**MATHEMATICS**

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# ON BEHAVIOUR NEAR THE BOUNDARY OF SOLUTIONS OF THE SECOND ORDER NON-UNIFORMLY DEGENERATE PARABOLIC EQUATIONS

**Abstract**

*In the paper a class of the second order parabolic equations of non-divergence structure, allowing the non-uniform power degeneration at boundary point of domain is considered. The sufficient regularity condition of this point with respect to the first boundary value problem for the mentioned equations is found.*

Let $\mathbb{R}_{n+1}$ be an $(n+1)-$ dimensional Euclidean space of the points $(x,t) = (x_1,...,x_n,t)$, $D$ be a bounded domain in $\mathbf{R}_{n+1}$, $\Gamma(D)$ be a parabolic boundary of $D$, $(0,0)\in\Gamma(D)$. Consider the following parabolic equation in $D$

$$\mathcal{L}u=\sum_{i,j=1}^{n}a_{ij}(x,t)\,u_{x_ix_j}+\sum_{i=1}^{n}b_i(x,t)\,u_{x_i}+c(x,t)\,u-u_t=0 \qquad (1)$$

in assumption that $\|a_{ij}(x,t)\|$ is a real symmetric matrix where for all $(x,t)\in D$ and any $n-$ dimensional vector $\xi$

$$\gamma\sum_{i=1}^{n}\lambda_i(x,t)\,\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\,\xi_i\xi_j\le\gamma^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\,\xi_i^2. \qquad (2)$$

Here $\gamma\in(0,1]$ is a constant, $\lambda_i(x,t)=\left(|x|_\alpha+\sqrt{|t|}\right)^{\alpha_i}$, $|x|_\alpha=\sum_{k=1}^{n}|x_k|^{\frac{2}{2+\alpha_k}}$, $-2<\alpha_i\le 2;\ i=1,...,n.$

Relative to minor coefficients of the operator $\mathcal{L}$ we'll assume the conditions

$$|b_i(x,t)|\le b_0;\ i=1,...,n;\quad -b_0\le c(x,t)\le 0;\quad (x,t)\in D \qquad (3)$$

are satisfied, where $b_0$ is non-negative constant.

The aim of the present paper is the determination of sufficient regularity conditions of the point $(0,0)$ with respect to the first boundary value problem for the equation (1). Note that the investigations on regularity of boundary points for the second order parabolic equations take the beginning with the classical works of I.G.Petrovsky [1] and A.N.Tikhonov [2]. The regularity condition of boundary point for the heat equation is obtained in [3] (see also [4]). For the equations with variable coefficients, the boundary properties of solutions are studied in [5-8]. Relative to the second order parabolic equations of divergence structure, we show in this connection papers [9-10].

At first let's agree to some denotations. For an $n$-dimensional vector $x^0$ and positive numbers $R$ and $k$ $\mathcal{E}_{R;k}\left(x^0\right)$ is the ellipsoid $\left\{x:\sum\limits_{i=1}^{n}\frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}<(kR)^2\right\}$ for $t^1<t^2$ $C_{R;k}^{t_1,t_2}\left(x^0\right)$ is the cylinder $\mathcal{E}_{R;k}\left(x^0\right)\times\left(t^1,t^2\right)$. For arbitrary cylinder $\mathbb{C}$, we'll denote its lateral surface and lower foundation by $S\left(\mathbb{C}\right)$ and $F\left(\mathbb{C}\right)$ respectively. Notation $C\left(\cdots\right)$ denotes that a positive constant $C$ depends only on the quantities appearing in parentheses.

Let $\alpha=\left(\alpha_1,...,\alpha_n\right)$, $\alpha^+=\max\left\{\alpha_1,...,\alpha_n\right\}$, $\alpha^-=\min\left\{\alpha_1,...,\alpha_n\right\}$. The function $u\left(x,t\right)\in C^{2,1}\left(D\right)$ is called $\mathcal{L}$-subparabolic in $D$ if $\mathcal{L}u\left(x,t\right)\geq 0$ for $(x,t)\in D$. The function $u\left(x,t\right)$ is called $\mathcal{L}-$superbolic in $D$, if the function $-u\left(x,t\right)$ is $\mathcal{L}$-subparabolic in $D$.

Let $\mathbb{C}^1=C_{R;17}^{-\frac{9bR^2}{8},0}\left(0\right)$, $\mathbb{C}^2=C_{R;1}^{-\frac{bR^2}{16},0}\left(0\right)$, $\mathbb{C}^3=\mathbb{C}^1\backslash\bar{\mathbb{C}}^2$, where the constant $b\in(0,1)$ will be choosen later. For $s>0$ and $\beta>0$ introduce the function

$$G_R^{s,\beta}\left(x,t\right)=\begin{cases} t^{-s}\exp\left[-\frac{1}{4\beta t}\sum\limits_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}\right], & \text{if } t>0,\\ 0, \quad\text{if} & t\leq 0.\end{cases}$$

Without loss of generality we'll assume that the coefficients of the operator $\mathcal{L}$ are extended in $\mathbb{R}_{n+1}\backslash D$ with preservation of the conditions (2)-(3).

