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COMPLEXITY AND ENTROPY OF COUNTABLE AMENABEL GROUPS ACTIONS

Abstract

We consider an ergodic measure-preserving action T of a countable amenabel group G on a standart probability space (X, M, μ). To a individual trajectory $\{T_g x\}_{g\in G}$ of some point $x \in X$ we put a correspondence a non-negative real number characterizing a degree (power) of complexity of a behaviour of this trajectory which is called as a trajectorial complexity of the point and investigate a connection of this notion with an entropy $h(T)$ of the action T with related to measure μ. Our goal is to demonstrate what is known for actions of Z that the trajectorial complexities of μ-a.e. points of X coincide with the entropy of action T.

1.Introduction

There are a lot of notions (such as wandering and non-wandering, periodic and recurrent and etc.) characterizing the variety of the behaviours of the individual trajectories of the ergodic dynamical systems. And it's a natural interest to determine a quantity tool dividing the individual trajectories into "a simple" and "a complex" parts related to the behaviours of these trajectories. It's also obvious that this notion must have a connection with an entropy which is considered as a measure of complexity and chaoticness of the dynamical system in whole.

After introducing a notion of complexity of a finite object, due to A.Kolmogorov [1] many authors tried to give the different variants of this quantity characteristics. For instance, V.M.Alekseev introduced a notion of "quasi-random" dynamical system [2], T.Kamae gave a definition of determinated trajectory and etc. In the abovementioned works a simple, determinated character of a behaviour of a trajectory was opposed to a complex, "quasi-random".

Mathematically strict definition of this notion called as a trajectorial complexity for an action of Z basing on the symbolic dynamics ideas from one hand and a notion of Kolmogorov complexity from other was given in works of A.Levin and A.A.Brudno [3, 4]. In [5] A.A.Brudno thoroughly investigated the connections of this notion with such notions of ergodic dynamical systems theory as an entropy, a partition into the ergodic sets and etc.

In [6] A.T.Tagi-zade gave an approach for construction the complexity notion in case of non-abelian groups actions.

The goal of this work is to lift a part of the results of A.A.Brudno and A.T.Tagi-zade to the actions of countable amenabel groups on a standart probability space, in particular to show that for such actions the trajectorial complexities of a.e. points coincide with the entropy. In our proof we essentially rely on a description of a geometric structure of amenabel groups known as the Ornstein-Weiss quasi-tiling theorem [7]. Another reason why we chosed the countable amenabel groups actions is the existence of the good analogue of the entropy theory, the Birkhoff mean ergodic thorem and the Shannon-McMillan theorem of Z actions for them.

In the next section we give the basic notions, denotions and results using in the work and introduce a trajectorial complexity for a countable amenabel group action. In section 3 we formulate and prove our main result.

2. Definitions and the used results

We say G is *a countable amenabel group* if it possesses a left-invariant Folner sequence of sets $F = \{F_n\}$, i.e. a sequence of the embeded each other finite sets $\{id\} \in F_1 \subset F_2 \subset ... \subset F_n \nearrow G$ such that

$$\lim_{n\to\infty} \frac{|gF_n \cap F_n|}{|F_n|} = 1 \quad \text{for all} \quad g \in G$$

where $|C|$ is the cardinality of C.

Here and further all definitions and statements depending on the left-invariant Folner sequences are also true for the right-invariant Folner sequences, i.e. finite sets $\{id\} \in F_1 \subset F_2 \subset ... \subset F_n \nearrow G$ such that

$$\lim_{n\to\infty} \frac{|F_n g \cap F_n|}{|F_n|} = 1 \quad \text{for all} \quad g \in G$$

A finite list of sets $H_1 \subset ... \subset H_k \subseteq G$ with $id \in H_1$ is said to ε-*quasi-tile* a finite set $D \subseteq G$ if there exists "centers" $C_i \subseteq G$, $i = 1, ..., k$ such that

(i) the $\{H_i c : c \in C_i\}$ are ε-disjoint for $i = 1, ..., k$.

(ii) the $H_i C_i$ are disjoint for $i = 1, ..., k$.

(iii) $\left|(\bigcup_{i=1}^{k} H_i C_i) \cap D\right| > (1-\varepsilon) \cdot |D|$.

