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COERCIVE PROPERTIES OF ANISOTROPIC DIFFERENTIAL-OPERATOR EQUATIONS

Abstract

The coercive solvability o boundary value problems for uniformly elliptic equation in bounded domains was considered in different papers.

In the present paper we study some class of differential-operator equations defined in R^n and having different derivatives by different variables it is main part, moreover, in general, with unbounded operator coefficients. The coercive solvability of the present paper in abstract L_p spaces is proved.

In half-space and on the whole space these questions were studied particularly in [3], [4] and others. The coercive properties were investigated in O.V.Besov's paper [1] for the system of differential operators in the Sobolev anisotropic spaces.

Introduce some definition.

Assume

$$S_\varphi = \{\lambda : \lambda \in C, \quad |\arg \lambda - \pi| \le \pi - \varphi, \quad 0 < \varphi \le \pi\},$$

where C is a set of complex numbers.

Definition 1. *The closed linear operator A is called positive in the Banach space E if $\overline{D(A)} = E$ and at $\lambda \in S_\varphi$ the estimation*

$$\left\| (A - \lambda J)^{-1} \right\|_{Z(E)} \le M (1 + |\lambda|)^{-1},$$

holds, where $D(A)$ is a domain of determination of the operator A, J is a unit operator in E, $Z(E)$ is a space of linearly bounded operators acting from E to E.

Let $\alpha = (\alpha_1, ..., \alpha_n)$, α_k be non-negative integers, $|\alpha| = \sum\limits_{k=1}^{n} \alpha_k$, $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$, u be abstract function with the values from E. For $-\infty < \theta < \infty$ assume

$$E\left(A^\theta\right) = \left\{u;\ u \in D\left(A^\theta\right), \quad \|u\|_{E(A^\theta)} = \left\|A^\theta u\right\|_E + \|u\|_E < \infty\right\}$$

Let $D(R^n)$ be a class of infinitely differentiable finite functions in R^n.

Definition 2. *The fucntion D^α summable by the Bokhner is called a generalized derivative of the abstract fucntion u on R^n if at any $\varphi \in D(R^n)$ the equality*

$$\int\limits_{R^n} D^\alpha u(x) \varphi(x)\, dx = (-1)^{|\alpha|} \int\limits_{R^n} u(x) D^\alpha \varphi(x)\, dx \ ,$$

holds, where the integral is understood by Bokhner.

Let E_0 and E be a Banach space and E_0 be continuously and densely embedded in E, $l=(l_1,...,l_n)$,, l_k natural numbers and the differentiation $D_k^{l_k}$, $k=\overline{1,n}$ is understood in terms of definition 2.

Definition 3.

$$W_p^l(R^n;E_0,E)=\Big\{f;\ f\in L_p(R^n;E_0),\ D_k^{l_k}f\in L_p(R^n;E_0),\ k=\overline{1,n}\ ,$$

$$\|f\|^p_{W_p^l(R^n;E_0;E)}=\int\limits_{R^n}\left[\|f(x)\|^p_{E_0}+\sum_{k=1}^n\left\|D_k^{l_k}f\right\|_E^p\right]dx\Big\}<\infty,$$

at $E=H$ the space $E\left(A^\theta\right)$ we'll denote by $H\left(A^\theta\right)$.

Assume $x=(x_1,...,x_n)$, $\xi=(\xi_1,...,\xi_n)$, $\xi^\alpha=\xi_1^{\alpha_1}...\xi_n^{\alpha_n}$.

Definition 4. *The fucntion* s$W_p^l(R^n;E_0,E)$ *satisfying the given equation on R^n almost everywhere on R^n is called a solution of the given equation.*

We can easily show that at $\forall z_1,z_2,\quad \arg z_1=\varphi_1,\ \arg z_2=\varphi_2,\quad |\varphi_1-\varphi_2|=\varphi$, $\varphi\in[\beta,\pi]$, $\beta>0$ there exists the constant C_φ such that at any such z_1,z_2 it holds the following inequality

$$|z_1+z_2|\geq C_\varphi\left(|z_1|+|z_2|\right)\ . \tag{$*$}$$

We'll assume that the constants appearing in estimations don't depend on variable of elements if there is no special stipulation.

