found such $r(\xi)$ that $0<r(\xi)<\varepsilon$ and

$$h^{-1}\left(r(\xi)\right)\int\limits_{\sigma(\xi,r(\xi))}f(y)dy>\delta.$$

By lemma 4 there will be found such set of points $\{\xi^{(j)}\}_{j=1}^{\infty} \subset E(\delta)$ that a class of closed balls $\{\bar{\sigma}(\xi^{(j)}, r(\xi^{(j)}))\}_{j=1}^{\infty}$ consists of mutually disjoint balls and

$$E(\delta) \subset \bigcup_{j=1}^{\infty} \bar{\sigma}\left(\xi^{(j)}, 5 \cdot 2^{\frac{|\lambda|}{\lambda n}} r\left(\xi^{(j)}\right)\right).$$

Then

$$\delta \sum_{j=1}^{\infty} h(r_j) \le \sum_{j=1}^{\infty} \int\limits_{\sigma(\xi^{(j)}, r(\xi^{(j)}))} f(y)dy = \int\limits_{\bigcup_{j=1}^{\infty} \sigma(\xi^{(j)}, r(\xi^{(j)}))} f(y)dy \le \int\limits_{R^n} f(y)dy < \infty. \quad (11)$$

Hence

$$\sum_{j=1}^{\infty} r_j^n \le \left(\sup_{0<r<\varepsilon} r^n h^{-1}(r)\right) \sum_{j=1}^{\infty} h(r_j) \le \left(\sup_{0<r<\varepsilon} r^n h^{-1}(r)\right) \delta^{-1} \int\limits_{R^n} f(y)dy.$$

Then $\lim\limits_{\varepsilon \to 0} \sum\limits_{j=1}^{\infty} r_j^n = 0$ and so $\lim\limits_{\varepsilon \to 0} m \left(\bigcup\limits_{j=1}^{\infty} \sigma(\xi^{(j)}, r(\xi^{(j)})) \right) = 0$, where $m(X)$ is a Lebesgue measure of the set X.

We get from (11)

$$H_h(E(\delta)) \le \sum_{j=1}^{\infty} h\left(5 \cdot 2^{\frac{|\lambda|}{\lambda n}} r\left(\xi^{(j)}\right)\right) \le C \sum_{j=1}^{\infty} h\left(r\left(\xi^{(j)}\right)\right) \le$$

$$\le C\delta^{-1} \int\limits_{\bigcup_{j=1}^{\infty} \sigma(\xi^{(j)}, r(\xi^{(j)}))} f(y)dy \underset{\varepsilon \to 0}{\to} 0.$$

So $H_h(E(\delta)) = 0$ and the lemma is proved.

Lemma 6. *Let $f \in \Phi_w$. Then there will be found such constant $M > 0$ that for any $a > 0$*

$$\int\limits_{\{y: f(y) \ge a\}} R_\alpha(x-y) f(y)dy \le M \left(\int\limits_{\{y: f(y) \ge a\}} f^p(y) w(f(y))dy \right)^{\frac{1}{p}} \left(\int\limits_a^{\infty} w^{-\frac{1}{p-1}}(t) t^{-1} dt \right)^{\frac{1}{p'}}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

The proof is carried out similar to the proofs of lemma 1 in [2] and lemma 1 in [4].

Lemma 7. *Let $f \in \Phi_w$. For $\beta < \alpha$ define*

$$E = \left\{ x \in R^n : \limsup_{r \downarrow 0} r^{\alpha - \beta - n} \int\limits_{\sigma(x,r)} |f(y) - f(x)|\, dy > 0 \right\}.$$

Then $H_{\beta p}(E) = 0$.