Recall that a couple A, B is called ε-disjoint, if there exist $\widetilde{A} \subset A$, $\widetilde{B} \subset B$ such that $\widetilde{A} \cap \widetilde{B} = \emptyset$ and $\left|\widetilde{A}\right| > (1-\varepsilon) \cdot |A|$, $\left|\widetilde{B}\right| > (1-\varepsilon) \cdot |B|$.

To say that a set H is *sufficiently invariant* means that for some (unspecified) $\delta > 0$ and finite $K \subseteq G$, H is $(K, \delta)-$invariant, i.e.

$$\left|KK^{-1}H \cap H\right| > (1-\delta) \cdot |H|$$

To say that a list of sets $H_1, ...H_k$ is *sufficiently invariant* to say that for some $\delta > 0$ and finite set $K \subseteq G$, setting $H_0 = K$, for each $j \in \{1, ..., k\}$ the set H_j is (H_{j-1}, δ)-invariant.

The existence of $(K, \delta)-$invariant sets for all $\delta > 0$ and finite sets K is equivalent to the amenability of the countable group.

We now present the Ornstein-Weiss quasi-tiling theorem [7].

Theorem 1. *Given $\varepsilon > 0$, there exists $l = l(\varepsilon)$such that in any countable amenabel group G, if $H_1, ...H_l$ is any sufficiently invariant list of sets, then for any*

$D \subseteq G$ that is sufficiently invariant (depending on the choice of $H_1, ...H_l$), D can be quasi-tiled by $H_1, ...H_l$.

Throughout the work (X, M, μ) will denote a probability space. By *a measurable action* $T = \{T_g\}_{g \in G}$ of G on X we mean a mapping from $G \times X \to X$ such that

(i) $T_{gh}x = T_g(T_h x)$ for all $g, h \in G, x \in X$ (an action)

(ii) $(T_g)^{-1}B \in M$ for all $B \in M$ (measurability).

A measurable action is *measure-preserving* if for each $g \in G$ $T_g : X \to X$ is measure-preserving, i.e. $\mu((T_g)^{-1}B) = \mu(B)$ for all $B \in M$.

We say that an action is *ergodic* if the only sets $B \in M$ that satisfy $(T_g)^{-1}B = B$ have a measure 0 or1.

We now present the mean ergodic theorem for the actions of countable amenabel groups [8].

Theorem 2. *For an ergodic measure-preserving action $T = \{T_g\}_{g \in G}$ of countable amenabel group G, a real-valued function $f \in L_1(X)$, a left-invariant Folner sequence $\digamma = \{F_n\}$ and μ-a.e. $x \in X$*

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x) = \int_X f(x) d\mu$$

For a measure-preserving action $T = \{T_g\}_{g \in G}$ of countable amenabel group G, a left-invariant Folner sequence $\digamma = \{F_n\}$ and a finite measurable partition $\beta = \{B_i : i \in I\}$ of X the entropy of action T we define as

$$h(T) = \sup_\beta h(T, \beta) \overset{def}{=} \sup_\beta \lim_{n \to \infty} \frac{1}{F_n} H(\underset{g \in F_n}{\vee} T_{g^{-1}}(\beta))$$

where $\underset{g \in F_n}{\vee} T_{g^{-1}}(\beta) = \{\underset{g \in F_n}{\cap} T_{g^{-1}} B_{i(g)} \neq \oslash : B_{i(g)} \in \beta\}$ and $H(\beta) = -\sum_{i \in I} \mu(B_i) \cdot \log \mu(B_i)$.

Note that this limit exists and is independent of the choice of Folner sequence [8].

For a countable group G let

$$\Lambda = \{\omega = \{\omega(g) : g \in G\} : \omega(g) \in I\}$$

be a sequence space equipped with a product topology, where I is a finite alphabet, σ be an action of G on Λ with the right-shifts

$$(\sigma_g \omega)_h = \omega_{hg} \text{ for all } g, h \in G$$

For a finite measurable partition $\beta = \{B_i : i \in I\}$ of X we set an one-to-one transformation $\varphi_\beta : X \to \Lambda$ as

$$\varphi_\beta(x) = \{\omega(g) : g \in G\} \text{ such that } \mathrm{T}_g x \in B_{\omega(g)} \text{ for all } g \in G$$

and consider the invariant (related to σ) closed subset $\Lambda_\beta = \varphi_\beta(X) = \{\omega \in \Lambda :$

$\bigcap\limits_{g\in G}(T_g)^{-1}B_{\omega(g)} \neq \oslash\}$ of Λ equipped with σ-algebra of the cylindrical sets $C(\omega^H) = \{\omega_1 = \{\omega_1(g) : g \in G\} : \omega_1(g) = \omega(g)$ for all $g \in H\}$, where $\omega^H = \{\omega(g) : g \in H\}$ is a finite word from alphabet I.

