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ON SOME QUALITATIVE PROPERTIES OF SOLUTIONS OF DEGENERATED QUASILINEAR EQUATIONS

Abstract

In the paper questions on the existence and uniqueness of generalized solution of the Dirichlet problem for the equation of the form

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\left(x,u,Du\right)\right)\frac{\partial u}{\partial x_j}=f\left(x\right) \tag{1}$$

in weighted Sobolev's classes and some qualitative properties of the obtained solution are investigated.

In the given paper questions of unique existence of generalized solution of the Dirichlet problem for equation of the form

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\left(x,u,Du\right)\right)\frac{\partial u}{\partial x_j}=f\left(x\right) \tag{1}$$

in weighted Sobolev's classes and some qualitative properties of the obtained solution (in unboundd domains and close to irregular points of the boundary) are studied.

Note that similar questions associated with the qualitative properties of solutions of equation (1) earlier were considered in papers [1, 2] and questions of existence of generalized solution- in [3, 4, 5].

We study questions of unique existence of generalized solution of the homogeneous Dirichlet problem for equation (1) in unbounded and nonsmooth domains.

Let $1<p<\infty$ and $\vartheta,\ \omega^{1-p'}$ be locally integrable functions reciving a.e. finite positive values in $R^n,\ n\geq 1$, where p'$=\frac{p}{p-1}$ at $1<p<\infty$.

In the cited below theorems 1, 2 we will suppose that $\vartheta\in A_\infty$: there exist $C>0$ and $\delta>0$ such that for any ball $Q\subset R^n$ and compact $e\subset Q$

$$\frac{\vartheta\left(e\right)}{\vartheta\left(Q\right)}\leq C\left(\frac{|e|}{|Q|}\right)^{\delta}$$

holds, where $|e|$ denotes the Lebesgue measure of the set e, and $\vartheta\left(e\right)=\int\limits_e\vartheta dx$; $(\vartheta,\omega)\in A_{pq}$ is a pair of functions (ϑ,ω):

$$\sup_{Q\subset R^n}|Q|^{\frac{1}{n}-1}\,\vartheta\left(Q\right)^{\frac{1}{q}}\left(\omega^{1-p'}\left(Q\right)\right)^{\frac{1}{p'}}<\infty.$$

Suppose, that D is an arbitrary open domain in R^n. Denote by $L^p_\vartheta\left(D\right)$ measurable in D functions $u\left(x\right)$ bounded by the norm

$$\|u\|_{L^p_\vartheta(D)}=\left(\int\limits_D|u\left(x\right)|^p\,\vartheta\left(x\right)dx\right)^{\frac{1}{p}}\quad\text{for}\ \ 1\leq p<\infty.$$

Belonging of domain $D\in K_{\varepsilon}$ means: there exists $\varepsilon\left(0,1\right)$ such that for any point $x\in D$ there exists ball $Q_{R(x)}^{x}$ with the center at the point x of the radius $R\left(x\right)$ satisfying the inequality

$$\left|Q_{R(x)}^{x}\right|D\right|>\varepsilon\left|Q_{R(x)}^{x}\right|.$$

Let $\dot{C}^{\infty}\left(D\right)$ denote infinitely differentiable in D functions vanishing on the boundary of domain. Define the semi norm in this space

$$\|f\|'=\|\nabla f\|_{L_{\omega}^{p}(D)}.$$

The following imbedding theorem was proved in [6, 7]: Let $1\leq p\leq q<\infty$, $D\in K_{\varepsilon}$ be unbounded domain, $\vartheta\in A_{\infty},\ (\vartheta,\omega)\in A_{pq},\ f\in C^{1}\left(D\right),\ f|_{\partial D}=0$, then the inequality

$$\|f\|_{L_{\vartheta}^{q}(D)}\leq C\,\|\nabla f\|_{L_{\omega}^{p}(D)}. \tag{2}$$

holds, where $C>0$ and it doesn't depend on f.

This inequality remains valid for any function $f\in Lip\left(D\right),\ f|_{\partial D}=0$. In fact, there exists a sequence of smooth functions f_{k}, such that $f_{k}\rightarrow f,\ \nabla f_{k}\rightarrow f$. For example, average of the function f with the smooth kernel [8] has this property. Then by virtue of the imbedding theorem (2)

$$\|f_{k}\|_{L_{\vartheta}^{q}(D)}\leq C\,\|\nabla f_{k}\|_{L_{\omega}^{p}(D)},$$

where $C>0$ doesn't depend on k.

Passing to the limit as $k\rightarrow\infty$ in this inequality, since $|\nabla f_{k}|<M$ and $\nabla f_{k}\rightarrow f$ a.e. in D, then by means of the Lebesgue theorem in the right-hand side one can pass to the limit under the integral sign.