Consider in $L_p(R^n;H)$ the equation

$$(L-\lambda)u=\sum_{|\alpha:l|=1}a_\alpha D^\alpha u+Au-\lambda u+\sum_{|\alpha:l|<1}A_\alpha(x)D^\alpha u=f \tag{1}$$

where A and $A_\alpha(x)$ in general are unbounded operators in H

$$\alpha=(\alpha_1,...,\alpha_n),\ l=(l_1,...,l_n),\ \ |\alpha:l|=\sum_{k=1}^n\frac{\alpha_k}{l_k}\ .$$

Denote by L_0 the differential operators defined by the equalities

$$D(L_0)=W_p^l(R^n;H(A),H),$$

$$L_0u=\sum_{|\alpha:l|=1}a_\alpha D^\alpha u+Au\ .$$

At first consider the problem

$$(L_0-\lambda)u=f \tag{2}$$

Assume

$$(L_0-\lambda)u=\sum_{|\alpha:l|=1}a_\alpha D^\alpha u-\lambda u\ ,$$

$$B(\xi)=\sum_{|\alpha:l|=1}(-1)^{|\alpha|}a_\alpha\xi_1^{\alpha_1}...\xi_n^{\alpha_n},\quad D^\alpha=D_1^{\alpha_1}...D_n^{\alpha_n},\quad D_k^{\alpha_k}=\left(i\frac{\partial}{\partial x_k}\right)^{\alpha_k}\ ,$$

where i is an imaginary unit.

Condition 1. *Let* $\overline{D(A)} = H$ *and at* $\forall\lambda \in S_\varphi$, $\varphi \in (0,\pi]$ *the estimation* $\left\|(A-\lambda J)^{-1}\right\|_{Z(H)} \leq C(1+|\lambda|)^{-1}$ *hold.*

Further let at $\forall\xi \in R^n$

$$\lambda - B(\xi) \in S_\varphi \ \text{ and } \ |B(\xi)| \geq c\sum_{k=1}^{n} |\xi_k|^{l_k} .$$

Theorem 1. *Let condition 1 be fulfilled. Then at* $\forall f \in L_p(R^n;H)$ *and* $\forall\lambda \in S_\varphi$, $\exists\varphi \in (0,\pi]$ *problem (2) has a unique solution* $u(x)$ *belonging to the space* $W_p^l(R^n;H(A),H)$ *and the estimation*

$$\|u\|_{W_p^l(R^n;H(A),H)} \leq c\|f\|_{L_p(R^n;H)} \tag{3}$$

holds.

Proof. Denote by F the Fourier transformation and by F^{-1} the inverse transformation. Since A is a closed operator having the bounded inverse not depending on the variables $x \in R^n$ then it commutes with F.

Allowing for this after the Fourier transformation we have

$$\left(\sum_{|\alpha:l|=1} (-1)^{|\alpha|} a_\alpha \xi^\alpha - \lambda\right)\hat{u} + A\ \hat{u} = \hat{f}$$

where $\hat{u} = Fu$, $u = u(\xi)$, $\forall\xi \in R^n$.

By the condition $\lambda - B(\xi) \in S_\varphi$, then

$$\hat{u} = (A-(\lambda - B(\xi)))^{-1}\hat{f} \tag{4}$$

Then for $\forall u \in W_p^l(R^n;H(A),H)$

$$\|u\|^p_{W_p^l(R^n;H(A),H)} = \left\|F^{-1}(A-(\lambda-B(\xi)))^{-1}\hat{f}\right\|^p_{L_p(R^n;H)} +$$
$$+\left\|F^{-1}A(A-(\lambda-B(\xi)))^{-1}\hat{f}\right\|^p_{L_p(R^n;H)} +$$
$$+\sum_{k=1}^{n}\left\|F^{-1}\xi_k^{l_k}(A-(\lambda-B(\xi)))^{-1}\hat{f}\right\|^p_{L_p(R^n;H)} . \tag{5}$$

In order to obtain estimation (3) it is sufficient to prove that the operators $(A-(\lambda-B(\xi)))^{-1}$, $A(A-(\lambda-B(\xi)))^{-1}$, $\xi_k^{l_k}(A-(\lambda-B(\xi)))^{-1}$ are uniformly bounded by $\xi \in R^n$, $\lambda \in S_\varphi$ in H and it is a multiplicator in the space $L_p(R^n;H)$.

By virtue of positiveness of the operator A their uniform boundedness is seen at once.