**Lemma 1.** *If relative to coefficients of the operator $\mathcal{L}$ the conditions (2)-(3) were satisfied, then there exist $s\left(\gamma,\alpha,n,b_0,b\right)$ and $\beta\left(\gamma,\alpha,n,b_0,b\right)$ such that for any fixed point $\left(y,\tau\right)\in\mathbb{C}^3$ the function $G_R^{s,\beta}\left(x-y,t-\tau\right)$ is $\mathcal{L}$-subparabolic in $\mathbb{C}^3\backslash\left\{\left(y,\tau\right)\right\}$ at $R\leq 1$.*

**Proof.** It's sufficient to consider the case $t>\tau$. For simplicity we denote the function $G_R^{s,\beta}\left(x,t\right)$ simply by $G\left(x,t\right)$. We have

$$J=\frac{\mathcal{L}G\left(x-y,t-\tau\right)}{G\left(x-y,t-\tau\right)}\left(t-\tau\right)=\frac{1}{4\beta^2\left(t-\tau\right)}\sum_{i,j=1}^{n}a_{ij}\left(x,t\right)\frac{\left(x_i-y_i\right)\left(x_j-y_j\right)}{R^{\alpha_i+\alpha_j}}-$$

$$-\frac{1}{2\beta}\sum_{i=1}^{n}\frac{a_{ii}\left(x,t\right)}{R^{\alpha_i}}+s-\frac{1}{4\beta\left(t-\tau\right)}\sum_{i=1}^{n}\frac{\left(x_i-y_i\right)^2}{R^{\alpha_i}}-\frac{1}{2\beta}\sum_{i=1}^{n}b_i\left(x,t\right)\frac{\left(x_i-y_i\right)}{R^{\alpha_i}}+$$

$$+c\left(x,t\right)\left(t-\tau\right)\geq\frac{1}{4\beta\left(t-\tau\right)}\left[\frac{\gamma}{\beta}\sum_{i=1}^{n}\frac{\lambda_i\left(x,t\right)}{R^{\alpha_i}}\frac{\left(x_i-y_i\right)^2}{R^{\alpha_i}}-\sum_{i=1}^{n}\frac{\left(x_i-y_i\right)^2}{R^{\alpha_i}}\right]+$$

$$+s-\frac{1}{2\beta\gamma}\sum_{i=1}^{n}\frac{\lambda_i\left(x,t\right)}{R^{\alpha_i}}-\frac{b_0}{2\beta}\sum_{i=1}^{n}\frac{\left|x_i-y_i\right|}{R^{\alpha_i}}-b_0\left(t-\tau\right). \tag{4}$$

If $\left(x,t\right)\in\mathbb{C}^3$ then $\left|x_i\right|\leq 17R^{1+\frac{\alpha_i}{2}}$; $i=1,...,n$.

Thus

$$\left|x\right|_\alpha\leq R\sum_{i=1}^{n}17^{\frac{2}{2+\alpha_i}}\leq 17^{\frac{2}{2+\alpha^-}}nR,$$

$$\sqrt{\left|t\right|}\leq\sqrt{\frac{9b}{8}}R\leq 2R.$$

So

$$|x|_{\alpha}+\sqrt{|t|}\leq\left(17^{\frac{2}{2+\alpha^{-}}}n+2\right)R. \tag{5}$$

On the other hand for $(x,t)\in\mathbb{C}^{3}$ either $\sum\limits_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}\geq R^2$ or $|t|\geq\frac{bR^2}{16}$.

Therefore there exists a natural number $i_0$, $1\leq i_0\leq n$, such that $|x_{i_0}|\geq n^{-\frac{1}{2}}R^{1+\frac{\alpha_{i_0}}{2}}$ or $\sqrt{|t|}\geq\frac{\sqrt{b}}{4}R$. So we have

$$|x|_{\alpha}+\sqrt{|t|}\geq\min\left\{|x|_{\alpha},\sqrt{|t|}\right\}\geq R\min\left\{n^{-\frac{1}{2+\alpha_{i_0}}},\frac{\sqrt{b}}{4}\right\}\geq$$

$$\geq R\min\left\{n^{-\frac{1}{2+\alpha^{-}}},\frac{\sqrt{b}}{4}\right\}=a_0(\alpha,n,b)\,R. \tag{6}$$

From (5)-(6) we conclude that when $\alpha_i\geq 0$

$$\lambda_i(x,t)\leq\left(17^{\frac{2}{2+\alpha^{-}}}n+2\right)^{\alpha_i}R^{\alpha_i}\leq\left(17^{\frac{2}{2+\alpha^{-}}}n+2\right)^{\alpha^{+}}R^{\alpha_i},$$

$$\lambda_i(x,t)\geq a_0^{\alpha_i}R^{\alpha_i}\geq a_0^{\alpha^{+}}R^{\alpha_i}.$$

If $\alpha_i<0$, then

$$\lambda_i(x,t)\leq a_0^{\alpha_i}R^{\alpha_i}\leq a_0^{\alpha^{-}}R^{\alpha_i},$$

$$\lambda_i(x,t)\geq\left(17^{\frac{2}{2+\alpha^{-}}}n+2\right)^{\alpha_i}R^{\alpha_i}\geq\left(17^{\frac{2}{2+\alpha^{-}}}n+2\right)^{\alpha^{-}}R^{\alpha_i}.$$