It follows from the previous inequality that for norm f_{k}

$$\varliminf_{k\rightarrow\infty}\|f_{k}\|_{L_{\vartheta}^{q}(D)}\leq C\varliminf_{k\rightarrow\infty}\|\nabla f_{k}\|_{L_{\omega}^{p}(D)}.$$

The right-hand side by virtue of aforesaid tends to $\|\nabla f_{k}\|_{L_{\omega}^{p}(D)}$, and the left-hand side by virtue of the Fateaux theorem

$$\left(\int_{D}\varliminf_{k\rightarrow\infty}|f_{k}|^{q}\,\vartheta\left(x\right)dx\right)^{\frac{1}{q}}=\left(\int_{D}|f|^{q}\,\vartheta\left(x\right)dx\right)^{\frac{1}{q}}.$$

Therefore,

$$\|f\|_{L_{\vartheta}^{q}(D)}\leq C\,\|\nabla f\|_{L_{\omega}^{p}(D)}$$

for any $f\in Lip\left(D\right),\ f|_{\partial D}=0$.

By virtue of (2) this seminorm defines also the norm in the space $C^{\infty}\left(D\right)$.

Denote the completion of functions $f\in Lip\left(D\right),\ f|_{\partial D}=0$ by the norm $\|f\|'$ by $\mathring{H}\left(D\right)$, then for any $f\in\mathring{H}\left(D\right)$

$$\|f\|_{L_{\vartheta}^{q}(D)}\leq C\,\|\nabla f\|_{L_{\omega}^{p}(D)}$$

holds.

Theorem 1. *Let* $2 < p < q$, $\vartheta(x) \le \omega(x)$ *for almost all* $x \in D$, D is a bounded domain or $p = q = 2$ and $D \in K_\varepsilon$:
$(\vartheta, \omega) \in A_{pq}$, $\vartheta \in A_\infty$, $V = \overset{\circ}{H}(D)$, $f \in V'$ *the conditions*

$$\left|\sum_{i=1}^{n} a_{ik}(x, u, Du)\, u_{x_i}\right| \le \omega\, |u_x|^{p-1} + g(x)\, \omega^{\frac{1}{p}}, \qquad (3a)$$

where

$$g(x) \in L_{p'}(D), \qquad \sum_{i,k=1}^{n} a_{ik} u_{x_i} u_{x_k} \ge \omega\, |u_x|^p, \qquad (3b)$$

$$\sum_{i=1}^{n} \left(a_{ik}(x, u, Du)\, u_{x_i} - a_{ik}(x, \vartheta, D\vartheta)\, \vartheta_{x_i}\right)\left(u_{x_k} - \vartheta_{x_k}\right) > 0 \qquad (3c)$$

be satisfied.

Then there exists a unique solution of equation (1) in the class V.

We shall solve equation (1) by the method of monotone operators in Banach space. Put $H = L_{2,\vartheta}$ *then* $V \subset H \subset V'$.

Let $u \in V$, $A : V \to V'$ *be an operator acting by the rule*

$$\langle A(u), \varphi\rangle = \int_D a_{ik}(x, u, Du)\, u_{x_i} \varphi_{x_k} dx$$

for any $\varphi \in V$, *where* $\langle A(u), \varphi\rangle$ *means value of the functional* $A(u)$ *on the element* $\varphi \in V$, V'*-conjugate to* V.

Then equation (1) is equivalent to the operator equation $Au = f$, $u \in V$, $f \in V'$.

Proof. In order to prove the theorem 1 it's sufficient to establish the following properties of the operator A: 1) boundedness; 2) coerciveness; 3) monotonicity; 4) semi-continuity (see [9]).

The boundedness of the operator A follows from (3a)

$$|\langle A(u), \varphi\rangle| \le C \left[\left|\int_D \omega\, |u_x|^p\, dx + \left(\int_D |g|^{p'}\, dx\right)\right|\right]^{\frac{1}{p'}} \left(\int_D \omega\, |\varphi_x|^p\, dx\right)^{\frac{1}{p}}.$$

Hence

$$\|A(u)\|_{V'} \le C \left(\|u\|_V^{p-1} + \|g\|_{L^{p'}(D)}\right).$$

Let $f \in V'$, and consider equation

$$A(u) = f,\ u \in V.$$

Then by virtue of (3b)

$$\frac{\langle A(u), u\rangle}{\|u\|_V} \ge \sum_{i,k=1}^{n} \int_D \frac{a_{ik} u_{x_i} u_{x_k}}{\|u\|_V} dx \ge C\, \|u\|_V^{p-1},$$

which means the coerciveness of the operator A.

The monotonicity of the operator A $(\langle Au - A\vartheta, u-\vartheta\rangle > 0$ if $u \neq \vartheta)$ follows from (3b). Let's show semi-continuity. By virtue of condition (3a) for any $u,\ \varphi,\ \psi \in V$ we have

$$\left|\sum_{i,k=1}^{n} a_{ik}\left(x, u+t\varphi,\ Du+tD\varphi\right)\left(u_{x_i}+t_{\varphi_{x_i}}\right)\psi_{x_k}\right| \leq$$

$$\leq C\left(|Du|^{p-1}+t^{p-1}\left|D\varphi\right|^{p-1}\right)|D\psi|\,\omega + g\omega^{\frac{1}{p}}\left|D\psi\right|,\quad |t|\leq t_0.$$

The left-hand side is continuous with respect to the parameter t, and the right-hand side is integrable function, then on the base of the Lebesgue theorem

$$\langle A\left(u+t\varphi\right),\psi\rangle \to \langle A\left(u\right),\psi\rangle \quad \text{as} \quad t\to 0.$$

The semi-continuity is proved. Theorem 1 is also proved.