It's known, that a measure defined on the cylinders $\{C(\omega^H) : \omega \in \Lambda_\beta\}$ as $\nu(C(\omega^H)) = \mu(\bigcap\limits_{g\in H}(T_g)^{-1}B_{\omega(g)})$ can be extended to a σ-invariant probability measure μ_β in the space Λ_β.

Now we present the Shannon-McMillan theorem [8].

Theorem 3. *For any $\varepsilon > 0$ if $H \subseteq G$ is sufficiently invariant then among the sets*

$\{\omega^H : \omega \in \Lambda_\beta\}$ is a collection Γ with

(i) $\mu_\beta(\bigcup\limits_{\omega^H\in\Gamma} C(\omega^H)) > 1 - \varepsilon$;

(ii) $|\Gamma| < 2^{(h(T,\beta)+\varepsilon))\cdot|H|}$;

(iii) for any $\omega^H \in \Gamma$, $\mu_\beta(C(\omega^H)) < 2^{-(h(T,\beta)-\varepsilon)\cdot|H|}$.

Now we begin to introduce a notion of trajectorial complexity for countable amenabel groups actions.

Let A be an algorithm defined on some subset of a space of all finite $0-1$ words and taking values in the set of all finite words of Λ_β and $l(p)$ be an amount of signs in a $0-1$ word p. Following the definition of the Kolmogorov asymptotic complexity [1] we define the complexity $K_A(\omega^H)$ of finite word ω^H related to algorithm A as

$$K_A(\omega^H) = \left\{ \begin{array}{ll} \inf\limits_p\{l(p) : A(p) = \omega^H\}, & \text{if } \{p : A(p) = \omega^H\} \neq \oslash \\ \infty & , \text{ if otherwise} \end{array} \right\}$$

In [1] the existence of such algorithm B called as optimal that for any algorithm A there exists constant C_{AB} with $K_B(\omega^H) \leq K_A(\omega^H) + C_{AB}$ for all ω^H was proved.

Definition 4. *The quantity*

$$K_A(\omega) = \sup_{\digamma} K_A^{\digamma}(\omega) \stackrel{def}{=} \sup_{\digamma} \overline{\lim_{n\to\infty}} \frac{1}{|F_n|} \cdot K_A(\omega^{F_n})$$

where sup *is taken over all left-invariant Folner sequences $\digamma = \{F_n\}$ in the countable amenabel group G, is called a complexity of sequence $\omega \in \Lambda_\beta$ is related to algorithm A.*

For two optimal algorithsms A and B taking into account the following inequality

$$\frac{1}{|F_n|} \cdot \left|K_A(\omega^{F_n}) - K_B(\omega^{F_n})\right| \leq \frac{1}{|F_n|} \cdot \max(C_{AB}, C_{BA}) \text{ for all } \omega^{F_n}$$

we get that the complexity of sequence doesn't depend on a choice of optimal algorithm. So further we will fix one optimal algorithm and the complexity of sequence ω will denote as $K(\omega)$ (and consequently $K^{\digamma}(\omega)$).

Definition 5. *For a finite measurable partition β of X the quantity*

$$K(x, T \mid \beta) = K(\varphi_\beta(x))$$

is called a trajectorial complexity of a point $x \in X$ is related to β.

3. The main result

Here is the goal of our work:

Theorem 6. *For an ergodic measure-preserving action T of a countable amenabel group G on a probability space (X, M, μ), any finite measurable partition $\beta = \{B_i : i \in I\}$ of X and μ-a.e. points $x \in X$*

$$K(x, T \mid \beta) = h(T, \beta)$$

To demonstrate Theorem 6 it's sufficient to get an analogous result on a symbolic system (Λ_β, σ) with a measure μ_β, i.e. to prove

Theorem 7. *For μ_β-a.e. sequences $\omega \in \Lambda_\beta$*

$$K(\omega) = h(\sigma)$$

Really, having in hands Theorem 7, taking into account the definitions of the trajectorial complexity and the complexity of sequence and from the known result about the coincidence of entropies $h(T, \beta) = h(\sigma)$ [9] we get Theorem 6.