Proof. Consider a function

$$g(y) = \begin{cases} f(y), & \text{for } \|x-y\|_\lambda \le r \\ 0, & \text{for } \|x-y\|_\lambda > r \end{cases}$$

then by lemma 6

$$r^{\alpha-\beta-n} \int\limits_{\sigma(x,r)} f(y)dy \le r^{-\beta} \int\limits_{\sigma(x,r)} \|x-y\|_\lambda^{\alpha-n} f(y)dy \le$$

$$\le \left[\int\limits_{\{y:f(y)<1\}\cap\sigma(x,r)} \|x-y\|_\lambda^{\alpha-n} f(y)dy + \int\limits_{\{y:f(y)\ge 1\}\cap\sigma(x,r)} \|x-y\|_\lambda^{\alpha-n} f(y)dy \right] \le$$

$$\le r^{-\beta} \left[\int\limits_{\{y:f(y)<1\}\cap\sigma(x,r)} \|x-y\|_\lambda^{\alpha-n} f(y)dy + M \left(\int\limits_{\{y:g(y)\ge 1\}} g^p(y) w(g(y))dy \right)^{\frac{1}{p}} \times \right.$$

$$\left. \times \left(\int\limits_1^\infty w^{-\frac{1}{p-1}}(t) t^{-1} dt \right)^{\frac{1}{p'}} \right] \le$$

$$\le M_1 r^{-\beta} \left[\int\limits_{\sigma(x,r)} \|x-y\|_\lambda^{\alpha-n} dy + \left(\int\limits_{\sigma(x,r)} f^p(y) w(f(y))dy \right)^{\frac{1}{p}} \right] \le$$

$$\le M_2 \left[r^{\alpha-\beta} + \left(r^{-\beta p} \int\limits_{\sigma(x,r)} f^p(y) w(f(y))dy \right)^{\frac{1}{p}} \right].$$

Then

$$\limsup_{r\downarrow 0} r^{\alpha-\beta-n} \int\limits_{\sigma(x,r)} f(y)dy \le M_2 \limsup_{r\downarrow 0} \left(r^{-\beta p} \int\limits_{\sigma(x,r)} f^p(y) w(f(y))dy \right)^{\frac{1}{p}}.$$

Consider a set

$$E' = \left\{ x \in R^n : \limsup_{r\downarrow 0} \left(r^{-\beta p} \int\limits_{\sigma(x,r)} f^p(y) w(f(y))dy \right)^{\frac{1}{p}} \right\}.$$

It is clear that $f^p(\cdot) w(f(\cdot)) \in L^1(R^n)$. Then by lemma 5 $H_{\beta p}(E') = 0$. It is easy to see $E \subset E'$. Hence $H_{\beta p}(E) = 0$. The lemma is proved.

Theorem 3. *Let $f \in \Phi_w$ and (4) be fulfilled. Assume that m is a natural number and*

1) $\frac{1}{\lambda_i} \in N$ or $\frac{1}{\lambda_i} > m$ for any $i = \overline{1,n}$.

2) $\overline{\lambda}\frac{m}{\alpha} < \frac{|\lambda|}{n} < \underline{\lambda}\frac{m+1}{m}$.

Then there exists a set $E \subset R^n$ that $R_{\alpha - \frac{\bar{\lambda} n m}{|\lambda|}, p}(E) = 0$ and $R_\alpha f$ is totally m times *differentiable outside of the set E, i.e. Riesz capacity of the totally non-differentiable set of the potential $R_\alpha f$ equals zero.*

Proof. Let $l = (l_1, l_2, ..., l_n)$ be a multiindex and

$$Q_{\alpha,m}(x,y) = R_\alpha(x-y) - \sum_{|l| \le m} \frac{(x-x_0)^l}{l!} \frac{\partial^l}{\partial x^l} R_\alpha(x_0 - y).$$

For $|l| \le m$ we define

$$A_l = \lim_{r \downarrow 0} \int\limits_{r^n \setminus \sigma(x_0, r)} \frac{\partial^l}{\partial x^l} R_\alpha(x_0 - y) f(y) dy.$$

By lemma 2

$$|A_l| \le M_l \left[\int\limits_{\|y-x_0\|_\lambda \ge 1} \|x_0 - y\|_\lambda^{\alpha - n} f(y) dy + \right.$$

$$\left. + \lim_{r \downarrow 0} \int\limits_{r < \|y-x_0\|_\lambda < 1} \|x_0 - y\|_\lambda^{\alpha - n - \frac{n \bar{\lambda} |l|}{|\lambda|}} f(y) dy \right] = M_l [I + II].$$

By the condition of theorem 1 $\quad I < \infty$.