Let $(\vartheta,\omega)\in A_{pq}$, where $1\leq p\leq q<\infty$ D is a bounded domain.

Theorem 2. *Let* $1\leq p\leq q<\infty,\ r>q',\ u,\ \alpha=\left(\frac{1}{r'}-\frac{1}{q}\right)\frac{1}{p-1}+\frac{1}{q'}>1,$ *u be a solution of equation (1) from the class* $V=\overset{\circ}{H}(D)$*. Then the following estimate holds*

$$\sup_D |u\left(x\right)|\leq C\left(\vartheta\left(D\right)^{\alpha-1}\right)\left\|\frac{f}{\vartheta}\right\|_{L^r_\vartheta(D)}^{\frac{1}{p-1}}.$$

Proof. Note that $f\in V'$. This follows from the finiteness of the norm $\|\,f|\,\vartheta\|_{L^r_\vartheta(D)}$. In fact

$$\left|\int_D f\varphi dx\right|\leq\left\|\frac{f}{\vartheta}\right\|_{L^r_\vartheta(D)}\cdot\|\varphi\|_{L^{r'}_\vartheta}\leq$$

$$\leq\left\|\frac{f}{\vartheta}\right\|_{L^r_\vartheta(D)}\cdot\|\varphi\|_{L^q_\vartheta}\cdot\vartheta\left(D\right)^{1-\frac{1}{r}-\frac{1}{q}}\leq\left\|\frac{f}{\vartheta}\right\|_{L^r_\vartheta(D)}\vartheta\left(D\right)^{1-\frac{1}{r}-\frac{1}{q}}\cdot\|\varphi\|_V\,.$$

or

$$\|f\|_{V'}\leq\left\|\frac{f}{\vartheta}\right\|_{L^r_\vartheta(D)}\cdot\vartheta\left(D\right)^{1-\frac{1}{r}-\frac{1}{q}}.$$

Therefore on the basis of theorem 1 there exists a unique solution $u\in V$ of equation (1). By virtue of the definition of solution $u\in V$

$$a\left(u,\varphi\right)=\int_D a_{ik}\left(x,u,Du\right)u_{x_i}\varphi_{x_j}dx=-\int_D f\varphi dx$$

for any $\varphi\in V$. Then there exists a sequence of function $\{u_j\}$ and $\{\varphi_k\}$ from $C^\infty(D)$, which are fundamental in the norm of the space V, $a\left(u_j,\varphi_k\right)\to a\left(u,\varphi\right)$ as $j,k\to\infty$.

Therefore

$$a\left(u_j,\varphi_k\right)=-\int_D f\varphi_k dx+\delta_{jk},$$

where $\delta_{jk}\to 0$ as $j,k\to\infty$.

Denote $u_j,\ \varphi_k$ and δ_{kj} again by $u,\ \varphi$ and δ respectively, then

$$a\left(u,\varphi\right)=-\int_D f\varphi dx+\delta.$$

Let's choose test function

$$\varphi = u_k(x) = \begin{cases} |u(x)| - k, & at \quad |u(x)| \geq k, \\ 0, & at \quad |u(x)| < k, \end{cases} \quad , \qquad k > 0,$$

then $\varphi \in V$ and therefore

$$\sum_{i,k=1}^{n} \int_{D_k} a_{ik}(x, u, Du)\, u_{x_i} u_{x_k} dx = -\int_{D_k} f u_k dx,$$

where $D_k = \{x \in D : |u(x)| > k\}$, $k \in R'$. Hence, subject to the condition we shall have

$$\int_{D_k} \omega |u_x|^p \, dx \leq \int_{D_k} u_k |f| \, dx + \delta.$$

Therefore

$$\int_{D_k} \omega |u_x|^p \, dx \leq \left(\int_{D_k} \vartheta |u_k|^{r'} dx \right)^{\frac{1}{r'}} \left(\int_{D_k} |f|^r \vartheta^{1-r} dx \right)^{\frac{1}{r}} + \delta,$$

$$\int_{D_k} \omega |u_x|^p \, dx \leq \left(\int_{D_k} \left(\frac{|f|}{\vartheta} \right)^r \vartheta dx \right)^{\frac{1}{r}} \left(\int_{D_k} \vartheta |u_k|^q \, dx \right)^{\frac{1}{q}} \vartheta(D_k)^{\frac{1}{r'} - \frac{1}{q}} + \delta.$$

Taking into account the imbedding theorem and Holder inequality, we obtain

$$\left(\int_{D_k} \vartheta |u_k|^q \, dx \right)^{\frac{1}{q}} \leq \vartheta(D_k)^{\frac{1}{r'(p-1)} - \frac{1}{q(p-1)}} \left(\int_{D_k} \frac{|f|}{|\vartheta|} \cdot \vartheta \; dx \right)^{\frac{1}{r(p-1)}} + \delta.$$