The proof of Theorem 7. we carry out in two parts. At first we will prove

Lemma 8. *For μ_β-a.e. sequences $\omega \in \Lambda_\beta$*

$$K(\omega) \leq h(\sigma)$$

Proof. Using Theorem 1 we choose the special left-invariant Folner sequences in G. Fix a listing of its elements $g_1, g_2, ... = G$, any $\varepsilon > 0, \delta_1 > 0$ and let $K_1 = \{g_1\}$. By Theorem 1 for this ε there exists a positive integer $l = l(\varepsilon)$ that a list of sufficiently invariant for (K_1, δ_1) sets $H_1^1, ..., H_l^1$ quasi-tile a finite sufficiently invariant for (H_l^1, δ_1) set F_1. Now let $K_2 = F_1 \cup \{g_2\}$ and $\delta_2 = \min\{\delta_1; \frac{1}{F_1}\}$. Analogously, a list of sufficiently invariant for (K_2, δ_2) sets $H_1^2, ..., H_l^2$ quasi-tile a finite sufficiently invariant for (H_l^2, δ_2) set F_2.

With this procedure we construct the left-invariant Folner sequences $\mathcal{F} = \{F_n\}_{n=1}^{\infty}, \mathcal{F}_m = \{H_m^n)\}_{n=1}^{\infty}, m = 1, ...l$ such that for each fix n the sets $H_1^n, ..., H_l^n$, $C_1^n, ..., C_l^n, F_n$ satisfy the conditions *(i)-(iii)* of the definition of quasi-tiling.

Now we will construct an algorithm A defined on a special subset of all finite $0-1$ words in the form of $p = p_1...p_{l+1}$, when the $0-1$ words $p_m, m = 1, 2, ..., l+1$ will be specified below, and taking values in the set of finite words $\{\omega^{F_n} : \omega \in \Lambda_\beta\}$.

We fix the listings of elements $g_1^m, ..., g_{|C_m^n|}^m = C_m^n$, $m = 1, ..., l$ and an arbitrary sequence $\omega \in \Lambda_\beta$.

Let's denote by A_m, $m = 1, ..., l$ the algorithsms to all finite $0-1$ words putting the correspondences accordingly the finite words $\omega^{H_m^n}$, i.e. such that $A_m(p) = \omega^{H_m^n}$, $m = 1, ..., l$.

The algorithm A we define by the following way:

For fix $m \in \{1, ..., l\}$ we write out the finite word $A_m(p)$, act on it by the right-shift $\sigma_{q_1^m}$ which the finite word $\omega^{H_m^n}$ transfers to finite word $\omega^{H_m^n q_1^m}$. Then once more write out $A_m(p)$, act on it by $\sigma_{q_2^m}$, erase the elements hit on the intersection with the set on which the previous configuration was defined and unite the obtained configurations. Analogously act until $|C_m^n|$-th step. Continuing this algorithm for all m, $m = 1,...,l$, we have got the finite words $\omega^{H_m^n C_m^n}, m = 1,...,l$. At last we unite these words (by the condition *(ii)* of quasi-tiling it can be done) and fill in the empty places with arbitrary way.

Let p_m, $m = 1,...,l$ be the minimal $0-1$ words corresponding accordingly to the finite words $\omega^{H_m^n C_m^n}, m = 1,...,l$ and p_{m+1} be a $0-1$ word corresponding to a configuration on the empty places. By *(iii)* of quasi-tiling the amount of empty places doesn't exceed $\varepsilon \cdot |F_n|$ and so to each configuration on the empty places we can put a correspondence a $0-1$ word with not more than $\left[\log(cardI)^{\varepsilon \cdot |F_n|}\right] + 1$ signs, where $[\alpha]$ is an integer part of α.

From this definition of algorithm A we get that for any $\omega \in \Lambda_\beta$ can be found a $0-1$ word p that

$$A(p) = \omega^{F_n}$$

with

$$K_A(\omega^{F_n}) \leq \sum_{m=1}^{l} l(p_m) + \left[\log(cardI)^{\varepsilon \cdot |F_n|}\right] + 1$$

and so

$$K^F(\omega) \leq K_A^F(\omega) = \overline{\lim_{n\to\infty}} \frac{1}{|F_n|} \cdot \sum_{m=1}^{l} l(p_m) + \varepsilon \cdot \log(cardI)$$