Let's show now that there are multiplicators in $L_p(R^n;H)$. For this by virtue of the theorem on multiplicators in $L_p(R^n;H)$ it is sufficient to show that at

$$\forall\xi \in R^n,\ \ \forall\xi_k \neq 0,\ k=\overline{1,n},\ \ \beta = (\beta_1,...,\beta_n),\ \beta_k \in \{0,1\},\ \xi^\beta = \xi_1^{\beta_1}...\xi_n^{\beta_n}$$

the estimations

$$\left|\xi^{\beta}\right| \ \left\|D_{\xi}^{\beta}\left(A-(\lambda-B(\xi))\right)^{-1}\right\|_{Z(H)} \leq$$

$$\leq C_1, \ \left|\xi^{\beta}\right| \ \left\|D_{\xi}^{\beta}\xi_k^{l_k}\left(A-(\lambda-B(\xi))\right)^{-1}\right\|_{Z(H)} \leq C_2,$$

$$\left|\xi^{\beta}\right| \ \left\|D_{\xi}^{\beta}A\left(A-(\lambda-B(\xi))\right)^{-1}\right\|_{Z(H)} \leq C_3$$

are true.

Really,

$$D_{\xi_j}\left[\left(A-(\lambda-B(\xi))\right)^{-1}\right]=-\left(A-(\lambda-B(\xi))\right)^{-1}\left(A-(\lambda-B(\xi))\right)'_{\xi_j}\times$$

$$\times\left(A-(\lambda-B(\xi))\right)^{-1}=-\left(A-(\lambda-B(\xi))\right)^{-2}\sum_{|\alpha:l|=1} i\alpha_j a_\alpha\left(i\xi_1\right)^{\alpha_1}...\left(i\xi_{j-1}\right)^{\alpha_{j-1}}\times$$

$$\times\left(i\xi_j\right)^{\alpha_j-1}\left(i\xi_{j+1}\right)^{\alpha_{j+1}}...\left(i\xi_n\right)^{\alpha_n}, \quad j=\overline{1,} \tag{6}$$

Using equality (6) we obtain

$$\left|\xi_j\right| \ \left\|D_{\xi_j}\left(A-(\lambda-B(\xi))\right)^{-1}\right\|_{Z(H)} \leq C\left(1+|\lambda-B(\xi)|\right)^{-2}\sum_{|\alpha:l|=1}|a_\alpha||\xi_1|^{\alpha_1}...|\xi_n|^{\alpha_n} \tag{7}$$

Further using estimation (*) and the known inequality

$$|\xi_1|^{\alpha_1}...|\xi_n|^{\alpha_n}\leq c\left(1+\sum_{k=1}^{n}|\xi_k|^{l_k}\right) \tag{8}$$

at $|\alpha;l|\leq 1$ from estimation (7) we obtain

$$\left|\xi_j\right|\left\|D_{\xi_j}\left(A-(\lambda-B(\xi))\right)^{-1}\right\|_{Z(H)}\leq C. \tag{9}$$

Analogously at $\forall\xi=(\xi_1,...,\xi_n)\,,\ \forall\xi_j\neq 0,\ \forall\beta,\ j=\overline{1,n}$ we obtain

$$\left|\xi^{\beta}\right|\left\|D_{\xi}\left(A-(\lambda-B(\xi))\right)^{-1}\right\|_{Z(H)}\leq C_1. \tag{10}$$

Let's prove now

$$\left|\xi^{\beta}\right|\left\|D_{\xi}^{\beta}\xi_k^{l_k}\left(A-(\lambda-B(\xi))\right)^{-1}\right\|_{Z(H)}\leq C_2. \tag{11}$$

Really it is clear that

$$D_{\xi_k}\left[\xi_k^{l_k}\left(A-(\lambda-B(\xi))\right)^{-1}\right]=$$

$$=l_k\xi_k^{l_k-1}\left(A-(\lambda-B(\xi))\right)^{-1}+\xi_k^{l_k}\left(A-(\lambda-B(\xi))\right)^{-2}B'_{\xi_k}(\xi) \tag{12}$$

By virtue of the condition of theorems and inequalities (*) and (8) we obtain

$$|\xi_k|\left\|D_{\xi_k}\left[\xi_k^{l_k}\left(A-(\lambda-B(\xi))\right)^{-1}\right]\right\|\leq|\xi_k|\left[l_k|\xi_k|^{l_k-1}\left\|\left(A-(\lambda-B(\xi))\right)^{-1}\right\|+\right.$$

$$+|\xi_k|^{l_k}\left\|(A-(\lambda-B(\xi)))^{-2}\right\|\ \left|B'_{\xi_k}(\xi)\right|\Big]\le c\Big[|\xi_k|^{l_k}(1+|\lambda|+|B(\xi)|)^{-1}+$$

$$+|\xi_k||B'_{\xi_k}(\xi)|1+|\lambda|+|B(\xi)|^{-1}|\xi_k|^{l_k}(1+|\lambda|+|B(\xi)|)^{-1}\Big]\le$$

$$\le c\left[|\xi_k^{l_k}|\left(1+|\lambda|+\sum_{k=1}^{n}|\xi_k|^{l_k}\right)^{-1}+\sum_{|\alpha:l|=1}|\xi_1|^{\alpha_1}...|\xi_n|^{\alpha_n}\times\right.$$

$$\left.\times\left(1+|\lambda|+\sum_{k=1}^{n}|\xi_k^{l_k}|\right)^{-1}|\xi_k|^{l_k}\left(1+|\lambda|+\sum_{k=1}^{n}|\xi_k|^{l_k}\right)^{-1}\right]\le c,\quad k=\overline{1,n}\ .$$