Denote by $C_0 = \left(\int_{D} \left| \frac{f}{\vartheta} \right|^r \vartheta dx \right)^{\frac{1}{r}}$ and by $z(k) = \int_{D_k} \vartheta |u_k| \, dx$, we obtain

$$z(k) = \vartheta(D_k)^{\left(\frac{1}{r'} - \frac{1}{q} \right) \frac{1}{p-1} + \frac{1}{q'}} \cdot C_0^{\frac{1}{p-1}}.$$

Let

$$\alpha = \left(\frac{1}{r'} - \frac{1}{q} \right) \frac{1}{p-1} + \frac{1}{q'} > 1,$$

then

$$-\frac{dz}{dk} = \vartheta(D_k) \quad \text{and} \quad z \leq C C_0^{\frac{1}{p-1}} \left(-\frac{dz}{dk} \right)^{\alpha}.$$

Integrating this inequality and taking into account the fact that

$$z(0) = \int_{D} |u| \, dx, \quad z \left(\sup_{D} |u| \right) = 0.$$

We obtain

$$\sup_D |u| \leq CC_0^{\frac{1}{p-1}} z^{1-\frac{1}{\alpha}}\Big|_0^{z(0)} = C_2 C_0^{\frac{1}{p-1}} \vartheta\,(D)^{\alpha-1}\,,$$

where C doesn't depend on $u,\ \vartheta,\ D$.

Theorem 2 is proved.

In the following results the locally summable function $\omega\,(x)$ satisfies the condition: there exist $\varepsilon > 0$ and $\alpha > 0$ for any compact E from the ball Q_R^0

$$|E|^{\frac{p}{n}-\varepsilon} \left(\frac{1}{|E|}\int\limits_E \omega\,(x)\,dx\right) \left(\frac{1}{|E|}\int\limits_E \omega^{-\frac{1}{p-1}}\,(x)\,dx\right)^{p-1} \leq \alpha R^{p-n\varepsilon} \tag{4}$$

is satisfied, where $|E|$ is n-dimensional Lebesgue measure of set E, number $1 < p < \infty$, α is a positive real number.

Note that power function of the form $|x|^s$ for $-p < s < 1$ satisfies the condition A'_p.

Denote by $\tilde{H}\,(D)$ the completion of functions $f \in Lip\,(D)$ by the norm

$$\|f\| = \|f\|_{L^p_\omega(D)} + \|\nabla f\|_{L^p_\omega(D)}\,,$$

where ω satisfies condition (4).

Element $u = \{u_j\}$ will be called the solution of equation

$$\frac{\partial}{\partial x_i}\left(\omega\,|\nabla u|^{p-2}\,\frac{\partial u}{\partial x_i}\right) = 0, \tag{5}$$

if

$$\sum_{i=1}^{n}\int\limits_D \omega\,|\nabla u|^{p-2}\,(u_j)_{x_i}\,(\varphi_k)_{x_i}\,dx = \delta_{jk}, \tag{6}$$

$u_j \in C^\infty\,(D)\,,\ \varphi_k \in C^\infty\,(D)\,,$ moreover $\{u_j\}\,,\ \{\varphi_k\}$ are fundamental in the norm $\tilde{H}\,(D)$, where $\delta_{jk} \to 0$ as $j,k \to \infty$.

Lemma 1. *Let Q_R^0 be a ball of radius R with the center at the point 0 and having limiting points on the surface of aphere S_R. Let Γ be that part of the boundary of domain D, which lies strictly inside of the ball Q_R. Let $u \equiv \{u_j\}$ be a nonnegative solution of equation (5) vanishing on Γ. It holds the condition (4) for the function $\omega\,(x)$. Then there exists constant M depending only on $n,\ \alpha,\ p$, such that if*

$$|D| < \frac{|Q_R|}{M}$$

then the following inequality is satisfied

$$u\,(0) < \frac{1}{2}\max_D u\,(x)\,.$$

In order to prove lemma 1 we shall follow the technique. (see [10])

In [10] it is applied to the linear equation without degeneration.

We denote by D_0 the set of points $x \in D \cap Q^0_{\frac{R}{2}}$ at which

$$u(x) \geq \frac{u(0)}{2}.$$

1) Suppose that

$$|D_0| > \frac{\left|Q^0_R\right|}{4^n M}. \tag{7}$$

Then we shall show that the statement of lemma 1 is valid. Later we shall show that the statement of lemma 1 is valid without assumption (7) but with changing the constant M to $4^n M$ in the inequality. For each $t > 0$ we denote the set of points $x \in D$, where $u_j(x) \geq t$ by D_t. For the simplicity we shall write u instead of u_j. The boundary of the set D_t consists of the points of level set $u(x) = t$ of the function $u(x)$, which will be denoted by E_t and of the points situated on the sphere S_R ($|x| = R$). Suppose $\overset{\curlywedge}{E}_t = E_t \cap Q^0_{\frac{R}{2}}$.

By the Kronrode formula [13] for level surface we have

$$\int\limits_0^{\frac{u(0)}{2}} \int\limits_{\overset{\curlywedge}{E}_t} \omega^{-\frac{1}{p-1}} \frac{d\sigma}{|\nabla u|} = \int\limits_{D_1} \omega^{-\frac{1}{p-1}} dx, \tag{8}$$

where $D_1 = \left\{ x \in D \cap Q^0_{\frac{R}{2}} \,\middle|\, u(x) < \dfrac{u(0)}{2} \right\}$.