Thus in any case

$$C_1(\alpha,n,b)\,R^{\alpha_i}\leq\lambda_i(x,t)\leq C_2(\alpha,n,b)\,R^{\alpha_i};\quad i=1,...,n. \tag{7}$$

Using (7) in (4) we obtain

$$J\geq\frac{1}{4\beta(t-\tau)}\left(\frac{\gamma C_1}{\beta}-1\right)\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}+s-\frac{nC_2}{2\beta\gamma}-\frac{b_0}{2\beta}\left(\sum_{i=1}^{n}R^{1-\frac{\alpha_i}{2}}\right)^{\frac{1}{2}}\times$$

$$\times\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}R^{-\frac{1}{2}}-\frac{9b}{8}b_0R^2. \tag{8}$$

Besides

$$\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\leq\left(\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n}\frac{y_i^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\leq 2\sqrt{17}R.$$

Therefore assuming in (8)

$$\beta=\gamma C_1 \tag{9}$$

and subject to the fact that $R\leq 1$ and $\alpha^{+}\leq 2$ we conclude

$$J\geq s-\frac{nC_2}{2\gamma^2 C_1}-\frac{b_0\sqrt{17n}}{\gamma C_1}-\frac{9b_0}{8}$$

Now it's sufficient to select

$$s=\frac{nC_2}{2\gamma^2C_1}+b_0\left(\frac{\sqrt{17n}}{\gamma C_1}+\frac{9}{8}\right) \tag{10}$$

and the lemma is proved.

Further, not specifying this we'll assume that the parameters $\beta$ and $s$ of the function $G_R^{s,\beta}(x,t)$ are chosen according to the equalities (9) and (10).

From the proof of the lemma it follows that

$$b\beta s\leq C_3(\gamma,\alpha,n,b_0)\,b^{1-\frac{|\alpha^-|}{2}}.$$

allowing for $|\alpha^-|<2$, we choose and fix an arbitrary number b$\in(0,1)$ satisfying the inequality

$$b\beta s\leq\frac{49}{4}. \tag{11}$$

Let $E$ be a $B$-set disposed in $\mathbb{C}^3$. We call the measure $\mu$ on $E$ $(s,\beta,R)$ - admissable, if

$$\int_E G_R^{s,\beta}(x-y,t-\tau)\,d\mu(y,\tau)\leq 1,\ \text{when}\ (x,t)\notin E.$$

The number $p_R^{s,\beta}(E)=\sup\mu(E)$, where the exact upper bound is taken on all the $(s,\beta,R)$- admissable measures, is called parabolic $(s,\beta,R)$- capacity of the set $E$.

Later on for shortening of notation we'll denote $p_R^{s,\beta}(E)$ simply by $p_R(E)$.

**Lemma 2.** *Let* $B=C_{R;\rho}^{t^0-\rho^2R^2,t^0}(x^0)$, *where* $\bar{B}\subset\mathbb{C}^3$. *Then*

$$C_4(s,\beta)(\rho R)^{2s}\leq p_R(B)\leq C_5(s,\beta)(\rho R)^{2s}.$$

**Proof.** We are restricted to proving of estimation of capacity below. Let $\mu$ be singular measure with the density $a$, concentrated at the point $\left(x^0,t^0-\frac{\rho^2R^2}{2}\right)$.

Consider the function

$$I(x,t)=\int_B G_R^{s,\beta}(x-y,t-\tau)\,d\mu(y,\tau)=$$

$$=\int_{\left\{\left(x^0,t^0-\frac{\rho^2R^2}{2}\right)\right\}} G_R^{s,\beta}\left(x-x^0,t-t^0+\rho^2R^2\right)d\mu\left(x^0,t^0-\frac{\rho^2R^2}{2}\right).$$

For $(x,t)\in S(B)$ we have

$$I(x,t)=\int_{\left\{\left(x^0,t^0-\frac{\rho^2R^2}{2}\right)\right\}}\left(t-t^0+\frac{\rho^2R^2}{2}\right)^{-s}\times$$

$$\times \exp\left[-\frac{\rho^2R^2}{4\beta\left(t-t^0+\frac{\rho^2R^2}{2}\right)}\right] d\mu\left(x^0,t^0-\frac{\rho^2R^2}{2}\right)\le$$

$$\le\left(\frac{\rho^2R^2}{4\beta s}\right)^{-s} e^{-4\beta s}a=(\rho R)^{-2s}(4\beta s)^s e^{-4\beta s}a, \tag{12}$$

since the function $z^{-s}\exp\left[-\frac{\rho^2R^2}{4\beta z}\right]$ defined on the semi-axis $(0,\infty)$ attains its maximum value at $z=\frac{\rho^2R^2}{4\beta s}$. If the point $(x,t)$ is disposed on upper foundation of $B$ then

$$I(x,t)\le(\rho R)^{-2s}\,2^s a. \tag{13}$$

Now assume $a=(\rho R)^{2s}\min\left\{(4\beta s)^{-s}e^{4\beta s},2^{-s}\right\}$.

