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A NONLINEAR PARABOLIC INVERSE COEFFICIENT PROBLEM

Abstract

In the paper a nonlinear second order parabolic inverse coefficent problem is considered. the theorem on iniqueness and stability of solution of the stated problem is proved.

In the present paper questions on uniqueness and stability of solution of nonlinear second order parabolic inverse coefficient problem are studied.

Let's accept the following denotations: D is a bounded domain from R^n with boundary $\partial D \in C^{2+\alpha}$, $Q_T = D \times (0,T]$, $S_T = \partial D \times [0,T]$, $0 < T = const$, $\|\cdot\|_{C^k} = \|\cdot\|_k$ spaces $C^k(\cdot)$, $C^{k+\alpha, \frac{(k+\alpha)}{2}}(\cdot)$, $k = 0,1,2;$ $0 < \alpha < 1$, and corresponding norms are defined for example in [1, p.16].

Let's consider the problem on determination of $\{u(x,t), g(u(x,t))\}$ from conditions

$$u_t - L(a,b,c)u = f(x,t)g(u), \ (x,t) \in Q_T, \tag{1}$$

$$u(x,0) = \varphi(x), \ x \in \overline{D} = D \cup \partial D; \ u(x,t) = \psi(x,t), \ (x,t) \in S_T, \tag{2}$$

$$u(x;t) = h(t), \ t \in [0,T], \tag{3}$$

where $L(a,b,c)u \equiv \sum\limits_{i,j=1}^{n} a_{ij}(x,t) \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum\limits_{i=1}^{n} b_i(x,t) \dfrac{\partial u}{\partial x_i} + c(x,t)u$, $u_t \equiv \dfrac{\partial u}{\partial t}$, $a_{ij}(x,t)$, $b_i(x,t)$, $\left(i,j=\overline{1,n}\right)$, $c(x,t)$, $f(x,t)$, $\varphi(x)$, $\psi(x,t)$, $h(t)$ are given functions, $x^* = (x_1^*, ..., x_n^*)$ is a fixed point from D.

The nonlinear parabolic inverse coefficient problems were considered in papers [1-5]. In papers [2, 3, 5] the uniqueness theorems of one-dimesional inverse problems were obtained under the assumption that function $g(\cdot)$ is known on a certain part of the range of function $u(x,t)$ (in [4] such a condition is absent) and temperature and heat flow are given on the "lateral wall" [2, 4, 5] (condition of the type (3) is given).

Let's make the following suppositions respective to the input data:

1^0.For an arbitrary real vector $\eta = (\eta_1, ..., \eta_n)$ and for any $(x,t) \in \overline{Q}_T$

$$m_0 \sum_{i=1}^{n} \eta_i^2 = \sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j \le m_1 \sum_{i=1}^{n} \eta_i^2, \ \ 0 < m_0 < m_1;$$

2^0. $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t) \in C^{\alpha,\frac{\alpha}{2}}\left(\overline{Q}_T\right)$, $i,j = \overline{1,n}$;

3^0. $f(x,t) \in C^{\alpha,\frac{\alpha}{2}}\left(\overline{Q}_T\right)$, $|f(x^*,t)| \ge m_2 > 0$, $t \in [0,T]$;

4^0. $\varphi(x) \in C^{2+\alpha}\left(\overline{D}\right)$, $\psi(x,t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(S_T)$, $\varphi(x) = \psi(x,0)$,

$$f(x^*,0)\left[\psi_t(x,0) - L(a,b,c)|_{t=0}\,\varphi(x)\right] =$$

$$= f(x,0)\left[h(0) - L(a,b,c)\,\varphi|_{\substack{x=x^*\\ t=0}}\right],\ x\in\partial D;$$

5^0. $h(t)\in C^{1+\alpha}[0,T]$, $h(0)=\varphi(x^*)$, $r_1=\min\limits_{[0,T]} h(t)$, $r_2=\max\limits_{[0,T]} h(t)$, $h(t)$ has the inverse function $H(h)\in C^{1+\alpha}[r_1,r_2]$ with the range in $[0,T]$.