It follows form (8) that there exists $0 \leq t_0 \leq \frac{u(0)}{2}$ which

$$\int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega^{-\frac{1}{p-1}} \frac{d\sigma}{|\nabla u|} \leq \frac{2}{u(0)} \int\limits_{D_1} \omega^{-\frac{1}{p-1}} dx.$$

Applying to the integral

$$\int\limits_{\overset{\curlywedge}{E}_{t_0}} d\sigma = \int\limits_{\overset{\curlywedge}{E}_{t_0}} \left(\omega^{\frac{1}{p}} |\nabla u|^{\frac{p-1}{p}}\right) \left(\omega^{-\frac{1}{p}} |\nabla u|^{-\frac{p-1}{p}}\right) d\sigma$$

the Holder inequality, we obtain

$$\left(\underset{n-1}{mes} \overset{\curlywedge}{E}_{t_0}\right)^p \leq \int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega |\nabla u|^{p-1} d\sigma \left(\int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega^{-\frac{1}{p-1}} \frac{d\sigma}{|\nabla u|} \right)^{p-1},$$

$$\int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega |\nabla u|^{p-1} d\sigma \geq \frac{\left(\underset{n-1}{mes} \overset{\curlywedge}{E}_{t_0}\right)^p}{\left(\int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega^{-\frac{1}{p-1}} \frac{d\sigma}{|\nabla u|} \right)^{p-1}} \geq \frac{\left(\underset{n-1}{mes} \overset{\curlywedge}{E}_{t_0}\right)^p}{\left(\int\limits_{D_1} \omega^{-\frac{1}{p-1}} dx \right)^{p-1}} \cdot \left(\frac{u(0)}{2}\right)^{p-1}, \tag{9}$$

since $\overset{\curlywedge}{E}_{t_0}$ separates the sets D_0 and $D_2 = Q^0_{\frac{R}{2}} \backslash D$ in the ball $Q^0_{\frac{R}{2}}$, moreover

$$|D_2| = \frac{\left|Q^0_R\right|}{2^n} - |D| \geq \frac{\left|Q^0_R\right|}{2^n} - \frac{\left|Q^0_R\right|}{M} = \left(\frac{1}{2^n} - \frac{1}{M}\right)\left|Q^0_R\right|.$$

Now let's choose M such that the following condition be fulfilled

$$\frac{1}{2^n} - \frac{1}{M} > \frac{1}{M}, \quad \text{i.e. } M > 2^{n+1}.$$

Then we have $|D_2| = \frac{|Q^0_R|}{M} > |D| > |D_0|$.

Everywhere below n-dimensional Lebesgue measure D will be denoted by $|D|$. By isoperimetric inequality

$$\underset{n-1}{mes} \overset{\curlywedge}{E}_{t_0} \geq \beta\left|D_0\right|^{\frac{n-1}{n}},$$

where β is the constant dependent only on dimension n of the space. Therefore (9) implies

$$\int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega\left|\nabla u\right|^{p-1} d\sigma \geq \frac{\beta^p}{2^{p-1}} \cdot \frac{\left|D_0\right|^{p \cdot \frac{n-1}{n}}}{\left(\int\limits_{D_1} \omega^{-\frac{1}{p-1}} dx\right)^{p-1}} u\left(0\right)^{p-1}. \tag{10}$$

Let $t > 0$, $h > 0$, $r > 0$.. Let's introduce the functions

$$\varphi_h = \begin{cases} \frac{u(x)-t}{h} & at \quad t \leq u < t+h, \\ 1 & at \quad u \geq t+h, \\ 0 & at \quad u < t, \end{cases}$$

$$\psi_h = \begin{cases} \frac{r+h-|x|}{h} & at \quad r \leq |x| < r+h, \\ 1 & at \quad |x| \leq r, \\ 0 & at \quad |x| > r+h. \end{cases}$$

Suppose $\varphi = \varphi_h \cdot \psi_h$ in identity (6) (therefore function $\varphi \in \overset{\circ}{H}(D)$ is a test function). Then

$$\int\limits_D \omega\left|\nabla u_j\right|^{p-2} \psi_h \nabla u_j \nabla\varphi_h dx + \int\limits_D \omega\left|\nabla u_j\right|^{p-2} \varphi_h \nabla u_j \nabla\varphi_h dx = \delta_j,$$

where $\delta_j \to 0$ as $j \to \infty$. (For the simplicity we omit the indices in u_j and δ_j). We have

$$\frac{1}{h} \int\limits_{\left\{\substack{t \leq u < t+h \\ |x| < r}\right\}} \omega\left|\nabla u\right|^p dx - \frac{1}{h} \int\limits_{\left\{\substack{t+x \leq u \\ r \leq |x| < r+h}\right\}} \omega\left|\nabla u\right|^{p-2} \nabla u \cdot \frac{x}{|x|} dx + \alpha_h = \delta,$$