Subject to the fact that the estimation (12) holds for $(x,t)\in\mathbb{R}_{n+1}$, if $x\notin\mathcal{E}_{R;\rho}\left(x^0\right)$, and the inequality (13) is valid for $(x,t)\in\mathbb{R}_{n+1}$ when $t\ge t^0$ and if we observe that $I(x,t)=0$ when $t\le t^0-\rho^2R^2$, we conclude

$$I(x,t)\le 1 \quad \text{when} \quad (x,t)\notin B.$$

From here it follows that the measure $\mu$ is $(s,\beta,R)$-admissable, and therefore

$$p_R(B)\ge\mu\left\{\left(x^0,t^0-\frac{a^2R^2}{2}\right)\right\}=(\rho R)^{2s}\min\left\{(4\beta s)^{-s}e^{4\beta s},2^{-s}\right\}.$$

The proof of the lemma is completed.

Let

$$\mathbb{C}^4=C_{R;9}^{-\frac{bR^2}{8},0}(0),\ \left(x^0,t^0\right)\in\Gamma\left(\mathbb{C}^4\right),\ \mathbb{C}^5\left(x^0,t^0\right)=C_{R;8}^{t^0-bR^2,t^0}\left(x^0\right),$$

$$\mathbb{C}^6\left(x^0,t^0\right)=C_{R;1}^{t^0-\frac{bR^2}{4},t^0}\left(x^0\right),\ \mathbb{C}^7\left(x^0,t^0\right)=C_{R;1}^{t^0-bR^2,t^0-\frac{bR^2}{2}}\left(x^0\right).$$

**Lemma 3.** *Let the domain $\mathbb{C}^5\left(x^0,t^0\right)$ having the limiting points on $\Gamma\left(\mathbb{C}^5\left(x^0,t^0\right)\right)$ and intersecting with $\mathbb{C}^6\left(x^0,t^0\right)$ is disposed in $D$. Let further the positive $\mathcal{L}$-subparabolic function $u(x,t)$, continuous in $\bar{D}$ vanishing on $\Gamma(D)\cap\mathbb{C}^5\left(x^0,t^0\right)$ be determined in $D$. Then if $E_R=\mathbb{C}^7\left(x^0,t^0\right)\backslash D$ and $R\le 1$ then*

$$\sup_D u\ge\left(1+\eta_1(\gamma,\alpha,n,b_0)R^{-2s}p_R(E_R)\right)\sup_{D\cap\mathbb{C}^6(x^0,t^0)}u. \tag{14}$$

**Proof.** For shortening of notation we'll denote the cylinder $\mathbb{C}^i\left(x^0,t^0\right)$ simple by $\mathbb{C}^i$, $i=5,6,7$. Without loss of generality we can assume that $p_R(E_R)>0$, otherwise the inequality (14) is obvious. Fix an arbitrary $\varepsilon\in(0,p_R(E_R))$ and let measure $\mu$ on $H_R$ be such that

$$U(x,t)=\int_{E_R}G(x-y,t-\tau)\,d\mu(y,\tau)\le 1 \ \text{ when } \ (x,t)\notin E_R, \tag{15}$$

$$\mu(E_R)>p_R(E_R)-\varepsilon, \tag{16}$$

where $G(x,t)=G_R^{s,\beta}(x,t)$. Fix the point $(y,\tau)\in\mathbb{C}^7$ and $n$-dimensional vector $x$ such that $x\in\partial\mathcal{E}_{R;8}\left(x^0\right)$. Here and further $\partial H$ is Euclidean boundary of the domain

$H$. Now we find that value $t>\tau$ for which the function $\vartheta(t)=G(x-y,t-\tau)$ attains its maximum. By equaling the derivative $\vartheta_t$ to zero, we obtain

$$t-\tau=\frac{1}{4\beta s}\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}.$$

On the other hand

$$\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\geq\left(\sum_{i=1}^{n}\frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}-\left(\sum_{i=1}^{n}\frac{\left(y_i-x_i^0\right)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\geq 8R-R=7R.$$

Thus $t-\tau\geq\frac{49}{4\beta s}R^2$. Then from the inequality (11) follows that $\tau-\tau\geq bR^2$. Subject to monotonicity of $\vartheta(t)$ upto the first maximum, we conclude

$$\sup_{\substack{(x,t)\in S(\mathbb{C}^5)\\(y,\tau)\in\mathbb{C}^7}}G(x-y,t-\tau)\leq\left(bR^2\right)^{-s}\exp\left[-\frac{49R^2}{4\beta bR^2}\right]=\left(bR^2\right)^{-s}\exp\left[-\frac{49}{4\beta b}\right].\quad(17)$$

Further we obtain

$$\inf_{\substack{(x,t)\in\mathbb{C}^6\\(y,\tau)\in\mathbb{C}^7}}G(x-y,t-\tau)\leq\left(bR^2\right)^{-s}\exp\left[-\frac{4R^2}{4\beta b\frac{R^2}{4}}\right]=\left(bR^2\right)^{-s}\exp\left[-\frac{4}{4\beta b}\right],\quad(18)$$

since if $x,y\in\mathcal{E}_{R;1}\left(x^0\right)$, then

$$\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\leq\left(\sum_{i=1}^{n}\frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n}\frac{\left(y_i-x_i^0\right)^2}{R^{\alpha_i}}\right)^{\frac{1}{2}}\leq 2R.$$