Consider the function

$$g(y) = \begin{cases} f(y), & \text{if } f(y) \ge 1, \\ 0, & \text{if } f(y) < 1. \end{cases}$$

Then

$$II \le \lim_{r \downarrow 0} \int\limits_{r < \|y-x_0\|_\lambda < 1} \|x_0 - y\|_\lambda^{\alpha - n - \frac{n \bar{\lambda} |l|}{|\lambda|}} dy +$$

$$+ \lim_{r \downarrow 0} \int\limits_{r < \|y-x_0\|_\lambda < 1} \|x - y\|_\lambda^{\alpha - n - \frac{n \bar{\lambda} |l|}{|\lambda|}} g(y) dy \le II_1 + II_2.$$

Consider II_1. We use a transformation $x_i = \theta_i \rho^{\frac{\lambda_i n}{|\lambda|}}$, where θ_i are the coordinates of the point θ on a unit sphere $S_{n-1} = \{x : \|x\|_\lambda = 1\}$. It is known that the Jacobian of this transformation equals $\rho^{n-1}\Omega(\theta)$, where $\Omega(\theta)$ depends only on the angle θ (see [1]). Let $C_1 = \int\limits_{S_{n-1}} \Omega(\theta) d\theta$. Then $II_1 \le C_1$.

Since

$$\int\limits_{R^n} g^p(x)dx \le w^{-1}(1)\int\limits_{R^n} f^p(x)w(f(x))dx < \infty,$$

then $g \in L^p(R^n)$.

Then there exists the set $E_1 \subset R^n$ that $A_l < \infty$ for $x_0 \in R^n \backslash E_1$ and $R_{\alpha-\frac{\bar{\lambda} nm}{|\lambda|},p}(E_1) = 0$. Thus, we'll assume $x_0 \in R^n/E_1$.

Consider the polynomial

$$P(x) = \sum_{|l|\le m} \frac{A_l}{l!}(x-x_0)^l.$$

Denote

$$b = 2^{\frac{|\lambda|}{\lambda n}+1}.$$

Then

$$\|x-x_0\|_\lambda^{-m}[R_\alpha(x)-P(x)] = \|x-x_0\|_\lambda^{-m}\int\limits_{R^n\backslash\sigma(x_0,1)} Q_{\alpha,m}(x,y)f(y)dy+$$

$$+\|x-x_0\|_\lambda^{-m}\int\limits_{\sigma(x_0,1)\backslash\sigma(x_0,b\|x-x_0\|_\lambda)} Q_{\alpha,m}(x,y)\left[f(y)-f(x_0)\right]dy-$$

$$-\|x-x_0\|_\lambda^{-m}\sum_{|l|\le m}\frac{(x-x_0)^l}{l!}\lim_{r\downarrow 0}\int\limits_{\sigma(x_0,b\|x-x_0\|_\lambda)\backslash\sigma(x_0,r)} \frac{\partial^l}{\partial x^l}R_\alpha(x_0-y)[f(y)-f(x_0)]dy+$$

$$+f(x_0)\|x-x_0\|_\lambda^{-m}\int\limits_{\sigma(x_0,1)} Q_{\alpha,m}(x,y)dy+$$

$$+\|x-x_0\|_\lambda^{-m}\int\limits_{\sigma(x_0,b)\|x-x_0\|_\lambda} R_\alpha(x-y)[f(y)-f(x_0)]dy = J_1+J_2+J_3+J_4+J_5.$$