If in equation (1) function $g(\cdot)$ is given then, naturally, condition (3) isn't given. The problem on determination of $u(x,t)$ from (1)-(2) in more general statement were considered, for example, in papers [1,6 and others].

Problem (1)-(3) relates to the class of incorrect by Hadamard problems therefore we have to regard it based on general conseptions of incorrect problems theory [7].

Definition 1. *We shall call functions $\{u(x,t), g(\cdot)\}$ the solution of problem (1)-(3) if: 1) $u(x,t)\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$; 2) $g(\cdot)\in C(R^1)$; 3) relations (1)-(3) are satisfied for them.*

Definition 2. *We shall say that solution of problem (1)-(3) belongs to the set K if: 1)*

$$u(x,t)\in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T);\ 2)g(\cdot)\in C^{\alpha}(R^1),\ \|g(\cdot)\|_{C(R^1)}\leq$$

$$\leq \|g(\cdot)\|_{C[r_1,r_2]}.$$

Theorem. *Let conditions 1^0-5^0 be satisfied. Then if solution of problem (1)-(3) exists and belongs to the set K, then it is unique and the following estimate of stability is valid*

$$\|u-\overline{u}\|+\|g(u)-\overline{g}(\overline{u})\|\leq M_1\left[\sum_{i,j=1}^{n}\|a_{ij}-\overline{a}_{ij}\|+\sum_{i=1}^{n}\left\|b_i-\overline{b}_i\right\|+\right.$$

$$\left.+\|c-\overline{c}\|+\left\|f-\overline{f}\right\|+\|\varphi-\overline{\varphi}\|_2+\left\|\psi-\overline{\psi}\right\|_{2,1}+\left\|h-\overline{h}\right\|_1\right],\tag{4}$$

where $\{\overline{g}(\overline{u}),\overline{u}(x,t)\}$ is the solution of problem (1)-(3) from the set K with the data $\overline{a}_{ij}$, $\overline{b}_i$, $\overline{c}$, $\overline{f}$, $\overline{\varphi}$, $\overline{\psi}$, $\overline{h}$, which satisfy the conditions 1^0-5^0respectively, $M_1>0$ depends on the problem data and the set K (everywhere below by M_i we shall denote positive constants which depend both on the problem data and the set K, and those which depend only on the problem data- by N_i).

Proof. Let's define the function [1, p.509]

$$p(x,t)\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T),\ p(x,0)=\varphi(x),\ x\in\overline{D},$$

$$p(x,t)=\psi(x,t),\ (x,t)\in S_T.\tag{5}$$

Let

$$z(x,t)=u(x,t)-\overline{u}(x,t),\ \lambda(u,\overline{u})=g(u)-\overline{g}(\overline{u}),$$

$$\delta_1(x,t)=\sum_{i,j=1}^{n}(a_{ij}(x,t)-\overline{a}_{ij}(x,t)),$$

$$\delta_2(x,t)=\sum_{i=1}^{n}\left(b_i(x,t)-\bar{b}_i(x,t)\right),\quad \delta_3(x,t)=c(x,t)-\bar{c}(x,t),$$

$$\delta_4(x,t)=f(x,t)-\bar{f}(x,t),\qquad \delta_5(x)=\varphi(x)-\overline{\varphi}(x),$$

$$\delta_6(x,t)=\psi(x,t)-\overline{\psi}(x,t),\ \delta_7(t)=h(t)-\overline{h}(t),\ \delta_8(x,t)=p(x,t)-\overline{p}(x,t).$$

