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# LIMIT THEOREMS FOR CONTINUOUS TIME BRANCHING PROCESSES

**Abstract**

*In the paper the continuous time branching processes are investigated. The limit theorems on convergence of investigated processes to the limit with defined generating function are obtained.*

In [1,2] we consider basically a class of branching processe which the existence time of one generation was fixed. Although a lot of phenomena in the various fields of life satisfy this condition, the continuous time branching processes represents certain interest.

Let $\xi(t)$, $t \in [0,\infty)$ be a random process getting the integer non-negative values with transient probabilities $P_{ij}(t) = P\{\xi(t) = j/\xi(0) = i\}$ such that at the existing at the initial moment $i$ particles at the time $t$ are converted in $j-$particles.

If for the transient probabilities $P_{ij}(t)$ the conditions

$$P_{ij}(t) = \begin{cases} \delta_{0j}, \quad i = 0, \\ \sum\limits_{j_1+j_2+\cdots+j_i=j} P_{1j_1}(t) \cdot P_{1j_2}(t) \cdots P_{1j_i}(t), \quad i \neq 0 \end{cases}$$

and

$$\lim_{t\to 0} P_{ij}(t) = \delta_{ij}$$

($\delta_{ij}$ is a Kronecker's symbol) are fulfilled, then $\xi(t)$ is a branching process with continuous time.

Now let there will be one particle at the initial moment, i.e. $\xi(0) = 1$ and the transient probabilities $P_{1j}(t)$ are representable in the following form

$$P_{11}(t) = 1 + p_1 t + o(t), \quad j = 1,$$

$$P_{1j}(t) = p_j(t) + o(t), \ j \neq 1.$$

Assume that the densities of the transient probabilities $p_j \geq 0$ at $j \neq 1$, $p_1 < 0$ and

$$\sum_{j=0}^{\infty} p_j = 0. \tag{1}$$

Denote the generating function of densities of transient probabilities by $f(s)$

$$f(s) = \sum_{j=0}^{\infty} p_j s^j, \ f'(1) = a, \ f''(1) = b, \ |s| \leq 1.$$

The characteristics of continuous time branching processes is given by generating function of numbers of direct descendants of a particle at the time $t$ $F(t;s)$

$$F(t;s) = \sum_{j=0}^{\infty} P_{1j}(t) s^j.$$

It's known [3, p.27] that the generating function $F(t;s)$ satisfies the ordinary differential equation

$$\frac{\partial F(t;s)}{\partial t} = f(F(t;s)) \qquad (2)$$

with the initial condition

$$F(0;s) = s. \qquad (3)$$

The continuous time branching processes is divided into three groups depending on the sign $a$: subcritical $(a < 0)$, critical $(a = 0, b > 0)$ and above-critical $(a > 0)$. We'll consider only subcritical case under the additional condition $b < \infty$.

Introduce the function

$$K(s) = \int_1^s \left[\frac{1}{f(x)} - \frac{1}{a(x-1)}\right] dx + \frac{1}{a}\ln(1-s). \qquad (4)$$

We prove the existence and finitness of the integral later on.

Now we formulate the theorem to the effect that how the generating function $F(t;s)$ is expressed by the function $K(s)$.

**Theorem 1.** *Let $\xi(t)$, $t \geq 0$ be a continuous time branching process and with generating function of the density $f(s)$. Assume that the the condition (1) is satisfied and $a < 0$. Then*

$$F(t;s) = K^{-1}(t + K(s)), \quad |s| \leq 1.$$

**Proof.** At first we show the finiteness of the integrand. Indeed expanding the function $f(s)$ in the neighbourhod of the point $s = 1$, we obtain the following

$$f(s) = f(1) + a(s-1) + \frac{b}{2}(s-1)^2, \quad s \leq 1.$$

Since by the condition (1) $f(1) = 0$, then

$$\frac{1}{f(s)} = \frac{1}{a(s-1) + \frac{b}{2}(s-1)^2} = \frac{1}{a(s-1)\left[1 + \frac{b}{2a}(s-1)\right]} =$$

$$= \frac{1}{a(s-1)} - \frac{b/2a^2}{1 + b(s-1)/2a}.$$

Hence

$$\frac{1}{f(s)} - \frac{1}{a(s-1)} = -\frac{b/2a^2}{1 + b(s-1)/2a} < \infty,$$

and consequently the integral in (4) exists and is finite. It's obvious from (4) that by virtue the assumption $f'(1) = a < 0$