From lemma 3

$$|J_1| \le C\|x-x_0\|_\lambda^{-m+\frac{\lambda n}{|\lambda|}(m+1)}\int\limits_{R^n\backslash\sigma(x_0,1)} \|y-x_0\|_\lambda^{\alpha-n} f(y)dy,$$

and from the condition of the theorem $|J_1| \underset{\|x-x_0\|_\lambda\to 0}{\to} 0.$

By lemma 3

$$|J_2| \le \sum_{|l|=m+1} C_l\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i - m}\times$$

$$\times\int\limits_{\sigma(x_0,1)\backslash\sigma(x_0,b\|x-x_0\|_\lambda)} \|y-x_0\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}\lambda_i l_i}|f(y)-f(x_0)|\,dy = \sum_{|l|=m+1} C_l J_2^l,$$

where C_l is a positive constant.

Take any δ provided $0<\delta<1$. Then

$$J_2^l \le \|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i - m} \int\limits_{\sigma(x_0,1)\backslash\sigma(x_0,\delta)} \|y-x_0\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i} |f(y)-f(x_0)|\,dy+$$

$$+\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i - m} \int\limits_{\sigma(x_0,\delta)\backslash\sigma\left(x_0,b\|x-x_0\|_\lambda\right)} \|y-x_0\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i} |f(y)-f(x_0)|\,dy = J_{2,1}^l + J_{2,2}^l.$$

Taking into account $|l|=m+1$ we obtian:

$$J_{2,1}^l \le \|x-x_0\|_\lambda^{\frac{n\lambda(m+1)}{|\lambda|}-m} \int\limits_{\sigma(x_0,1)\backslash\sigma(x_0,\delta)} \|y-x_0\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i} |f(y)-f(x_0)|\,dy.$$

Hence $J_{2,1}^l \to 0$ as $\|x-x_0\|_\lambda \to 0$.
If $\|y-x_0\|_\lambda \ge b\,\|x-x_0\|_\lambda$, then
$\|y-x_0\|_\lambda \ge \frac{1}{2}\|y-x_0\|_\lambda + \frac{b}{2}\|x-x_0\|_\lambda \ge \frac{1}{2}\left(\|y-x_0\|_\lambda + \|x-x_0\|_\lambda\right).$
Then

$$J_{2,2}^l \le \|x-x_0\|_\lambda^{\frac{n}{\lambda}\sum\limits_{i=1}\lambda_i l_i - m} \int\limits_{\sigma(x_0,\delta)} \left(\frac{1}{2}\left(\|y-x_0\|_\lambda + \|x-x_0\|_\lambda\right)\right)^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}\lambda_i l_i} \times$$

$$\times |f(y)-f(x_0)|\,dy = M_1\,\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i - m} \int\limits_0^\delta \left(\rho + \|x-x_0\|_\lambda\right)^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i} \times$$

$$\times d\left(\int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\,dy\right) = M_1\left\{\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i - m} \times\right.$$

$$\times\left[\left(\rho+\|x-x_0\|_\lambda\right)^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i} \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\,dy\right]\Bigg|_0^\delta +$$

$$+\|x-x_0\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i, l_i - m}\left(n+\frac{n}{|\lambda|}\sum_{i=1}^n \lambda_i l_i - \alpha\right)\int\limits_0^\delta \left(\rho+\|x-x_0\|_\lambda\right)^{\alpha-n-\frac{n}{|\lambda|}\lambda_i l_i - 1} \times$$

$$\left.\times \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\,dyd\rho\right\} = M_1\{J_{2,2,1}^l + J_{2,2,2}^l\},$$

$$J_{2,2,1}^l = \|x-x_0\|_\lambda^{\frac{n}{\lambda}\sum\limits_{i=1}^n \lambda_i l_i - m} \left(\delta + \|x-x_0\|_\lambda\right)^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^n \lambda_i l_i} \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\,dy \le$$

$$\leq \delta^{\alpha-n-m} \int\limits_{\sigma(x_0,\delta)} |f(y)-f(x_0)|\, dy.$$

Let $t = \dfrac{\lambda n}{|\lambda|}(m+1) - m$. It is clear that $t > 0$.