$$\alpha_h = \frac{1}{h}\int\limits_{G_h} \omega\left|\nabla u\right|^p \psi_h dx - \frac{1}{h}\int\limits_{G_h} \omega\left|\nabla u\right|^{p-2} \nabla u \cdot \frac{x}{|x|}\varphi_h dx,$$

where

$$G_h = \{x \in D : t \le u(x) < t+h,\ r \le |x| < r+h\}.$$

Passing to the limit as $h \to 0$ by virtue of Lebesgue theorem for almost all $t > 0,\ r > 0$ we shall have

$$\int\limits_{\left\{ {u=t \atop |x|<r} \right\}} \omega |\nabla u|^{p-2} \frac{\partial u}{\partial n} d\sigma - \int\limits_{\left\{ {u>t \atop |x|=r} \right\}} \omega |\nabla u|^{p-2} \cdot \frac{\partial u}{\partial r} d\sigma = \delta. \tag{11}$$

In fact, by means of Federer formula (see [13])

$$\frac{1}{h} \int\limits_{\left\{ {t\le u<t+h \atop |x|<r} \right\}} \omega |\nabla u|^{p} dx = \frac{1}{h} \int\limits_{t}^{t+h} ds \left(\int\limits_{\left\{ {u\equiv s \atop |x|<r} \right\}} \omega |\nabla u|^{p-1} d\sigma \right) \to$$

$$\to \int\limits_{\left\{ {u\equiv t \atop |x|<r} \right\}} \omega |\nabla u|^{p-1} d\sigma = \int\limits_{\left\{ {u\equiv t \atop |x|<r} \right\}} \omega |\nabla u|^{p-2} \frac{\partial u}{\partial n} d\sigma,$$

as $h \to 0$ for a.e. $t \in R'$.

(We have used the fact that $\left|\frac{\partial u}{\partial n}\right| = |\nabla u|$ will be fulfilled on the level surface of the function u)

$$\frac{1}{h} \int\limits_{\left\{ {u>t+h \atop |x|<r+h} \right\}} \omega |\nabla u|^{p-2} \frac{\partial u}{\partial r} = \frac{1}{h} \int\limits_{r}^{r+h} ds \left(\int\limits_{\left\{ {|x|=s \atop u>t+h} \right\}} \omega |\nabla u|^{p-2} \frac{\partial u}{\partial r} d\sigma \right) \to \int\limits_{\left\{ {|x|=r \atop u>t} \right\}} \omega |\nabla u|^{p-2} \frac{\partial u}{\partial r} d\sigma,$$

as $h \to 0$ for a.e. $r > 0$.

Analogously one can prove that $\lim\limits_{h\to 0} \alpha_h = 0$.

By means of equality (11) we shall show the estimate

$$\int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega |\nabla u|^{p-2} \left|\frac{\partial u}{\partial n}\right| d\sigma \le 6\omega\left(F_{t_0 R}\right) \cdot \frac{\left(\underset{D_{t_0}}{osc u}\right)^{p-1}}{R^p} \le$$

$$\le K \frac{\omega(D)}{R^p} \left(\max_{D} u\right)^{p-1},$$

where $F_{t_0 R} = \left\{x \in D \cap Q_R^{x_0} : u(x) > t_0,\ \frac{R}{2} < |x| < R\right\}$.

Integrating (11) with respect to $\left[\frac{5}{4}R, \frac{4}{3}R\right]$ at $t = t_0$, we obtain

$$\frac{R}{6} \int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega |\nabla u|^{p-2} \frac{\partial u}{\partial n} d\sigma \le \int\limits_{\Omega_{t_0 R}} \omega |\nabla u|^{p-1} dx + \delta \cdot \frac{R}{6} \le$$

$$\leq \omega\left(\Omega_{t_0R}\right)^{\frac{1}{p}} \cdot \left(\int\limits_{\Omega_{t_0R}} \omega\left|\nabla u\right|^p dx\right)^{\frac{1}{p'}} + \delta\cdot\frac{R}{6}, \tag{12}$$

where $\Omega_{t_0R} = \left\{x : \frac{2}{3}R < |x| < \frac{5}{6}R,\ u(x) > t_0\right\}$.

It remains to show the estimate

$$\int\limits_{\Omega_{t_0R}} \omega\left|\nabla u\right|^p dx \leq \frac{\left(\underset{Dt_0}{oscu}\right)^p}{R^p}\left(\int\limits_{F_{t_0R}} \omega dx\right);$$

Put $\varphi = (u - t_0)\,\xi^p\left(\frac{|x|}{R}\right)$ in identity (6) where $\xi \in \left[\frac{1}{2}, 1\right]$, $\xi(t) = 1$ for $t \in \left[\frac{2}{3}, \frac{5}{6}\right]$, $\left|\xi'\right| < C_0$. Then

$$\int\limits_{F_{t_0R}} \omega\left|\nabla u\right|^p \xi^p dx + p\int\limits_{F_{t_0R}} \xi^{p-1}\omega\left(u - t_0\right)\left|\nabla u\right|^{p-2}\cdot\nabla u\cdot\frac{x}{|x|}\cdot\frac{\xi'}{R}dx = 0,$$