Now we consider the auxiliary function

$$W(x,t)=M\left[1-U(x,t)+\left(bR^2\right)^{-s}\exp\left[-\frac{49}{4\beta R}\right]p_R(E_R)\right]-u(x,t),$$

where $M=\sup\limits_{D}u$. It's clear that the function $W(x,t)$ is $\mathcal{L}$-superparabolic in $D$ by virtue of lemma 1 and the condition $c(x,t)\leq 0$. According to the inequality (17) $W(x,t)\geq 0$ for $(x,t)\in\Gamma(D)\cap S\left(\mathbb{C}^5\right)$. Besides $W(x,t)$ for $(x,t)\in\Gamma(D)\cap\mathbb{C}^5$ by virtue of the inequality (15). Finally $W(x,t)\geq 0$ for $(x,t)\in F\left(\mathbb{C}^5\right)$ outside of $E_R$, because there $U(x,t)=0$. Thus $W(x,t)\geq 0$ for $(x,t)\in\Gamma(D)$. By the maximum principle $W(x,t)\geq 0$ for $(x,t)\in D$ and in particular allowing for (18) and (16)

$$\sup_{D\cap\mathbb{C}^6}\leq M\left[1-\inf_{(x,t)\in\mathbb{C}^7}U(x,t)+\left(bR^2\right)^{-s}\exp\left[-\frac{49}{4\beta b}\right]p_R(E_R)\right]\leq$$

$$\leq M\left[1-\left(bR^2\right)^{-s}\left(\exp\left[-\frac{4}{\beta b}\right]-\exp\left[-\frac{49}{4\beta b}\right]\right)p_R(E_R)+\varepsilon\left(bR^2\right)^{-s}\exp\left[-\frac{4}{\beta b}\right]\right].$$

Now subject to arbitrariness of $\varepsilon$ and denoting $b^{-s}\left(\exp\left[-\frac{4}{\beta b}\right]-\exp\left[\frac{49}{4\beta b}\right]\right)$ by $\eta_1$ we obtain

$$\sup_{D\cap\mathbb{C}^6}u\leq M\left[1-\eta_1R^{-2s}p_R(E_R)\right].$$

Hence the required estimation (14) follows. The lemma is proved.

Let $\mathbb{C}^8 = C_{R;9}^{-bR^2,-\frac{3bR^2}{4}}(0)$.

**Theorem 1.** *Let the domain $\mathbb{C}^3$ having the limiting points on parabolic boundaries of the both cylinders $\mathbb{C}^1$ and $\mathbb{C}^2$ be disposed in $D$. Let further the positive $\mathcal{L}$-subparabolic function $u(x,t)$ continuous in $D$ and vanishing on $\Gamma(D)\cap\mathbb{C}^3$ be determined in $D$. Then if $H_R = \mathbb{C}^8\backslash D$ and $R \le 1$, then*

$$\sup_D u \ge \left(1+\eta_2(\gamma,\alpha,n,b_0)\,R^{-2s}p_R(H_R)\right)\sup_{D\cap\Gamma(\mathbb{C}^4)} u.$$

**Proof.** Without loss of generality we can assume that $\sup\limits_{D\cap\Gamma(\mathbb{C}^4)} u = 1$. Let $(x^*,t^*) \in \in D\cap\Gamma\left(\mathbb{C}^4\right)$ be a point in which $u(x^*,t^*) = 1$. At first assume that $(x^*,t^*) \in \bar{F}\left(\mathbb{C}^4\right)$, i.e. $(x^*,t^*) = \left(x^*,t^0\right)$, where $t^0 = -\frac{bR^2}{8}$. We choose on $\bar{F}\left(\mathbb{C}^4\right)$ a minimum number of the points $\left(x^1,t^0\right),...,\left(x^m,t^0\right)$ such that

i) $\bar{\mathbb{C}}^4 \subset \bigcup\limits_{i=1}^{m}\mathbb{C}^7\left(x^i,t^0\right)$;

ii) one of the points $\left(x^i,t^0\right)$ coincides with the point $\left(x^*,t^0\right)$;

iii) for any $i,\ 1\le i\le m$ there be found $j,\ 1\le 1\le m$ such that $x^j \in \partial\mathcal{E}_{\frac{R}{A^m};1}\left(x^i\right)$, where the constant $A>1$ will be chosen later.

It's clear that the number $m$ depends only on $\alpha$ and $n$. From properties of covering it follows that for any $i_0,\ 1\le i_0\le m$ there exists the chain $\left(x^{i_1},t^0\right),...,\left(x^{i_k},t^0\right)$ such that $\left(x^{i_k},t^0\right) = \left(x^*,t^0\right)$, $x^{i_l+1} \in \partial\mathcal{E}_{\frac{R}{A^m};1}\left(x^{i_l}\right)$; $l = 0,...,k-1$. From subadditivity of parabolic capacity we conclude on existence of $i_0,\ 1\le i_0\le m$, such that

$$p_R\left(H_R\cap\mathbb{C}^7\left(x^{i_0},t^o\right)\right) \ge \frac{p_R(H_R)}{m}.$$