It remains to calculate the lengthes of $0-1$ words p_m, $m = 1,...,l$. At first for a fixed integer $m \in \{1,...,l\}$, an element $g_0 \in H_m^n$ and a sequence $\omega \in \Lambda_\beta$ we will find the minimal account of information necessary for restoration of finite word $\omega^{H_m^n C_m^n g_0}$. This word we write in the following form:

$$\omega^{H_m^n C_m^n g_0} = \omega^{H_m^n q_1^n g_0} ... \omega^{H_m^n q_{|C_m^n|}^n g_0}$$

Let M be an account of all possible words of alphabet I with length $|H_m^n|$. Number them arbitrarily and let j-th of them meets among $\omega^{H_m^n q_i^n g_0}$, $i = 1,..,|C_m^n|$ exactly s_j time. It's obvious that

$$\sum_{j=1}^{M} s_j = |C_m^n| \tag{1}$$

Denote by $p(\omega^{H_m^n C_m^n g_0})$ a natural logarithm of the account of words $\{\omega_1^{H_m^n C_m^n g_0} : \omega_1 \in \Lambda_\beta\}$ having the same collection $\{s_j : j = 1,...,M\}$ that and so $\omega^{H_m^n C_m^n g_0}$. From the known example of combinatorics

$$p(\omega^{H_m^n C_m^n g_0}) = \ln \frac{|C_m^n|!}{s_1! \cdot ... \cdot s_M!}$$

Let $NM(\omega^{H_m^n C_m^n g_0})$ be a number of $\omega^{H_m^n C_m^n g_0}$ among all such $\omega_1^{H_m^n C_m^n g_0}$.

So a finite word $\omega^{H_m^n C_m^n g_0}$ is restored synonymously by the collection of integers $\{|H_m^n|, |C_m^n|, s_1, ..., s_M, NM(\omega^{H_m^n C_m^n g_0})\}$ and analogously

$$l(\widetilde{p_m}) \leq l(|H_m^n|) + l(|C_m^n|) + \sum_{j=1}^{M} l(s_j) + l(NM(\omega^{H_m^n C_m^n g_0}))$$

where in the brackets at right side of inequality we understand $0-1$ words corresponding these integers by the natural way $p \longleftrightarrow k$, $l(p) \leq [\log k]+1$ and $l(\widetilde{p_m}) = K(\omega^{H_m^n C_m^n g_0})$.

By this correspondence and the obvious inequality

$$\log NM(\omega^{H_m^n C_m^n g_0}) \leq p(\omega^{H_m^n C_m^n g_0})$$

we have

$$l(\widetilde{p_m}) \leq \log(|H_m^n| \cdot |C_m^n|) + \sum_{j=1}^{M} l(s_j) + p(\omega^{H_m^n C_m^n g_0}) + const$$

and taking into account (1) and $H_m^n C_m^n \subset F_n$ we can write

$$\lim_{n\to\infty} \frac{1}{F_n} \cdot \left\{ \log(|H_m^n| \cdot |C_m^n|) + \sum_{j=1}^{M} l(s_j) + p(\omega^{H_m^n C_m^n g_0}) + const \right\} = 0$$

Using the Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1+\overline{o}(1))$$

we obtain

$$p(\omega^{H_m^n C_m^n g_0}) \leq -|C_m^n| \cdot \sum_{j=1}^{M} \frac{s_j}{|C_m^n|} \log \frac{s_j}{|C_m^n|}$$

Let's denote by

$$\beta^{H_m^n} = \left\{ C(\omega^{H_m^n} : \omega \in \Lambda_\beta \right\}$$

finite measurable partition of Λ_β into all possible cylindrical sets on the place H_m^n. Since the number of such cylinders is equal to M the elements of this partition we number in the same order as we did earlier with the configurations of length $|H_m^n|$:

$$\beta^{H_m^n} = \{B_1, ..., B_M\}$$

We take any $B_j \in \beta^{H_m^n}$ and consider the shift $\sigma_{g_0 q}^{-1}$, $q \in C_m^n$. Then

$$\sum_{q\in C_m^n} \chi_{B_j}(\sigma_{g_0 q}^{-1}\omega)$$

(where χ_B is the characteristic function of B) will mean the number of finite words among $\omega^{H_m^n q s g_0}$, $s = 1, ..., |C_m^n|$ which belong to B_j or otherwords is equal to s_j, i.e.