$$J^l_{2,2,2} = M_2 \left\|x - x_0\right\|_\lambda^{\frac{n}{\lambda}\sum\limits_{i=1}^{n}\lambda_i l_i - m} \sup_{\rho\in(0,\delta)} \left[\left(\rho + \left\|x-x_0\right\|_\lambda\right)^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^{n}\lambda_i l_i + t} \right.$$

$$\left. \times \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\, dy \right] \int\limits_0^\delta \left(\rho + \left\|x-x_0\right\|_\lambda\right)^{-t-1} d\rho = tM_2 \left\|x-x_0\right\|_\lambda^{\frac{n}{\lambda}\sum\limits_{i=1}^{n}\lambda_i l_i - m} \times$$

$$\times \left(\left\|x-x_0\right\|_\lambda^{-t} - \left(\delta - \left\|x-x_0\right\|_\lambda\right)^{-t} \right) \sup_{\rho\in(0,\delta)} \left[\left(\rho + \left\|x-x_0\right\|_\lambda\right)^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^{n}\lambda_i l_i + t} \times \right.$$

$$\left. \times \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\, dy \right] \leq M_3 \left(1 - \left(\frac{\left\|x-x_0\right\|_\lambda}{\delta + \left\|x-x_0\right\|_\lambda} \right)^t \right) \times$$

$$\times \sup_{\rho\in(0,\delta)} \rho^{\alpha-n-m} \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\, dy.$$

By lemma 7 there exists a set $E_2 \in R^n$ that $H_{mp}(E_2) = 0$ and for any $x_0 \in R^n \backslash E_2$ we have $J_2 \to 0$ as $\left\|x-x_0\right\|_\lambda \to 0$.

By lemma 2

$$|J_3| \leq \left\|x-x_0\right\|_\lambda^{-m} \sum_{|l|\leq m} \frac{\left|(x-x_0)^l\right|}{l!} \lim_{r\downarrow 0} \int\limits_{\sigma(x_0,b\|x-x_0\|_\lambda)\sigma(x_0,r)} \left| \frac{\partial^l}{\partial x^l} R_\alpha(x_0-y) \right| \times$$

$$\times |f(y)-f(x_0)|\, dy \leq \sum_{|l|\leq m} M_l \left\|x-x_0\right\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^{n}\lambda_i l_i - m} \times$$

$$\times \lim_{r\downarrow 0} \int\limits_{\sigma(x_0,b\|x-x_0\|_\lambda)\backslash\sigma(x_0,r)} \left\|y-x_0\right\|_\lambda^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i} |f(y)-f(x_0)|\, dy = \sum_{|l|\leq m} M_l J^l_3,$$

$$J^l_3 = \left\|x-x_0\right\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i - m} \left[\rho^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i} \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\, dy \right] \Bigg|_0^{b\|x-x_0\|_\lambda} +$$

$$+ \left(n + \frac{n}{|\lambda|}\sum_{i=1}^{n} l_i\lambda_i - \alpha \right) \left\|x-x_0\right\|_\lambda^{\frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i - m} \int\limits_0^{b\|x-x_0\|} \rho^{\alpha-n-\frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i - 1} \times$$

$$\times \int\limits_{\sigma(x_0,\rho)} |f(y)-f(x_0)|\, dy d\rho = J^l_{3,1} + J^l_{3,2}.$$

Since $|l| \leq m$ then

$$J_{3,1}^{l} = \|x - x_0\|_{\lambda}^{\frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i - m} \left[(b\|x - x_0\|_{\lambda})^{\alpha - n - \frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i} \int\limits_{\sigma(x_0,\rho)} |f(y) - f(x_0)|\, dy - \right.$$

$$\left. -\lim_{r\downarrow 0} \rho^{\alpha - n - \frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i} \int\limits_{\sigma(x_0,\rho)} |f(y) - f(x_0)|\, dy \right] =$$

