Bilal T.BILALOV, Mehmet Kh.KARAKASH

THE HAUSDORFF-YOUNG TYPE THEOREM FOR ONE SYSTEM OF EXPONENTS

Abstract

In the paper the system of exponents with linear disturbance is considered and the analogue of the Hausdorff-Young theorem is proved for this system.

Consider the following system of the exponents

$$\left\{e^{i[(n+\beta_1)t+\gamma_1]}; e^{-[(k+\beta_2)t+\gamma_2]}\right\}_{n\geq 0,k\geq 1}, \tag{1}$$

where $\beta_i = \operatorname{Re}\beta_i + i\operatorname{Im}\beta_i,\ \gamma_i = \operatorname{Re}\gamma_i + i\operatorname{Im}\gamma_i,\ \ i = 1,2;$ are complex parameters. It is easy to note that this system is a generalization of classical system of exponents, and whose particular case is the system of exponents $\left\{e^{i(n+sign\alpha)t}\right\}, n = 0, \pm 1, \ldots.$ This system is a definite disturbance of classical system of exponents therefore the study of basic properties of such systems in different functional spaces is of particular interest with relation to application in spectral theory of differential operators. In this connection the known mathematicians N. Levinson [1], N. Wiener, R. Paley [2], A.V. Bitzadze [3] and others studied the properties of this system. Generally in these investigations the completeness, basicity, uniform convergence of biorthogonal series by these systems were studied. On the other hand the Hausdorff-Young theorem is well known relative to the classical system of exponents determining the relation between biorthogonal coefficients and the corresponding function from the spaces $L_p\,(l_p)$.

In the present paper the analogous theorem relative to system (1) is obtained.

Note that earlier basicity system (1) in L_p was studied in [4]. Recall the theorem on basicity of system (1) in L_p. Let $\beta = \beta_1 + \beta_2$.

Theorem A. *System (1) forms the basis in $L_p(-\pi,\pi)$, $p \in (1,+\infty)$ (at $p = 2$ the Riesz basis), if $-\frac{1}{q} < \operatorname{Re}\beta < \frac{1}{p}, \frac{1}{q} + \frac{1}{p} = 1$. At $\operatorname{Re}\beta \leq -\frac{1}{q}$ it is complete in $L_p(-\pi,\pi)$, but not minimal; at $\operatorname{Re}\beta > \frac{1}{p}$ it isn't complete, but minimal; and at $\operatorname{Re}\beta = \frac{1}{p}$ it is complete, minimal, but the basis in $L_p(-\pi,\pi)$ doesn't form a basis. In addition in case of minimality the biorthogonal system $\left\{h_n^1(t); h_k^2(t)\right\}_{n\geq 0,k\geq 1}$ takes the form:*

$$h_n^1(t) = \frac{\left(1+e^{it}\right)^\beta}{2\pi} e^{-i(\beta_1 t+\gamma_1)} \sum_{k=0}^{n} C_{-\beta}^{n-k} e^{-ikt},\ n \geq 0;$$

$$h_n^2(t) = -\frac{\left(1+e^{it}\right)^\beta}{2\pi} e^{-i(\beta_1 t-\gamma_2)} \sum_{k=0}^{m} C_{-\beta}^{m-k} e^{ikt},\ m \geq 1$$

where $C_\beta^n = \frac{\beta(\beta-1)\cdot\ldots\cdot(\beta-n+1)}{n!}$ are binomial coefficients.

Let's give the definition of basicity of "double" $\{x_n^+; x_n^-\}_{n\geq 0}$ system in some Banach space B with the norm $\|\cdot\|$.

Definition. *The "double" system* $\{x_n^+; x_n^-\}_{n\geq 0} \subset B$ *generates the basis in* B *if for* $\forall x \in B$ *there exists a unique sequence of numbers*

$$\left\{a_n^+; a_n^-\right\}_{n\geq 0} : \lim_{N^{\pm}\to\infty} \left\| \sum_{n=0}^{N^+} a_n^+ x_n^+ + \sum_{n=0}^{N^-} a_n^- x_n^- - x \right\| = 0.$$

So, the following is true.