Let $\delta = \frac{\eta_1 R^{-2s}p_R(H_R)}{2m\left(1+\frac{\eta_1}{m}C_6\right)}$, where the constant $C_6(\gamma,\alpha,n,b_0)$ is such that $p_R(H_R) \le C_6R^{2s}$ (see lemma 2). Denote $\mathbb{C}^7\left(x^{i_0},t^0\right)\backslash D$ by $H$. Then

$$p_R(H) \ge \frac{p_R(H_R)}{m}. \tag{19}$$

Assume that $\sup\limits_{D\cap\mathbb{C}^6\left(x^{i_0},t^0\right)} u \ge 1-\delta$. Then according to lemma 3 and the inequality (19)

$$\sup_D u \ge \sup_{D\cap\mathbb{C}^5\left(x^{i_0},t^0\right)} u \ge \left(1+\eta_1R^{-2s}p_R(H)\right)(1-\delta) \ge$$

$$\ge\left(1+\frac{\eta_1R^{-2s}p_R(H_R)}{m}\right)\left(1-\frac{\eta_1R^{-2s}p_R(H_R)}{2m\left(1+\frac{\eta_1R^{-2s}p_R(H_R)}{m}\right)}\right) = 1+\frac{\eta_1}{2m}R^{-2s}p_R(H_R),$$

and in this case the statement of the theorem is proved.

Let now $u(x,t) < 1-\delta$ for $(x,t)\in\mathbb{C}^6\left(x^{i_0},t^0\right)\cap D$. Consider the function $\vartheta_1(x,t) = u(x,t)-1+\delta$. It's easy to see that the function $\vartheta_1(x,t)$ is $\mathcal{L}$- subparabolic in $D$, since $\delta<1$.

Let $D_1=\{(x,t):(x,t)\in D, \vartheta_1(x,t)>0\}$. By assumption $\mathbb{C}^6\left(x^{i_0},t^0\right)\subset\mathbb{R}_{n+1}\backslash D_1$. For $(x',t')\in\Gamma\left(\mathbb{C}^4\right)$ denote by $\mathbb{C}^i_{R'}(x',t')$ the cylinder $\mathbb{C}^i(x',t')$; $i=4,...,8$, emphasizing that in it $R=R'$. Now we find such $A>1$ that $\mathbb{C}^5_R(x',t')\subset\mathbb{C}^6_{AR}(x',t')$. It's clear that for the validity of inclusion it's sufficient that

$$\frac{b\,(AR)^2}{4}\geq bR^2,\quad AR\,(AR)^{\frac{\alpha_i}{2}}\geq 8R^{1+\frac{\alpha_i}{2}};\ \ i=1,...,n.$$

The last inequalities are satisfied, if we fix $A=8^{\frac{2}{2+\alpha^-}}$. Then the statement of lemma 3 is valid, if all its conditions are satisfied, but the domain $D$ is disposed in the cylinder $\mathbb{C}^6_{AR}\left(x^0,t^0\right)$.

Let $\left(x^{i_1},t^0\right),...,\left(x^{i_k},t^0\right)-$ be abovementioned chain. By construction $\mathbb{C}^7_{\frac{R}{A}}\left(x^{i_1},t^0\right)\backslash D_1$ contains the cylinder $C^{t'-b\left(\frac{R}{A}\rho\right)^2,t'}_{\frac{x}{A};\rho}(x')$, and the parabolic $\left(s,\beta,\frac{R}{A}\right)$-capacity of which according to lemma 2, is not less than $C_7(\rho,\gamma,\alpha,n,b_0)\left(\frac{R}{A}\right)^{2s}$. For this $\rho$ depends only on $\alpha$ and $n$. Let

$$\sigma=\frac{\eta_1 C_7}{2\,(1+\eta_1 C_7)}.$$

Assume that $\sup\limits_{D_1\cap\mathbb{C}^6_{\frac{R}{A}}\left(x^{i_1},t^0\right)}\vartheta_1\geq\delta\,(1-\sigma)$, i.e. $\sup\limits_{D_1\cap\mathbb{C}^6_{\frac{R}{A}}\left(x^{i_1},t^0\right)}u\geq 1-\delta\sigma$. Using lemma 3 we obtain $\sup\limits_{D_1\cap\mathbb{C}^6_{R}\left(x^{i_1},t^0\right)}\vartheta_1\geq(1+\eta_1C_7)\,\delta\,(1-\sigma)$.

Thus

$$\sup_D u\geq\sup_{D\cap\mathbb{C}^6_R\left(x^{i_1},t^0\right)}u\geq 1-\delta+(1+\eta_1C_7)\,\delta\,(1-\sigma)=1+\delta\eta_1C_7-$$

$$-\delta\sigma\,(1+\eta_1C_7)=1+\frac{\delta\eta_1C_7}{2}$$

and in this case the statement of the theorem is proved.