$$s_j = \sum_{q\in C_m^n} \chi_{B_j}(\sigma_{g_0 q}^{-1}\omega)$$

Summing up this equality on $g_0 \in H_m^n$ we have

$$\frac{s_j}{|C_m^n|} = \frac{1}{|H_m^n|} \cdot \frac{1}{|C_m^n|} \cdot \sum_{g_0 \in H_m^n} \sum_{q \in C_m^n} \chi_{B_j} \left(\sigma_{g_0 q}^{-1} \omega\right)$$

By *(i)* of quasi-tiling there exists the set $\widetilde{H_m^n} \subset H_m^n$ with $\left|\widetilde{H_m^n}\right| > (1-\varepsilon) \cdot |H_m^n|$ such that the sets $\left\{\widetilde{H_m^n} c : c \in C_m^n\right\}$are disjoint and so

$$\frac{s_j}{|C_m^n|} = \frac{1}{|H_m^n|} \cdot \frac{1}{|C_m^n|} \cdot \sum_{g_0 \in \widetilde{H_m^n}} \sum_{q \in C_m^n} \chi_{B_j} \left(\sigma_{g_0 q}^{-1} \omega\right) +$$

$$\frac{1}{|H_m^n|} \cdot \frac{1}{|C_m^n|} \cdot \sum_{g_0 \in H_m^n \setminus \widetilde{H_m^n}} \sum_{q \in C_m^n} \chi_{B_j} \left(\sigma_{g_0 q}^{-1} \omega\right) =$$

$$\frac{|H_m^n C_m^n|}{|H_m^n| \cdot |C_m^n|} \cdot \frac{1}{|H_m^n C_m^n|} \sum_{h \in H_m^n C_m^n} \chi_{B_j} \left(\sigma_h^{-1} \omega\right) +$$

$$\frac{1}{|H_m^n|} \cdot \frac{1}{|C_m^n|} \cdot \left\{ \sum_{g_0 \in H_m^n \setminus \widetilde{H_m^n}} \sum_{q \in C_m^n} \chi_{B_j} \left(\sigma_{g_0 q}^{-1} \omega\right) - \sum_{h \in H_m^n \setminus \widetilde{H_m^n} C_m^n} \right\} \chi_{B_j} \left(\sigma_h^{-1} \omega\right) \tag{2}$$

For the left-invariant Folner sequence $\{H_m^n C_m^n\}_{n=1}^{\infty}$ by virtue of Theorem 2 we have that for μ_β-a.e. sequences $\omega \in \Lambda_\beta$

$$\lim_{n\to\infty} \frac{1}{|H_m^n C_m^n|} \cdot \sum_{h \in H_m^n C_m^n} \chi_{B_j} \left(\sigma_h^{-1} \omega\right) = \mu_\beta \left(B_j\right)$$

Also using the equivalent equalities

$$\lim_{n\to\infty} \frac{|H_m^n C_m^n|}{|H_m^n| \cdot |C_m^n|} = 1 \text{ and } \lim_{n\to\infty} \frac{\left|H_m^n \setminus \widetilde{H_m^n}\right|}{|H_m^n|} = 0$$

(because otherwise it contradicts to the arbitrarity and not depending on n of ε in the quasi-tiling theorem) and taking into consideration that the maximal value of second addendum in (2) is equal to $\frac{\left|H_m^n \setminus \widetilde{H_m^n}\right| \cdot (|C_m^n| - 1)}{|H_m^n| \cdot |C_m^n|}$ we have

$$\lim_{n\to\infty} \frac{s_j}{|C_m^n|} = \mu_\beta \left(B_j\right)$$

and so

$$\lim_{n\to\infty} \frac{1}{|H_m^n|} \cdot \left\{ -|C_m^n| \cdot \sum_{j=1}^{M} \frac{s_j}{|C_m^n|} \log \frac{s_j}{|C_m^n|} \right\} = h(\sigma)$$

Note that here we take into account that $\{H_m^n C_m^n\}_{n=1}^{\infty}$ is the left-invariant Folner sequence, $\beta = \left\{ \left[{1 \atop id}\right], ..., \left[{I \atop id}\right] \right\}$ is the generating partition of Λ_β and so $h(\sigma) = h(\sigma, \beta)$.

(Recall that the finite measurable partition β is the generating if $diam\beta^{F_n} \underset{n\to\infty}{\to} 0$ for any Folner sequence $\{F_n\}$).