Theorem 1. *Let* $-\frac{1}{q} < \operatorname{Re}\beta < \frac{1}{p}$, $1 < p \leq 2$, $\frac{1}{q} + \frac{1}{p} = 1$. *Then* $\exists M > 0$*:*
1) From $f \in L_p$ *it follows that*

$$\left\|a_n^{\pm}\right\|_{l_q} \overset{def}{\equiv} \left(\sum_{k=0}^{\infty} \left|a_k^+\right|^q + \sum_{k=1}^{\infty} \left|a_k^-\right|^q\right)^{\frac{1}{q}} \leq M \left\|f\right\|_p ;$$

where $a_k^{\pm} = \int\limits_{-\pi}^{\pi} f(t) \overline{h_k^{\pm}(t)} dt$*;*

2) From $\left\{a_k^+; a_n^-\right\}_{k\geq 0, n\geq 1} \in l_p$ *it follows that* $\exists f \in L_q$*:*

$$\left\|f\right\|_q \leq M \left\|\left\{a_n^{\pm}\right\}\right\|_{l_p},$$

where $\left\|\cdot\right\|_p$ *is a norm in* L_p*.*

This theorem is a natural generalization of the Hausdorff-Young classic theorems for trigonometric systems and of F.Riesz for general, uniformly by module bounded systems.

The short scheme of proof. Let there be the case 1). The following Riemann problem of theory of analytical functions in the Hardy classes $H_p^{\pm}$ is considered:

$$\left\{ \begin{array}{l} F^+\left(e^{it}\right) + e^{-i(\beta t+\gamma)} F^-\left(e^{it}\right) = e^{-i(\beta_1 t+\gamma_1)} f(t), \\ F^-(\infty) = 0, \quad \gamma = \gamma_1 + \gamma_2, \ -\pi < t < \pi; \end{array} \right. \tag{2}$$

i.e. a pair of analytical functions $F^+(z)$; $F^-(z)$ belonging to the classes H_p^+ and H_p^-, respectively, whose non-tangent boundary values almost everywhere satisfy conditions (2) is found. The theory of such problems is detailed (see for example [5]). From assumptions of theorem it follows that this problem in noted classes has a unique solution, moreover, the biorthogonal coefficients $\{a_n^+\}_{n\geq 0}$, $\{a_n^-\}_{n\geq 1}$ of the function f by system (1) are Fourier coefficients of the functions $F^+\left(e^{it}\right)$, $F^-\left(e^{it}\right)$ respectively by the classic system of exponents, i.e.,

$$a_n^+ = \int\limits_{-\pi}^{\pi} f(t) \overline{h_n^+(t)} dt = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} F^+\left(e^{it}\right) e^{-\mathrm{int}} dt, \ n \geq 0,$$

and

$$a_n^- = \int\limits_{-\pi}^{\pi} f(t) \overline{h_n^-(t)} dt = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} F^-\left(e^{it}\right) e^{\mathrm{int}} dt, \ n \geq 1.$$

Besides from $F^+ \in H_p^+$ and $F^- \in H_p^-$ it follows that

$$a_n^+ = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} F^+\left(e^{it}\right) e^{-\mathrm{int}} dt = 0, \ n \geq -1,$$

$$a_n^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^-\left(e^{it}\right) e^{\mathrm{int}} dt = 0,\ n \leq 0.$$

Consequently by Hausdorff-Young theorem [6] we have:

$$\left\| \left\{ a_n^+ \right\}_{n\geq 0} \right\|_{l_q} \leq M_1 \left\| F^+\left(e^{it}\right)\right\|_p ,$$

$$\left\| \left\{ a_n^+ \right\}_{n\geq 0} \right\|_{l_q} \leq M_1 \left\| F^-\left(e^{it}\right)\right\|_p .$$

Denoting $a^+ \equiv \{a_n^+\}_{n\geq 0}$, $a^- \equiv \{a_n^-\}_{n\geq 0}$ and applying Minkovsky inequality we easily obtain

$$\left\| \left\{ a_n^+ ; a_k^- \right\}_{n\geq 0, k\geq 1} \right\|_{l_q} \leq \left\| \left\{ a_n^+ \right\}_{n\geq 0} \right\|_{l_q} + \left\| \left\{ a_n^- \right\}_{n\geq 1} \right\|_{l_q}$$

As a result we have:

$$\left\| \left\{ a_n^+ ; a_{n+1}^- \right\}_{n\geq 0} \right\|_{l_q} \leq M_2 \left(\left\| F^+\left(e^{it}\right)\right\|_p + \left\| F^-\left(e^{it}\right)\right\|_p \right). \tag{3}$$