Assume that $u(x,t)<1-\delta\sigma$ for $(x,t)\in D\cap\mathbb{C}^6_{\frac{R}{A}}\left(x^{i_1},t^0\right)$. Consider the $\mathcal{L}$-superparabolic function $\vartheta_2(x,t)=u(x,t)-1+\delta\sigma$ in $D$. Let

$$D_2=\{(x,t):(x,t)\in D,\vartheta_2(x,t)>0\}.$$

By assumption $\mathbb{C}^6_{\frac{R}{A}}\left(x^{i_1},t^0\right)\subset\mathbb{R}_{n+1}\backslash D_2$. If now

$$\sup_{D_2\cap\mathbb{C}^6_{\frac{R}{A^2}}\left(x^{i_2},t^0\right)}\vartheta_2\geq\delta\sigma\,(1-\sigma)\,,\ \text{ i.e. }\sup_{D\cap\mathbb{C}^6_{\frac{R}{A^2}}\left(x^{i_2},t^0\right)}u\geq 1-\delta\sigma^2,$$

then using lemma 3 we obtain

$$\sup_D u\geq\sup_{D\cap\mathbb{C}^6_{\frac{R}{A}}\left(x^{i_2},t^0\right)}u\geq 1-\delta\sigma+(1+\eta_1C_7)\,\delta\sigma\,(1-\sigma)=1+\frac{\delta\sigma\eta_1C_7}{2},$$

and in this case the statement of the theorem is proved. If $u(x,t)<1-\delta\sigma^2$ for $(x,t)\in\mathbb{C}^6_{\frac{R}{A^2}}\left(x^{i_2},t^0\right)$, then we continue this process analogously. At the latest than in $k$-th step we'll prove the theorem, since $u\left(x^{i_k},t^0\right)=u\left(x^*,t^0\right)=1$. Thus the theorem is proved, if $(x^*,t^*)\in\bar{F}\left(\mathbb{C}^4\right)$. Let now $(x^*,t^*)\in S\left(\mathbb{C}^4\right)$ and $t^*>t^0$. It's clear that $x^*\in\partial\mathcal{E}_{R;9}(0)$. From above mentioned reasonings it follows that either the theorem is proved or $u(x,t)<1-\delta\sigma^m$ for $(x,t)\in D\cap\mathbb{C}^6_{\frac{R}{A^m}}\left(x^*,t^0\right)$.

We choose the minimum number $p$ of points $\left(x^*,t^1\right),...,(x^*,t^p)$ on the segment $l$ connecting the points $\left(x^*,t^0\right)$ and $(x^*,t^*)$ such that

j) $l\subset\bigcup\limits_{i=1}^{p}\mathbb{C}^6_{\frac{R}{A^m}}\left(x^*,t^i\right);\ t^p=t^*$;

jj) the cylinder $\mathbb{C}^7_{\frac{R}{A^{m+1}}}\left(x^*,t^{i+1}\right);\ i=0,...,p-1$ is contained in the intersection $\mathbb{C}^6_{\frac{R}{A^m}}\left(x^*,t^i\right)\cap\mathbb{C}^6_{\frac{R}{A^m}}\left(x^*,t^{i+1}\right)$.

It's clear that $p$ depends only on $\alpha$ and $n$. By construction and lemma 2 $p_{\frac{R}{A^m}}\left(\mathbb{C}^7_{\frac{R}{A}}x^*,t^i\backslash D\right)\geq C_8(\gamma,\alpha,n,b_0)\left(\frac{R}{A^m}\right)^{2s}$. Consider the $\mathcal{L}$-subparabolic function $w_1(x,t)=u(x,t)-1+\delta\sigma_1^m$ in $D$, where $\sigma_1=\min\left\{\sigma,\frac{\eta_1C_8}{2(1+\eta_1C_8)}\right\}$. Let $D^1=\{(x,t):(x,t)\in\ \in D, w_1(x,t)>0\}$. By assumption $\mathbb{C}^6_{\frac{R}{A^m}}\left(x^*,t^i\right)\subset\mathbb{R}_{n+1}\backslash D$. If

$$\sup_{D^1\cap\mathbb{C}^6_{\frac{R}{A^m}}(x^*,t^1)}w_1\geq\delta\sigma_1^m(1-\sigma_1)\,,\ \text{i.e.}\ \sup_{D\cap\mathbb{C}^6_{\frac{R}{A^m}}(x^*,t^1)}u\geq1-\delta\sigma_1^{m+1},$$

then using lemma 3 we obtain

$$\sup_{D^1\cap\mathbb{C}^6_{\frac{R}{A^{m-1}}}(x^*,t^1)}w_1\geq(1+\eta_1C_8)\,\delta\sigma_1^m(1-\sigma_1)\,.$$

Thus

$$\sup_D u\geq\sup_{D\cap\mathbb{C}^6_{\frac{R}{A^{m-1}}}(x^*,t^1)}u\geq1-\delta\sigma_1^m+(1+\eta_1C_8)\,\delta\sigma_1^m(1-\sigma_1)\geq1+\frac{\delta\sigma_1^m\eta_1C_8}{2},$$

and in this case the statement of the theorem is proved. If $u(x,t)<1-\delta\sigma_1^{m+1}$ for $(x,t)\in D\cap\mathbb{C}^6_{\frac{R}{A^m}}\left(x^*,t^1\right)$, then we continue the process analogously. At the latest than on $p$-th step we prove the theorem, since $u(x^*,t^p)=u(x^*,t^*)=1$. The theorem is completely proved.