Thus we found the estimation of $l(\widetilde{p_m})$

$$l(\widetilde{p_m}) \leq |H_m^n| \cdot |C_m^n| \cdot (h(\sigma) + \gamma_n) + \widetilde{\gamma_n}$$

where γ_n and $\widetilde{\gamma_n}$ converge to zero as $n \to \infty$, i.e. the minimal account of information necessary for restoration of finite word $\omega^{H_m^n C_m^n g_0}$ doesn't exceed the quantity $|H_m^n| \cdot |C_m^n| \cdot (h(\sigma) + \gamma_n) + \widetilde{\gamma_n}$.

As

$$\omega^{H_m^n C_m^n} = \sigma_{g_0^{-1}} \omega^{H_m^n C_m^n g_0}$$

then the analogous result is also true for the finite word $\omega^{H_m^n C_m^n}$. Really, if we define an algorithm B by the following way:

1. Write out the finite word $\omega^{H_m^n C_m^n g_0}$
2. Act on it by the shift $\sigma_{g_0^{-1}}$

then we have

$$K_{B(}\omega^{H_m^n C_m^n}) \leq l(\widetilde{p_m})$$

which give us

$$l(p_m) \leq l(\widetilde{p_m}) \leq |H_m^n| \cdot |C_m^n| \cdot (h(\sigma) + \gamma_n) + \widetilde{\gamma_n}$$

Since this equality is true for all m, $m = 1, ..., l$ we have at last

$$K^F(\omega) \leq \overline{\lim_{n\to\infty}} \frac{1}{|F_n|} \cdot \sum_{m=1}^{l} \{|H_m^n| \cdot |C_m^n| \cdot (h(\sigma) + \gamma_n) + \widetilde{\gamma_n}\} + \varepsilon \cdot \log(cardI)$$

and using *(iii)* of quasi-tiling

$$K^F(\omega) \leq h(\sigma) + \varepsilon \cdot \log(cardI)$$

Taking into account the arbitrarity of ε and independence of the right-side on F we have that for μ_β-a.e. sequences $\omega \in \Lambda_\beta$

$$K(\omega) \leq h(\sigma)$$

Our last step is to prove an opposite inequality.

Lemma 9. *For μ_β-a.e. sequences $\omega \in \Lambda_\beta$*

$$h(\sigma) \leq K(\omega)$$

Proof It will be carried out by contrarity method. Suppose that the set $Q = \{\omega \in \Lambda_\beta : K(\omega) < h(\sigma)\}$ has a positive measure. The set is a measurable and invariant. Measurability of Q is obvious from the decomposition

$$Q = \bigcup_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n>N} \left\{\omega \in \Lambda_\beta : K(\omega^{F_n}) < (h(\sigma) - \frac{1}{k}) \cdot |F_n|\right\}$$

where $\{F_n\}_{n=1}^{\infty}$ is an arbitrary left-invariant Folner sequence, taking into account that for fix k and n the set in brackets is the union of finite number of cylindrical sets.

Now, we prove the invariantness of Q. In reality, we prove the stronger result that is the complexities of sequences lying on the same trajectory coincide. In other words, if $\omega_1 = \sigma_g \omega_2$ for any $g \in G$, then $K(\omega_1) = K(\omega_2)$. For this purpose, we fix the sequences $\omega_1, \omega_2 \in \Lambda_\beta$, the element $g \in G$ satisfying the above corrolation and the right-invariant Folner sequence $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$. We will prove at first the inequality $K(\omega_1) \le K(\omega_2)$. Let A be an optimal algorithm for which there is a $0-1$ word p with

$$A(p) = \omega_2^{F_n \cup F_n g^{-1}}$$

We define an algorithm $A_1 = A_1(p)$ taking values in the set $\left\{\omega_1^{F_n} : \omega \in \Lambda_\beta\right\}$ by the following rule:

1. Write out the finite word $A(p)$
2. Act on it by the shift σ_g
3. Finally, the algorithm A_1 erases the elements going out of its value set.