$$= b^{m - \frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i} (b\|x - x_0\|_{\lambda})^{\alpha - n - m} \int\limits_{\sigma(x_0,\rho)} |f(y) - f(x_0)|\, dy \underset{\|x - x_0\|_{\lambda} \to 0}{\longrightarrow} 0,$$

$$J_{3,2}^{l} \leq M_4 \|x - x_0\|_{\lambda}^{\frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i - m} \sup_{\rho \in (0, b\|x - x_0\|_{\lambda})} \left(\rho^{-n} \int\limits_{\sigma(x_0,\rho)} |f(y) - f(x_0)|\, dy \right) \times$$

$$\times \int\limits_{0}^{b\|x - x_0\|_{\lambda}} \rho^{\alpha - \frac{n}{|\lambda|}\sum\limits_{i=1}^{n} l_i\lambda_i - 1} d\rho =$$

$$= M_5 \|x - x_0\|_{\lambda}^{\alpha - m} \sup_{\rho \in (0, b\|x - x_0\|_{\lambda})} \left(\rho^{-n} \int\limits_{\sigma(x_0,\rho)} |f(y) - f(x_0)|\, dy \right).$$

Then, it is clear that $J_{3,2}^{l} \to 0$ as $\|x - x_0\|_{\lambda} \to 0$. So $J_3 \to 0$ as $\|x - x_0\|_{\lambda} \to 0$.

Prove that a function $u(x) = \int\limits_{\sigma(x_0,1)} R_{\alpha}(x - y) dy$ is infinitely differentiable in $\sigma(x_0, 1)$. Let $x \in \sigma(x_0 1)$. Take $0 < \xi < 1$ so that $x \in \sigma(x_0, \xi)$. We also take the function φ so that it equals zero in $R^n \backslash \sigma(x_0, 1)$, equals unit in $\sigma(x_0, \xi)$ and it is infinitely differentiable on R^n. Then

$$u(x) = \int\limits_{R^n} \|x - x_0\|_{\lambda}^{\alpha - n} \varphi(y) dy + \int\limits_{\sigma(x_0,1)} \|x - y\|_{\lambda}^{\alpha - n} (1 + \varphi(y)) dy.$$

Hence, it is clear that u is infinitely differentiable at the point x so in $\sigma(x_0, 1)$.

Then

$$J_4 = f(x_0) \|x - x_0\|_{\lambda}^{-m} \left[u(x) - \sum_{|l| \leq m} \frac{\|x - x_0\|_{\lambda}^{l}}{l!} \frac{\partial^l}{\partial x^l} u(x_0) \right] \to 0, \text{ as } \|x - x_0\|_{\lambda} \to 0.$$

Take a function $g(y) = |f(y) - f(x)|$ and any $a > 0$. Then

$$J_5 = \|x - x_0\|_{\lambda}^{-m} \int\limits_{\{y: y \in \sigma(x_0, b\|x - x_0\|_{\lambda}), g(y) \leq a\}} R_{\alpha}(x - y) g(y) dy + \|x - x_0\|_{\lambda}^{-m} \times$$

$$\times \int\limits_{\{y: y \in \sigma(x_0, b\|x - x_0\|_{\lambda}), g(y) > a\}} R_{\alpha}(x - y) g(y) dy = J_{5,1} + J_{5,2},$$

$$J_{5,1} \leq aC_1 \left\| x - x_0 \right\|_\lambda^{-m} \rho^\alpha \Big|_0^{b\|x-x_0\|_\lambda} + a \left\| x - x_0 \right\|_\lambda^{-m} b \left\| x - x_0 \right\|_\lambda^{\alpha - m},$$

$$\int\limits_{\|y-x_0\|_\lambda < b\|x-x_0\|_\lambda} dy \leq C_2 a \left\| x - x_0 \right\|_\lambda^{\alpha-m} \to 0 \text{ as } \left\| x - x_0 \right\|_\lambda \to 0.$$