It's easy to check that the functions $\{\lambda(u,\overline{u}),\vartheta(x,t)=z(x,t)-\delta_8(x,t)\}$ satisfy the system

$$\vartheta_t-L(a,b,c)\vartheta=f(x,t)\lambda(u,\overline{u})+F(x,t),\ (x,t)\in\overline{Q}_T, \tag{6}$$

$$\vartheta(x,0)=0,\ x\in\overline{D};\ \vartheta(x,t)=0,\ (x,t)\in S_T, \tag{7}$$

$$\lambda\left(h,\overline{h}\right)=H(x^*,t)-\left(L(a,b,c)\,z|_{x=x^*}\right)/f(x^*,t)\ ,\quad t\in[0,T] \tag{8}$$

where

$$F(x,t)=L(\delta_1,\delta_2,\delta_3)\overline{u}+L(a,b,c)\delta_8-\delta_{8t}+\delta_4\overline{g}(\overline{u}),\ H(x^*,t)=[\delta_7-$$

$$-\ L(\delta_1,\delta_2,\delta_3)\overline{u}|_{x=x^*}]/\left[f(x^*,t)\cdot\overline{f}(x^*,t)\right]-\delta_4(x^*,t)\times$$

$$\times\left[\bar{h}'-L\left(\bar{a},\bar{b},\bar{c}\right)\bar{u}\big|_{x=x^*}\right]/\left[f(x^*,t)\cdot\bar{f}(x^*,t)\right].$$

Under the conditions of the theorem and from the definition of the set K it follows that the coefficients and the right- handside of equation (6) satisfy the Hōlder condition. it means that there exists a classical solution of problem on determination of $\vartheta(x,t)$ from conditions (6), (7) and it can be represented in the form [1, p.468]:

$$\vartheta(x,t)=\int_0^t\int_D G(x,t;\xi,\tau)\left[f(\xi,\tau)\lambda(u,\overline{u})+F(\xi,\tau)\right]d\xi d\tau, \tag{9}$$

where $d\xi=d\xi_1 d\xi_2...d\xi_n$, $G(x,t;\xi,\tau)$ is Green's function of problem (6), (7) for which the following estimates are valid [1, ch.IV]

$$|G(x,t;\xi,\tau)|\le N_1(t-\tau)^{-n/2}\exp\left(-N_2|x-\xi|^2/(t-\tau)\right),$$

$$\int_D\left|D_x^k G(x,t;\xi,\tau)\right|d\xi\le N_3(t-\tau)^{-(k-\alpha)/2},\ \ k=0,1,2. \tag{10}$$

Here D_x^k are all possible derivatives with respect to x_i of the order k. Taking into account that $\vartheta(x,t)=z(x,t)-\delta_8(x,t)$ from (9) we'll obtain

$$z(x,t) = \delta_8(x,t) + \int_0^t \int_D G(x,t;\xi,\tau)\left[f(\xi,\tau)\lambda(u,\overline{u}) + F(\xi,\tau)\right] d\xi d\tau. \qquad (11)$$

Suppose

$$æ = \|u - \overline{u}\| + \|g(u) - \overline{g}(\overline{u})\|.$$

Under the conditions of the theorem from the definition of the set K subject to estimates (10) we'll obtain

$$|z(x,t)| \le M_2\left[\sum_{i=1}^{4}\|\delta_i\| + \|\delta_8\|_{2,1}\right] + M_3 æ t, \quad (x,t) \in \overline{Q}_T, \qquad (12)$$

$$|\lambda(u,\overline{u})| \le \left|\lambda\left(h,\overline{h}\right)\right| \le M_4\left[\sum_{i=1}^{4}\|\delta_i\| + \|\delta_7\|_1 + \|\delta_8\|_{2,1}\right] + M_5 æ t^{\alpha/2},$$

$$(x,t) \in \overline{Q}_T. \qquad (13)$$

Inequalities (12), (13) are satisfied for any values of $(x,t) \in \overline{Q}_T$. Therefore they must be satisfied also for the maximal values of the left-hand sides. Consequently,

$$æ \le M_6\left[\sum_{i=1}^{4}\|\delta_i\| + \|\delta_7\|_1 + \|\delta_8\|_{2,1}\right] + M_7 æ t^{\alpha/2}. \qquad (14)$$

Let T_1 $(0 < T_1 \le T)$ be such a number that $M_7 T_1^{\alpha/2} < 1$. Then from (14) we'll obtain that for $(x,t) \in \overline{Q}_{T_1}$ the estimate of stability (4) is valid for the solution of problem (1)-(3).