Analogously at $\forall\xi=(\xi_1,...,\xi_n)\,,\ \ \forall\xi_j\ne0,\ \ \forall\beta=(\beta_1,...,\beta_n)$ we obtain (11).

Now let's show that the operator of the function $A(A-(\lambda-B(\xi)))^{-1}$ is a multiplicator from $L_p(R^n;H)$ in $L_p(R^n;H)$.

Since at $\forall\xi_j\ne0,\ \ \xi=(\xi_1,...,\xi_n)\,,\ \ k=\overline{1,n}$

$$D_{\xi_k}A(A-(\lambda-B(\xi)))^{-1}=-A(A-(\lambda-B(\xi)))^{-2}B'_{\xi_k}(\xi)\ .$$

By virtue of definition of resolvent we have

$$A(A-(\lambda-B(\xi)))^{-1}=J+(\lambda-B(\xi))(A-(\lambda-B(\xi)))^{-1}\tag{13}$$

where J is a unit operator in $L_p(R^n;H)$.

Then by virtue of estimation (13) and previous arguments we have

$$\left\|D_{\xi_k}A(A-(\lambda-B(\xi)))^{-1}\right\|\le\left\|A(A-(\lambda-B(\xi)))^{-1}\right\|\left|B'_{\xi_k}\right|\times$$

$$\times\left\|(A-(\lambda-B(\xi)))^{-1}\right\|\le c\left(\|J\|+\frac{|\lambda-B(\xi)|}{1+|\lambda-B(\xi)|}\right)\times$$

$$\times\sum_{|\alpha:l|=1}|\xi_1^{\alpha_1}|...|\xi_n^{\alpha_n}|\ (1+|\lambda-B(\xi)|)^{-1}|\xi_k|^{-1}\le c|\xi_k|^{-1}\ .$$

Also at $\beta=(\beta_1,...,\beta_n)\,,\ \ \beta_j\in\{0,1\}\,,\ \forall\xi_k\ne0,\ \xi=(\xi_1,...,\xi_n)\,,\ k=\overline{1,n}$ we obtain

$$\left\|D_{\xi_k}A(A-(\lambda-B(\xi)))^{-1}\right\|\le c|\xi_1|^{-\beta_1}...|\xi_n|^{-\beta_n}\ .$$

So, we proved that the operators $(A-(\lambda-B(\xi)))^{-1}$, $A(A-(\lambda-B(\xi)))^{-1}$, $\xi_k^{l_k}(A-(\lambda-B(\xi)))^{-1}$ are multiplicators in $L_p(R^n;H)$. Then hence estimation (3) follow.

Denote by $L-\lambda$ the operator determined by the equalities

$$D(L)=W_p^l(R^n;H(A),H)\,,$$

$$(L-\lambda)u=\sum_{|\alpha:l|=1}a_\alpha D^\alpha u+Au-\lambda u+\sum_{|\alpha:l|<1}A_\alpha(x)D^\alpha u\ .\tag{14}$$

Theorem 2. *Let condition 1 be fulfilled*

$$A_\alpha(x)\, A^{-(1-|\alpha:l|-\mu)} \in L_\infty(R^n; Z(H))$$

at some $0<\mu<1-|\alpha:l|$. *Then at sufficient large* λ_0, $\forall|\lambda|\geq\lambda_0$, $\lambda\in S_\varphi$, $\exists\varphi\in(0,\pi]$ *at* $\forall f\in L_p(R^n;H)$ *there exists a unique solution of problem (1) belonging to the space* $W_p^1(R^n;H(A),H)$ *and the estimation*

$$\|u\|_{W_p^l(R^n;H(A),H)} \leq c\,\|f\|_{L_p(R^n;H)} \tag{15}$$

holds.