By lemma 6 we can write

$$J_{5,2} \leq C_2 \left\| x - x_0 \right\|_\lambda^{-m} \left(\int\limits_{\sigma(x_0, b\|x-x_0\|_\lambda} g^p(y) w(g(y)) dy \right)^{\frac{1}{p}} \left(\int\limits_0^\infty w^{-\frac{1}{p-1}}(t) t^{-1} dt \right)^{\frac{1}{p'}},$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Define a set

$$E_3 = \left\{ x \in R^n : \limsup_{r \downarrow 0} r^{-mp} \int\limits_{\sigma(x,r)} \left| f^p(y) w(f(y)) - f^p(x) w(f(x)) \right| dy > 0 \right\}.$$

It is seen from lemma 6 that $H_{mp}(E_3) = 0$.

By the condition of the theorem w is a positive non-decreasing function. Therefore $(t+s)^p w(t+s) - t^p w(t) \geq s^p w(s)$ for $t, s \geq 0$.

Then

$$\left\| x - x_0 \right\|_\lambda^{-m} \left(\int\limits_{\sigma\left(x_0, b\|x-x_0\|_\lambda\right)} g^p(y) w(g(y)) dy \right)^{\frac{1}{p}} =$$

$$= \left[\left\| x - x_0 \right\|_\lambda^{-mp} \int\limits_{\sigma(x_0, b\|x-x_0\|_\lambda)} \left| f(y) - f(x) \right|^p w\left(\left| f(y) - f(x) \right| \right) dy \right]^{\frac{1}{p}} \leq$$

$$\geq \left[\left\| x - x_0 \right\|_\lambda^{-mp} \int\limits_{\sigma(x_0, b\|x-x_0\|_\lambda)} \left| f^p(y) w(f(y)) - f^p(x) w(f(x)) \right| dy \right]^{\frac{1}{p}}.$$

Then for $x_0 \in R^n \backslash E_3$

$$\lim_{\|x-x_0\|_\lambda \to 0} J_5 < C_3 a.$$

Finally let $E = E_1 \cup E_2 \cup E_3$. It is clear that $R_\alpha f$ is totally differentiable m times in $x_0 \in R^n \backslash E$. It is seen from theorem 3.6 proved in [7] that $R_{\alpha - \frac{\bar{\lambda} n m}{|\lambda|}, p}(E) = 0$. The theorem is proved.

The author expresses his deep gratitude to acad. A.D.Gadjiev for the statement of the problem, useful advices, discussions and helpful comments.

References

[1]. O.V. Besov, V.P.Il'in, S.M.Nikolsky. *Integral representations of functions and embedding theorems.* M., Nauka, 1975. (Russian)

[2]. A.D. Gadjiev, O.Doğru. *On conbination of Riesz potentials with non-isotropic kernels.* Indian J. Pure appl. Math. 30, 1999, No 6, pp.546-556.

[3]. N.G. Meyers. *A theory of capacities for potentials of functions in Lebesgue classes.* Math. Scand. 26, 1970, No2, pp.255-292.

[4]. Y.Mizuta. *Continuity properties of Riesz potentials and boundary limits of Beppo-Levi functions.* Math.Scand. 63,1988, pp.238-260.

[5]. E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Univ.Press, Princeton, 1970.

[6]. S.K. Vodopyanov. *Maximum principle in potential theory and embedding theorem for anisotropic spaces of differentiable functions.* Sib math.J.,29, 1988, No2, pp.17-33. (Russian)

[7]. S.K. Vodopyanov. *L_p potential theory and quasiconformal mappings onto homogeneous groups. In modern problems of geometry and analysis.* Proceedings of the Inst. of Mathematics of NAS of SSSR. Siberian division. 14, 1989, pp.45-89. (Russian)

[8]. N.Wiener. *The ergodic theorem.* Duke Math. J. 5, 1939, pp.1-18.

Mubariz G. Hajibayov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 4394 720 (off.).
E-mail: mubarizh@box.az.

Received January 12, 2004; Revised May 22, 2004.
Translated by Mamedova Sh.N.