Further allowing for Riesz property [7, p.113]

$$\left\| \sum_{k=0}^{m} a_k e^{ikt} \right\|_p + \left\| \sum_{k=-m}^{-1} a_k e^{ikt} \right\|_p \leq c \left\| \sum_{k=-m}^{m} a_k e^{ikt} \right\|_p$$

from (3) we obtain the required

$$\left\| \left\{ a_n^+ ; a_{n+1}^- \right\}_{n\geq 0} \right\|_{l_q} \leq M \left\| f\left(t\right)\right\|_p .$$

And now let there be the case 2). Take $\forall \left\{ a_n^+ ; a_{n+1}^- \right\}_{n\geq 0} \in l_p$. Consider the following functions

$$F^+\left(z\right) = \sum_{n\geq 0} a_n^+ z^n ; F^-\left(z\right) = \sum_{n\geq 1} a_n^- z^{-n}$$

As it follows from the results of the paper [8] the functions $F^+\left(z\right), F^-\left(z\right)$ belong to the classes H_q^+, H_q^- $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ respectively, moreover there are inequalities:

$$\left\| F^\pm \right\|_{H_q^\pm} \leq M \left\| \left\{ a_n^\pm \right\} \right\|_{l_p} \tag{4}$$

where $\left\|\cdot\right\|_{H_q^\pm}$ is a norm in $H_q^\pm$. Denote

$$f\left(t\right) \equiv e^{i\left[\beta_1 t+\gamma_1\right]} F^+\left(e^{it}\right) + e^{-\left[\beta_2 t+\gamma_2\right]} F^-\left(e^{it}\right).$$

It is obvious that $f\left(t\right) \in L_q\left(-\pi,\pi\right)$ moreover $\left\{ a_n^+ ; a_{n+1}^- \right\}_{n\geq 0}$ are biorthogonal coefficients of this function by system (1). On the other hand allowing for (4) we have:

$$\left\| f \right\|_q \leq M_1 \left(\left\| F^+\left(e^{it}\right)\right\|_q + \left\| F^-\left(e^{it}\right)\right\|_q \right) \leq M_2 \left\| F^+\left(e^{it}\right) + F^-\left(e^{it}\right)\right\|_q$$

Here we again apply the Riesz property. Further the required result follows from the Hausdorff-Young theorem for trigonometric system.

The theorem is proved.

From this theorem the following corollary follows.

Corollary. *Let* $-\frac{1}{2q} < \alpha < \frac{1}{2p}$, $1 < p \leq 2$, $\frac{1}{q} + \frac{1}{p} = 1$.. *Then* $\exists M > 0$:

1) From $f \in L_p$ *it follows that*

$$\left\| \left\{ a_n^+ ; a_n^- \right\} \right\|_{l_p} \leq M \left\| f \right\|_p ;$$

2) From $\left\{ a_n^+ ; a_{n+1}^- \right\}_{n \geq 0} \in l_p$ *it follows that* $\exists f \in L_q$*:*

$$\left\| f \right\|_q \leq M \left\| \left\{ a_n^+ ; a_{n+1}^- \right\} \right\|_{l_p},$$

where $\left\{ a_n^+ ; a_{n+1}^- \right\}_{n \geq 0}$ *are biorthogonal coefficients of the function* $f(t)$ *by the system* $\left\{ e^{i(n + sign n \cdot \alpha)t} \right\}_{-\infty}^{+\infty}$.

References

[1]. Levinson N. ,*Gap and density theorems.* New York, Publ. Amer. Math. Soc., 1940, 246p.

[2]. Wiener N., Paley R. *Fourier transformation in a complex domain.* M.: "Nauka", 1964. (Russian)

[3]. Bitsadze A.V. On one system of functions. UMN, 1950, v.5, No4(38), pp.150-151. (Russian)

[4]. Bilalov B.T. *Basicity of some exponents, cosines and sines systems.* Differ. Uravneniya, 1990, v.26, No1, pp.10-16. (Russian)

[5]. Danilyuk I.I. *Nonregular boundary value problems on plane.* M.: "Nauka", 1975. (Russian)

[6]. Kachmazh S., Steinhous G. *Theory of orthogonal series.* M.: GUFML, 1958, 508p. (Russian)

[7]. Garnett J. *Bounded analytic functions.* M.: "Mir", 1984. (Russian)

[8]. Privalov I.I. *Boundary properties of analytical functions.* M.-L., 1950. (Russian)

Bilal T. Bilalov
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 498 91 90 (off.)
E-mail: b_bilalov@mail.ru

Mehmet Karakash Kh.
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 498 91 90 (off.)

Received December 21, 2003; Revised April 22, 2004.
Translated by Mamedova V.A.