$$\int\limits_{F_{t_0R}} \omega\left|\nabla u\right|^p \xi^p dx \leq \frac{pC_0}{R}\int\limits_{F_{t_0R}} \left|u - t_0\right|\omega\left|\nabla u\right|^{p-1} dx \leq$$

$$\leq \frac{pC_0}{R}\left(\int\limits_{F_{t_0R}} \omega\left|\nabla u\right|^p \xi^p dx\right)^{\frac{1}{p'}}\left(\int\limits_{F_{t_0R}} \omega\left|u - t_0\right|^p dx\right)^{\frac{1}{p}},$$

hence,

$$\int\limits_{\Omega_{t_0R}} \omega\left|\nabla u\right|^p dx \leq \left(\frac{pC_0}{R}\right)^p\int\limits_{F_{t_0R}} \omega\left|u - t_0\right|^p dx \leq \frac{C}{R^p}\left(\underset{D_{t_0}}{oscu}\right)^p\omega\left(F_{t_0R}\right), \tag{13}$$

Taking into account the last estimate in (12) we obtain

$$\int\limits_{\overset{\curlywedge}{E}_{t_0}} \omega\left|\nabla u\right|^{p-2}\left|\frac{\partial u}{\partial n}\right| d\sigma \leq 6\omega\left(F_{t_0R}\right)\cdot\frac{\left(\underset{D_{t_0}}{oscu}\right)^{p-1}}{R^p} + 6\delta \leq K\frac{\omega(D)}{R^p}\left(\underset{D}{\max}u\right)^{p-1},$$

Hence subject to (10) we obtain

$$\underset{D}{\max}u(x) \geq \left[\frac{\left|D_0\right|^{p\frac{n-1}{n}} R^p}{\left(\left(\int\limits_D \omega dx\right)\left(\int\limits_{D_1} \omega^{-\frac{1}{p-1}}\right)^{p-1}\right.}\right]^{\frac{1}{p-1}} u(0) - \delta,$$

where $C = \dfrac{\beta^{\frac{p}{p-1}}}{2k^1}$.

By virtue of condition (4) we obtain

$$\max_D u(x) \geq \frac{C}{\alpha^{\frac{1}{p-1}}} \left(\frac{|D_0|}{|D|^p}\right)^{\frac{p'}{n}} \left(\frac{R^n}{|D|}\right)^{\frac{\varepsilon}{p-1}} u(0) - \delta. \tag{14}$$

Taking into account conditions $|D_0| \geq \frac{|Q_R^0|}{4^n M}$, $|D| \leq \frac{|Q_R^0|}{M}$ and the arbitrariness of δ we derive from (14) that

$$\max_D u(x) \geq C M^{\frac{\varepsilon}{p-1}} u(0),$$

where C doesn't depend on M.

Now if we choose such M that the condition

$$C M^{\frac{\varepsilon}{p-1}} > 2$$

be fulfilled, then the first part of lemma 1 will be complete.

2) Let now condition (7) be violated. Lemma 1 will be also valid without assumption (7) if one chooses rather greater constant $\tilde{M} = 2^n M$ instead of M.

Suppose that for domain D and solution $u(x)$ of equation (5), condition (7) isn't satisfied.

Assume

$$M(r) = \max_{\substack{|x|=r \\ x \in D}} u(x) = u(x_r), \qquad (0 < r \leq R).$$

Let's prove that for any r $\left(0 < r < \frac{R}{2}\right)$ it will be found Δ $\left(0 < \Delta < \frac{R}{2}\right)$ such that

$$M(r+\Delta) > \left(1 + \frac{2\Delta}{R}\right) M(r). \tag{15}$$

Let's introduce the following denotations: Q_r^m is ball with the radius $2^{\frac{R}{m+1}}$ with the center at the point x_r; D_r^m is a component of the point set $x \in D$, where $u(x) > \frac{2^m-1}{2^m} M(r)$, containing the point x_r

$$u_r^m = u(x) - \frac{2^m - 1}{2^m} M(r).$$

Function u_r^m is the solution of equation (5). For it inequality (7) have the following form in the ball Q_r^m

$$\left|D_r^{m+1}\right| > \frac{\left|Q_r^{m+1}\right|}{\tilde{M}}.$$

Let $m = 0$, if inequality (7) is valid then we can apply the first point of lemma 1 to the function $u(x)$ and we shall obtain the statement of lemma 1. If inequality (7) isn't satisfied, then

$$\left|D_r'\right| < \frac{|Q_r'|}{\tilde{M}}.$$

But for function u_r^1 it will take the form

$$\left|D_r^2\right| > \frac{\left|Q_r^2\right|}{\tilde{M}} \quad \text{and} \quad \text{etc.}$$

We assert that there exists number m_1 such that inequality (7) is satisfied for function $u_{r_1}^{m_1}$. We can show it by the following way. At the point x_r we have $|\nabla u| \neq 0$.

In fact, for sufficiently large m either inequality (7) is fulfilled or there exists the sequence of numbers $m_j \to \infty$ as $j \to \infty$ such that for each m_j the level surface

$$u(x) = \frac{2^{m_j} - 1}{2^{m_j}} \cdot M(r)$$

will intersect the corresponding ball $Q_r^{m_j}$. But then on the surface of each ball $Q_r^{m_j}$ a point y_j will exist for which

$$\frac{|u(y_j) - u(x_r)|}{|y_j - x_r|} \geq \frac{\frac{M(r)}{2^{m_j}}}{\frac{R}{2^{m_j+1}}} = \frac{M(r)}{2R}$$

is satisfied.