We'll show by induction method that estimate (4) holds for all $t \in [0,T]$. Let estimate (4) be valid for $0 \le t \le kT_1$. We'll show it's also valid for $kT_1 \le t \le$ $\le (k+1)T_1$.

If we'll consider system (1)-(3) in the domain $\left\{x \in \overline{D},\ kT_1 \le t \le (k+1)T_1\right\}$ then we'll obtain that functions $\{\vartheta(x,t) = z(x,t) - \delta_8(x,t), \lambda(u,\overline{u}) = g(u) - \overline{g}(\overline{u})\}$ satisfy the system

$$\vartheta_t - L(a,b,c)\vartheta = f(x,t)\lambda(u,\overline{u}) + F(x,t), \ x \in D, \ t \in [kT_1, (k+1)T_1], \qquad (15)$$

$$\vartheta(x,kT_1) = z(x,kT_1) - \delta_8(x,kT_1), \ x \in \overline{D}; \ \vartheta(x,t) = 0,$$

$$(x,t) \in D \times [kT_1, (k+1)T_1], \qquad (16)$$

$$\lambda\left(h,\overline{h}\right) = H(x^*,t) - \left(L(a,b,c)z|_{x=x^*}\right)/f(x^*,t), \quad t \in [kT_1, (k+1)T_1], \qquad (17)$$

where functions $F(x,t)$, $H(x^*,t)$ have the same form as in problem (6)-(8).

Under the conditions of the theorem and from definition 2 it follows that solution of problem (15), (16) can be represented in the form [1, p.468]:

$$\vartheta(x,t)=\int_{kT_1}^{t}\int_D G(x,t;\xi,\tau)\left[f(\xi,\tau)\lambda(u,\overline{u})+F(\xi,\tau)\right]d\xi d\tau+$$

$$+\int_D G(x,t;\xi,kT_1)\left[z(\xi,kT_1)-\delta_8(\xi,kT_1)\right]d\xi.$$

Hence we obtain

$$z(x,t)=\delta_8(x,t)+\int_{kT_1}^{t}\int_D G(x,t;\xi,\tau)\left[f(\xi,\tau)\lambda(u,\overline{u})+F(\xi,\tau)\right]d\xi d\tau+$$

$$+\int_D G(x,t;\xi,kT_1)\left[z(\xi,kT_1)-\delta_8(\xi,kT_1)\right]d\xi.$$

Substituting expression for $z(x,t)$ from the last equality into (17) and reasoning in the same way as at derivation of inequality (14) we'll obtain:

$$x\le M_8\left[\sum_{i=1}^{4}\|\delta_i\|+\|\delta_7\|_1+\|\delta_8\|_{2,1}\right]+M_7(t-kT_1)^{\alpha/2},$$

$$kT_1\le t\le(k+1)T_1. \tag{18}$$

Here the constant number $M_7>0$ is the same as in inequality (14). therefore it follows from (18) that estimation (4) holds for $kT_1\le t\le(k+1)T_1$.

Thus, we have proved that estimate of stability (4) holds for all $(x,t)\in\overline{Q}_T$. The Uniqueness of the solution of problem (1)-(3) follows from estimation (4) at $a_{ij}=\overline{a}_{ij},\ b_i=\overline{b}_i,\ i,j=\overline{1,n},\ c=\overline{c},\ f=\overline{f},\ \varphi=\overline{\varphi},\ \psi=\overline{\psi},\ h=\overline{h}$.

Thus, the theorem is completely proved.

Remark. The analogous results are valid also in the cases if

1) domain D is unbounded and noncylindrical one;

2) instead of boundary condition of the **I** kind the more general boundary condition of the form $\dfrac{\partial u}{\partial\nu}+q(x,t)u=\psi(x,t)$, $(x,t)\in S$ is given.

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