$$K'(s) = \frac{1}{f(s)} > 0 \ \ at \ \ 0 \leq s < 1.$$

It means that $K(s)$ is a stricly increasing continuous function. Consequently, the mapping

$$W = K(s) \tag{5}$$

has the inverse which is the continuous strictly increasing function

$$s = K^{-1}(W) = L(W), \quad L(K(s)) = s. \tag{6}$$

This inverse function has such properties that at variation of $s$ on the interval $[0,1)$ the function $W$ varies on the interval $[K(0), \infty)$, $K(0) < 0$.

Now we have the all necessary for the solving the equation (2) with the initial condition (3).

Separating the variable and integrating from (2) we obtain the exact formula with respect to $F(t;s)$

$$\int_s^{F(t,s)} \frac{dx}{f(x)} = t.$$

By accomplishing the simple transformations we find

$$t = \int_s^{F(t,s)} \frac{dx}{f(x)} = \int_s^{F(t,s)} \left[\frac{1}{f(x)} - \frac{1}{a(x-1)}\right] dx +$$

$$+\frac{1}{a}\ln[1 - F(t;s)] - \frac{1}{a}\ln(1-s) =$$

$$= \int_1^{F(t,s)} \left[\frac{1}{f(x)} - \frac{1}{a(x-1)}\right] dx + \frac{1}{a}\ln[1 - F(t;s)] -$$

$$-\int_1^{s} \left[\frac{1}{f(x)} - \frac{1}{a(x-1)}\right] dx - \frac{1}{a}\ln(1-s) == K(F(t;s)) - K(s)$$

or

$$K(F(t;s)) = t + K(s).$$

Since there exists the inverse function, then from here the required is obtained.

If in branching process at some moment of time $t$ the number of particles $\xi(t) = 0$, then it's said that the process is degeneration to the moment $t$, and $P_{10}(t) = P\{\xi(t) = 0/\xi(0) = 1\}$ is called the probability of degeneration to the moment $t$.

Consequently $1 - P_{10}(t) = P\{\xi(t) > 0\}$ will be the probability of continuation of process. Note that for the subcritical processes $(a < 0)$ the probability of degeneration is

$$q = \lim_{t\to\infty} P_{10}(t) = 1.$$

For the subcritical processes we can also obtain some asymptotical results with respect to probability of degeneration to the moment $t$.

**Theorem 2.** *Let $\xi(t)$, $t \geq 0$ be a subcritical branching process with the generating function $f(s)$ and $b = f''(1) < \infty$. Then the probability that the generation doesn't occur to time $t$, tends to zero as an exponent in accordance with the relation*

$$\lim_{t\to\infty}\left[1-P_{10}(t)\right]e^{-a[t+K(0)]}=1$$

*and the finite dimensional distribution* $\xi(t)$ *weakly converges to its limit with the generating function*

$$g(s)=\exp\left[a\int_0^s\frac{dx}{f(x)}\right]\qquad \text{as}\ \ t\to\infty.$$

**Proof.** Since the integrand function in (4) is bounded at $0\le s<1$, we can write

$$K(s)=\frac{1}{a}\ln(1-s)+C(1-s)+0(1-s), \tag{7}$$

where

$$C=\lim_{x\to 1}\left[\frac{1}{f(x)}-\frac{1}{a(x-1)}\right].$$

We rewrite the equality (7) in the form of

$$\ln(1-s)=aK(s)-Ca(1-s)+0(1-s),$$

whence

$$1-s=\exp\left[aK(s)\right]\exp\left[-Ca(1-s)\right]\exp\left[0(1-s)\right].$$

Taking into account that

$$\exp\left[0(1-s)\right]=1+0(1-s),$$

$$\exp\left[-Ca(1-s)\right]=1-Ca(1-s)+0(1-s),$$

from the last equality we obtain

$$1-s=\exp\left[aK(s)\right]\cdot\left[1-Ca(1-s)+0(1-s)\right]. \tag{8}$$

Consequently,

$$\lim_{s\to\infty}\frac{1-s}{\exp\left[aK(s)\right]}=1.$$

We can write (8) with the help of this limit relation in the following form

$$1-s=\left\{\exp\left[aK(s)\right]\right\}\left\{1-Ca\exp\left[aK(s)\right]+0\left(\exp\left[aK(s)\right]\right)\right\}.$$

Substituting $s$ by $K^{-1}(W)$ in the relation (8) we obtain

$$1-K^{-1}(W)=\exp\left[aW\right]\left\{1-Ca\exp\left[aW\right]+0\left(\exp\left[aW\right]\right)\right\}, \tag{9}$$

in addition, the relation $s\to 1$ is equivalent to $W\to\infty$.