Hence,

$$|\nabla u||_{x_r} > \frac{M(r)}{2R} \neq 0.$$

Now it follows from $|\nabla u| \neq 0$ that the surface of level of the function $u(x)$ passing through the point x_r in the vicinity of x_r has a bounded curvature.

Therefore there will be found m_0 such that there is a ball with the radius $\frac{R}{2^{m_0+3}}$ on whose surface lies the point x_r, and whose rest points lie in domain $u(x) > M(r)$.

But then this ball is contained in $D_r^{m_0+1}$ and therefore

$$\left|D_r^{m_0+1}\right| > \left(\frac{1}{4} \cdot \frac{R}{2^{m_0+1}}\right)^n |\Omega_n|$$

and it means that there exists number m_1 $(0 < m_1 < m_2)$, for which inequality (7) is satisfied, where Ω_n is a unit ball in R^n. Therefore applying lemma 1 to the function $u_r^{m_1}$, we obtain

$$\max_{D_r^{m_1}} u_r^{m_1}(x) > 2u_r^{m_1}(x_r) = \frac{M(r)}{2^{m_1-1}}$$

or

$$M\left(r + \frac{R}{2^{m_1+1}}\right) \geq \max_{D_r^{m_1}} u_r^{m_1}(x) + \frac{2^{m_1} - 1}{2^{m_1}} M(r) \geq \left(1 + \frac{1}{2^{m_1}}\right) M(r).$$

It remains to put $\Delta = \frac{R}{2^{m_1+1}}$. But inequality (15) implies

$$\max_D u(x) > 2u(0).$$

In fact, let r_1 be an upper bound of $r \leq R$ such that

$$M(r) \geq u(0)\left(1 + \frac{2r}{R}\right).$$

If $r_1 < \dfrac{R}{2}$, then

$$M(r+\Delta) > M(r_1)\left(1+\frac{2\Delta}{R}\right) > u(0)\left(1+\frac{2(r_1+\Delta)}{R}\right)$$

and we get the contradiction to the fact that r_1 is an upper bound of r for which (15) is fulfilled.

Thereby it means

$$M(r_1) > 2u(0).$$

Thus lemma 1 is completely proved.

Now after proving lemma 1 we can refer to the standard reasonings of paper [12] and prove the following theorems.

Theorem 3. *Let Q_R be the ball with the radius R with the center at the point 0. Let D be domain situated inside the ball containing point 0 and having limiting points on the boundary of the ball. Let Γ be that part of the boundary of domain D which is situated strictly inside the ball. Furthermore,*

$$|D| = \sigma < \frac{|Q|}{M},$$

where M is a constant of lemma 1.

Suppose that $u(x) \in \tilde{H}(D)$ is a solution positive in D and vanishing on Γ. All the assumptions of lemma 1 are satisfied with respect to $\omega(x)$.

Then

$$u(0) \leq \exp\left\{-\frac{R^{\frac{n}{n-1}}}{C\sigma^{\frac{1}{n-1}}}\right\}\max_{\bar{D}} u(x),$$

where constant C depends only on n, p and α from condition (4).

Theorem 4 (of Fragmen-Lindelöf type). *Let D be a bounded domain of solid angle type of dimension η, and exactly for all natural m starting with some m_0 the inequality*

$$|D \cap Q_{2^m}| < \eta\,|Q_{2^m}|$$

be satisfied, where Q_{2^m} is the ball with the radius 2^m with the center at the point 0.

Further let

$$\eta < \frac{2}{2^n M},$$

where M is a constant of lemma 1.

Suppose that $u(x) \in \tilde{H}(D)$ is positive in D solution of equation (5) vanishing on the boundary ∂D of the domain D. Condition (4) is fulfilled with respect to $\omega(x)$. Then either

1) $u(x) \leq 0$ everywhere in D or, 2) if we assume

$$M(R) = \max_{|x|=R} u(x)$$

then $\lim\limits_{R\to\infty} \dfrac{M(R)}{R^{\frac{1}{k\eta^{\frac{1}{n-1}}}}}$, where k is constant dependent only on α, p, γ and on dimension of the space.

Theorem 5. *Let D be the domain with the limiting point* 0. *Let equation (5) be defined in D. Condition (4) is satisfied with respect to $\omega(x)$. Further let there exist number η satisfying the inequality*

$$\eta < \frac{1}{2^n M}$$

(here M is the constant of lemma 1, such that

$$\left|D \cap Q^0_{2^{-m}}\right| \leq \eta \left|Q^0_{2^{-m}}\right|$$

for all numbers m starting with same m_0. Then if we suppose $M(r) = \max\limits_{\substack{|x|=r \\ x\in D}} u(x)$, then $M(r) \leq Cr^{\frac{1}{k\eta^{\frac{1}{m-1}}}}$. Constants C and k depend only on P, n, α and C depends also on m and m_0.

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