Proof. By virtue of definition of the space $W_p^1(R^n;H(A),H)$ and embedding at $\forall u\in W_p^1(R^n;H(A),H)$ we have

$$\|(L-\lambda)u\|_{L_p(R^n;H)} \leq c\sum_{|\alpha:l|=1}\|D^\alpha u\|_{L_p(R^n;H)} + \|Au\|_{L_p(R^n;H)} + |\lambda|\ \|u\|_{L_p(R^n;H)} +$$

$$+\sum_{|\alpha:l|<1}\|A_\alpha(x)\,D^\alpha u\|_{L_p(R^n;H)} \leq c_\lambda\|u\|_{W_p^l(R^n;H(A),H)} +$$

$$+\sum_{|\alpha:l|<1}\left\|A^{1-|\alpha:l|-\mu}(x)\,D^\alpha u\right\|_{L_p(R^n;H)} \leq c_\lambda\|u\|_{W_p^l(R^n;H(A),H)} \tag{16}$$

i.e. the operator $L-\lambda$ at the fixed λ is bounded from $W_p^1(R^n;H(A),H)$ in $L_p(R^n;H)$.

Now for proving the first part of theorem 2 it is sufficient to show that the operator $L-\lambda$ has an inverse operator in $L_p(R^n;H)$ determined on whole space $L_p(R^n;H)$, moreover this inverse operator $(L-\lambda)^{-1}$ is bounded from $L_p(R^n;H)$ in $W_p^1(R^n;H(A),H)$.

By virtue of theorem 1 the operator $(L_0-\lambda)$ is reversible from $L_p(R^n;H)$ in $W_p^1(R^n;H(A),H)$. From equalities (2) and (14) it follows that

$$(L-\lambda)u = (L_0-\lambda)u + L_1u = \left[J + L_1(L_0-\lambda)^{-1}\right](L_0-\lambda)u\ , \tag{17}$$

where $L_1u = \sum\limits_{|\alpha:l|<1} A_\alpha(x)\,D^\alpha u$.

Let f be an arbitrary element from $L_p(R^n;H)$. Then by virtue of embedding theorem we have

$$\left\|L_1(L_0-\lambda)^{-1}f\right\|_{L_p(R^n;H)} \leq \sum_{|\alpha:l|<1}\left\|A_\alpha(x)\,D^\alpha(L_0-\lambda)^{-1}f\right\|_{L_p(R^n;H)} \leq$$

$$\leq c\sum_{|\alpha:l|<1}\left\|A^{1-|\alpha:l|-\mu}(x)\,D^\alpha(L_0-\lambda)^{-1}f\right\|_{L_p(R^n;H)} \leq$$

$$\leq\varepsilon\left\|(L_0-\lambda)^{-1}f\right\|_{W_p^1(R^n;H(A),H)} + c(\varepsilon)\left\|(L_0-\lambda)^{-1}f\right\|_{L_p(R^n;H)}, \tag{18}$$

where $\varepsilon > 0$ is sufficiently small $c(\varepsilon) > 0$, $c(\varepsilon)$ is a continuous fucntion from ε.

Since problem (2) is coercively solvable in $L_p(R^n; H)$ and the operator is positive in $L_p(R^n; H)$ then from (18) at

$$\forall u \in W_p^l(R^n; H(A), H), \quad \lambda \geq \lambda_0$$

we obtain

$$\left\| L_1 (L_0 - \lambda)^{-1} f \right\|_{L_p(R^n;H)} \leq \varepsilon \|f\|_{L_p(R^n;H)} + \frac{c(\varepsilon)}{\lambda} \|f\|_{L_p(R^n;H)} \tag{19}$$

Choosing λ such that $\lambda > 2c(\varepsilon)$, $\varepsilon < \frac{1}{2}$ from estimation (19) at $\forall f \in L_p(R^n; H)$ and at sufficiently large $\lambda > 0$ we obtain

$$\left\| L_1 (L_0 - \lambda)^{-1} f \right\|_{Z(L_p(R^n;H))} < 1 . \tag{20}$$

Then from estimation (20) it follows that at sufficiently large λ, $\forall|\lambda| \geq \lambda_0$ the operator $\left[J + L_1 (L_0 - \lambda)^{-1}\right]$ is invertible to $L_p(R^n; H)$.

Thus from equality (17) and (20) we obtain the operator $(L - \lambda)$ is invertible in the space $L_p(R^n; H)$, i.e.

$$(L - \lambda)^{-1} = \left[J + L_1 (L_0 - \lambda)^{-1}\right]^{-1} (L_0 - \lambda)^{-1}, \tag{21}$$

and at any $f \in L_p(R^n; H)$

$$\left\| (L - \lambda)^{-1} f \right\|_{W_p((R^n;H(A),H))} \leq C \|f\|_{L_p(R^n;H)} .$$

Hence the assertion of theorem 2 follows.

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