Now with the help of (9) and the equality $F(t;s)=K^{-1}(t+K(s))$ we can find the probability such that the generation doesn't occur to the moment $t$

$$1-P_{10}(t)=1-F(t;0)=1-K^{-1}(t+K(0))=$$

$$=\left\{\exp\left[a(K(0)+t)\right]\right\}\left\{1-Ca\exp\left[a(K(0)+t)\right]+0\left[a(K(0)+t)\right]\right\}=$$

$$= \exp\left[a\left(K\left(0\right)+t\right)\right] + O\left(\exp\left(2at\right)\right) + 0\left(\exp\left(at\right)\right)$$

or

$$1 - P_{10}\left(t\right) = \exp\left[a\left(K\left(0\right)+t\right)\right] + 0\left(\exp\left(at\right)\right).$$

The first part of the theorem is proved. Prove the second part.

The conditional generating function at the process $\xi(t)$ provided $\xi(t) \neq 0$, is defined by the equality

$$g\left(s,t\right) = \sum_{k=0}^{\infty} P\left\{\xi\left(t\right)=k \middle| \xi\left(t\right)\neq 0\right\} s^k, \qquad 0 \leq s < 1.$$

Since

$$P\left\{\xi\left(t\right)=k \middle| \xi\left(t\right)\neq 0\right\} = \frac{P\left\{\xi\left(t\right)=k\right\}}{1-P\left\{\xi\left(t\right)=0\right\}},\ k \neq 0,$$

then by using the formula for $F(t;s)$ we can write

$$g\left(s;t\right) = \sum_{k=1}^{\infty} \frac{P\left\{\xi\left(t\right)=k\right\}}{1-P\left\{\xi\left(t\right)=0\right\}} s^k = \frac{F\left(t;s\right)-F\left(t;0\right)}{1-F\left(t;0\right)} =$$

$$= \frac{K^{-1}\left(t+K\left(s\right)\right)-K^{-1}\left(t+K\left(0\right)\right)}{1-K^{-1}\left(t+K\left(0\right)\right)}$$

$$= \frac{\left[1-K^{-1}\left(t+K\left(0\right)\right)\right]-\left[1-K^{-1}\left(t+K\left(s\right)\right)\right]}{1-K^{-1}\left(t+K\left(0\right)\right)}.$$

By the formula (9) from here we obtain

$$g\left(s,t\right) = \frac{e^{a(t+K(0))}\left[1+o\left(e^{a(t+K(0))}\right)\right]-e^{a(t+K(s))}\left[1+o\left(e^{a(t+K(s))}\right)\right]}{e^{a(t+K(0))}\left[1+o\left(e^{a(t+K(0))}\right)\right]}+$$

$$= 1 - e^{a[K(s)-K(0)]}\frac{1+o\left(e^{a(t+K(s))}\right)}{1+o\left(e^{a(t+K(0))}\right)}.$$

Taking into account that $a < 0$, if we tend $t$ to $\infty$, then the relation in the right hand side of the last relation tends to 1.

Consequently,

$$\lim_{t\to\infty} g\left(s,t\right) = g\left(s\right) = 1 - \exp\left[a\left(K\left(s\right)-K\left(0\right)\right)\right].$$

On the other hand from the relation (4)

$$K\left(s\right)-K\left(0\right) = \int_0^s \left[\frac{1}{f\left(x\right)} - \frac{1}{a\left(x-1\right)}\right]dx + \frac{1}{a}\ln\left(1-s\right) = \int_0^s \frac{dx}{f\left(x\right)}.$$

Hence it follows that when $t\to\infty$ the limit generating function has the following form

$$g\left(s\right) = 1 - \exp\left[a\int_0^s \frac{dx}{f\left(x\right)}\right].$$

**References**

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