**Corollary 1.** *The statement of theorem remains valid, if all its conditions are satisfied, but the domain $D$ disposed in $\mathbb{C}^1$, has the limiting points on $\Gamma\left(\mathbb{C}^1\right)$, intersects $\mathbb{C}^4$ and $u|_{\Gamma(D)\cap\mathbb{C}^1}=0$. In addition $\sup\limits_{D\cap\Gamma(\mathbb{C}^4)}u=\sup\limits_{D\cap\mathbb{C}^4}u$.*

This corollary follows from theorem 1 and maximum principle.

**Corollary 2.** *Let $A_1=\max\left\{3,\frac{17^{\frac{2}{2+\alpha^-}}}{9}\right\}$. Then the statement of the theorem remains valid, if all its conditions are satisfied, but the domain $D$ disposed in $\mathbb{C}^4_{A_1R}$, has the limiting points on $\Gamma\left(\mathbb{C}^4_{A_1R}\right)$, intersects $\mathbb{C}^4$ and $u|_{\Gamma(D)\cap\mathbb{C}^4_{A_1R}}=0$.*

For the proving it's sufficient to note that $\mathbb{C}^1\subset\mathbb{C}^4_{A_1R}$.

Consider the first boundary value problem for the equation (1)

$$\mathcal{L}u=0,\quad(x,t)\in D;\quad u|_{\Gamma(D)}=\varphi,\quad\varphi\in[\Gamma(D)]\,.\tag{20}$$

Let $u_\varphi(x,t)$ be a generalized solution by Wiener-Landis [5] of this problem. We shall assume its existence not specifying this.

The point $(0,0)$ is called regular with respect to the first boundary value problem (19), if for any $\varphi(x,t) \in C[\Gamma(D)]$ the limiting equality

$$\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in D}} u_\varphi(x,t) = \varphi(0,0)$$

is valid.

Let for natural numbers $j$ $H(j) = \mathbb{C}^8_{A_1^{-j}} \backslash D,\ p_j = p_{A_1^{-j}}(H(j))$.

**Theorem 2.** I*f relative to the coefficients of the operator $\mathcal{L}$ the conditions (2)-(3) are satisfied in the domain $D$, then for regularity of the point $(0,0)$ with respect to the first boundary value problem (19) it's sufficient that*

$$\sum_{j=1}^{\infty} A_1^{2sj} p_j = \infty. \tag{21}$$

**Proof.** According to [5] it's sufficient to show the following: whatever were the numbers $\varepsilon_1 > 0,\ \varepsilon_2 > 0$, the subdomain $D'$ of the domain $D$ completely disposed in the halfspace $t < 0$ and $\mathcal{L}$-subparabolic function $u(x,t) \le 1$ in $D'$, there exists $\delta > 0$ such that from $u|_{\Gamma(D')\cap\mathbb{C}^4_{\varepsilon_1}} \le 0$ it follows $u|_{D'\cap\mathbb{C}^4_\delta} \le \varepsilon_2$ .

Let $j_0$ be the least natural number for which $A_1^{-j_0} < \varepsilon_1$, and $j > j_0$ be a natural number such that there exists a point $(x',t')$ in $D' \cap \mathbb{C}^4_{A_1^{-j}}$, where $u(x',t') \ge \varepsilon_2$. Allowing for $H(j) \subset \mathbb{C}^8_{A_1^{-j}} \backslash D'$ and using corollary 2 from from the theorem 1 we obtain

$$1 \ge M_{j_0} \ge \left(1 + \eta_2 A_1^{2s(j_0+1)} p_{j_0+1}\right) M_{j_0+1} \ge \dots \ge \prod_{i=j_0}^{j-1} \left(1 + \eta_2 A_1^{2s(i+1)} p_{i+1}\right) M_j \ge$$

$$\ge \prod_{i=j_0+1}^{j} \left(1 + \eta_2 A_1^{2si} p_i\right) \varepsilon_2,$$

where $M_i = \sup\limits_{D'\cap\mathbb{C}^4_{A_1^{-i}}} u^+;\ i = j_0, ..., j;\ u^+(x,t) = \max\{u(x,t), 0\}$. Hence it follows that

$$\sum_{i=j_0+1}^{j} \ln\left(1 + \eta_2 A_1^{2si} p_i\right) \le \ln \frac{1}{\varepsilon_2}. \tag{22}$$

On the other hand, according lemma 2

$$p_i \le C_9(\gamma, \alpha, n, b_0) A_1^{-2si}; \quad i = j_0, ..., j.$$

So we have

$$\ln\left(1 + \eta_2 A_1^{2si} p_i\right) \ge C_{10}(\gamma, \alpha, n, b_0) A_1^{2si} p_i; \quad i = j_0, ..., j.$$

Therefore, we conclude from (21)

$$\sum_{i=j_0+1}^{j} A_1^{2si} p_i \leq C_{10}^{-1} \ln \frac{1}{\varepsilon_2}.$$

By virtue of the condition (20) the last inequality can't be satisfied when $j \geq j^* (\varepsilon_1, \varepsilon_2, \gamma, \alpha, n, b_0)$. Now it's sufficient to choose $\delta = A_1^{-j^*}$, and the theorem is proved.

**Remark.** We can write the condition (20) in integral form. Namely, for regularity of the point $(0,0)$ with respect to the first boundary value problem (19) it's sufficient that

$$\int_0^1 \frac{p_z (H_z)}{z^{2s+1}} dz = \infty.$$

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