So writing out the finite word $A(p)$, executing consequently the steps 2 and 3, taking into account $\omega_1 = \sigma_g \omega_2$ we get $A_1(p) = \omega_1^{F_n}$. Hence we have

$$K_{A_1}(\omega_1^{F_n}) \le K_A(\omega_2^{F_n \cup F_n g^{-1}})$$

Therefore

$$K^{\mathcal{F}}(\omega_1) \le K^{\mathcal{F}}_{A_1}(\omega_1) = \overline{\lim_{n\to\infty}} |F_n|^{-1} \cdot K_{A_1}(\omega_1^{F_n}) \le \overline{\lim_{n\to\infty}} |F_n|^{-1} \cdot K_A(\omega_2^{F_n \cup F_n g^{-1}}) =$$

$$\overline{\lim_{n\to\infty}} |F_n|^{-1} \cdot \left|F_n \cup F_n g^{-1}\right| \cdot \left|F_n \cup F_n g^{-1}\right|^{-1} \cdot K_A(\omega_2^{F_n \cup F_n g^{-1}}) = K^{\mathcal{F}_1}(\omega_2) \le K(\omega_2)$$

where $\mathcal{F}_1 = \left\{F_n \cup F_n g^{-1}\right\}$ is also the right-invariant Folner sequence by virtue of definition of $\mathcal{F}$.

In view of arbitrarity of choosing $\mathcal{F}$

$$K(\omega_1) \le K(\omega_2)$$

The opposite inequality is proved analogously. Thus the invariantness of Q is proved.

By force of measurabiliy and invariantness of Q and ergodicity of the measure μ we get $\mu(Q) = 1$. For the following decomposition

$$Q = \bigcup_{r\in Z^+} \bigcup_{k=1}^{\infty} Q_{r,k} \overset{def}{=} \bigcup_r \bigcup_k \left\{\omega \in \Lambda_\beta : K(\omega^{F_n}) < (h(\sigma) - \frac{1}{r}) \cdot |F_n| \text{ for all } n > k\right\}$$

exist such R and K that for $k > K$ we have

$$\mu_\beta(Q_{R,k}) > 1 - \delta \quad (\delta > 0 \text{ is arbitrary})$$

Let $\varepsilon < \min\left\{\frac{1}{R}, 1-\delta\right\}$, the number $N = N(\varepsilon)$, the collection $\Gamma = \left\{\omega^{F_k} : \ \omega \in \Lambda_\beta\right\}$ $(k > N(\varepsilon))$ satisfy the Shannon-McMillan theorem. Put

$$Q^{\Gamma}_{R,k} = Q_{R,k} \cap \Gamma \text{ and } Q^{\Lambda_\beta \backslash \Gamma}_{R,k} = Q_{R,k} \cap (\Lambda_\beta \backslash \Gamma)$$

As $Q^{\Lambda_\beta \backslash \Gamma}_{R,k} \subseteq \Lambda_\beta \backslash \Gamma$ then for all $k > \max\{K; N(\varepsilon)\}$

$$\mu_\beta(Q^{\Lambda_\beta \backslash \Gamma}_{R,k}) \leq \mu_\beta(\Lambda_\beta \backslash \Gamma) < \varepsilon$$

and

$$\mu_\beta(Q^{\Gamma}_{R,k}) = \mu_\beta(Q_{R,k}) - \mu_\beta(Q^{\Lambda_\beta \backslash \Gamma}_{R,k}) > 1 - \delta - \varepsilon > 0 \tag{3}$$

On the other hand, if $\omega^{F_k} \in Q^{\Gamma}_{R,k}$ then

$$K(\omega^{F_k}) \leq |F_k| \cdot (h(\sigma) - \frac{1}{R})$$

hence

$$card\left\{\omega^{F_k} \in \Lambda^{F_k}_\beta : \omega^{F_k} \in Q^{\Gamma}_{R,k}\right\} \leq 2^{|F_k| \cdot (h(\sigma) - \frac{1}{R}) + 1}$$

Moreover, if $\omega^{F_k} \in Q^{\Gamma}_{R,k}$, then $\omega^{F_k} \in \Gamma$ and by force of latter inequality

$$\mu_\beta(Q^{\Gamma}_{R,k}) \leq 2^{|F_k| \cdot (h(\sigma) - \frac{1}{R}) + 1} \cdot 2^{-|F_k| \cdot (h(\sigma) - \varepsilon)} = 2^{|F_k| \cdot (\varepsilon - \frac{1}{R}) + 1}$$

Consequently,

$$\lim_{k \to \infty} \mu_\beta(Q^{\Gamma}_{R,k}) = 0$$

that contradicts (3).

The obtained contradiction completes the lemma 3.4.

Remark 10. *The aboveused method gives an opportunity to obtain analogous results in case of local-compact unimodular amenabel groups actions.*

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