Sabir S. MIRZOYEV, Khanum V. YAGUBOVA

ON THE COMPLETENESS OF ONE DERIVATIVE CHAIN OF THE ROOT VECTORS OF THE SECOND ORDER OPERATOR BUNCHES

Abstract

In the paper sufficient conditions are obtained on the coefficient of the second order operator bunch which provide the completeness of some system of derivatives chains of root vectors in the Hilbert space. This system of derivatives chains saisfies some initial boundary condition which has unbounded operator in the boundary condition for the second order operator differential equation.

Let H be a separable Hilbert space, A is a positively defined self-adjoint operator in H, and H_γ $(\gamma \geq 0)$ is a scale of Hilbert spaces, degenerated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$.

let's consider the second order operator bunch in the space H

$$P(\lambda) = -(\lambda E - \omega_1 A)(\lambda E - \omega_2 A) + \lambda A_1, \tag{1}$$

where $\omega_1 < 0,\ \omega_2 > 0,\ A_1$ is a linear operator in H.

We connect the bunch (1) with initially boundary value problem

$$P(d/dt)\, u(t) = 0, \quad t \in R_+ = (0, \infty) \tag{2}$$

$$u'(0) - Ku(0) = \varphi, \tag{3}$$

where K is a linear bounded operator, acting from the space $H_{3/2}$ to $H_{1/2}$, i.e. $K \in L\left(H_{3/2}, H_{1/2}\right)$, $\varphi \in H_{1/2}$, and $u(t) \in W_2^2(R_+; H)$.

We remind that the space $W_2^2(R_+; H)$ is defined by the following way [1]:

$$W_2^2(R_+; H) = \left\{ u(t) \mid u"(t) \in L_2(R_+; H),\ A^2 u \in L_2(R_+; H) \right\}$$

with the norm

$$||u||_{W_2^2(R_+;H)} = \left(||u"||^2_{L_2(R_+;H)} + \left|\left|A^2 u\right|\right|^2_{L_2(R_+;H)} \right)^{1/2}.$$

Here and further derivatives are considered in sense of generalized functions theory. So, denote by

$$W_2^2(R_+; H; K) = \left\{ u \mid u \in W_2^2(R_+; H),\ u'(0) - Ku(0) = 0 \right\}$$

and

$$W_2^2(R_+; H; 0; 1) = \left\{ u \mid u \in W_2^2(R_+; H),\ u(0) = u(1) = 0 \right\}.$$

From the theorem on tracks [1] and from the condition $K \in L\left(H_{3/2}, H_{1/2}\right)$ it follows that $W_2^2(R_+; H; K)$ and $W_2^2(R_+; H; 0; 1)$ are closed subspaces of the space H.

Definition 1. *Let for any $\varphi \in H_{1/2}$ there is $u(t) \in W_2^2(R_+;H)$, which satisfies the equation (2) almost everywhere in R_+, the boundary condition (3) in sense of convergence of the norm of the space $H_{1/2}$ and satisfies the inequality*

$$||u||_{W_2^2(R_+;H)} \leq const\, ||\varphi||_{1/2}\,.$$

Then the problem (2), (3) is called reqularly solvable.

Suppose that A^{-1} is absolutely continuous, and $B_1 = A_1 A^{-1}$ is a continuous operator. Then it is easy to see, that the operator bunch $P(\lambda)$ has discrete spectrum, having unique limit point at the infinity.

Let $\lambda_n,\;\; n = \overline{1,\infty}$ be the eigen values of the bunch $P(\lambda)$ from the left semiplane, and $\{e_{0,n},...,e_{m,n}\}_{n=1}^{\infty}$ are the root vectors of the bunch $P(\lambda)$, corresponding to eigen values λ_n, i.e. $P(\lambda_n)\, e_{0,n} = 0,\;\; ((e_{0,n}) \neq 0)$,

$$P(\lambda_n)\, e_{1,n} + \frac{P'(\lambda_n)}{1!} e_{0,n} = 0, \quad P(\lambda_n)\, e_{k,n} + \frac{P'(\lambda_n)}{1!} e_{k-1,n} +$$

$$+ \frac{P"(\lambda_n)}{2!} e_{k-2,n} = 0, \quad k = 2,3,...,m$$

Then obviously, vector-function

$$u_{p,n}(t) = e^{\lambda_n t}\left(\frac{t^p}{p!} e_{0,n} + \frac{t^{p-1}}{(p-1)!} e_{1,n} + ... + e_{p,n}\right), \quad p = 0,...,m$$

belongs to the space $W_2^2(R_+;H)$ and satisfies the equation (2). We'll call these solutions elementary. Obviously, elementary solutions $u_{p,n}(t)$ satisfy the boundary conditions

$$u'_{0,n}(0) - K u_{0,n}(0) \equiv \lambda_n e_{0,n} - K e_{0,n} \equiv \psi_{0,n}. \tag{4}$$

System $\left\{\psi_{p,n}\right\}_{n=1}^{\infty}$ we'll call the system of the root vectors chain, corresponding to the problem (2), (3).

The aim of the present paper is to prove the completeness of the chains system $\left\{\psi_{p,n}\right\}$ in the space $H_{1/2}$. Note, that analogous problem for $\omega_1 = -1,\; \omega_2 = 1$ is studied in work [2].

We'll prove some auxiliary theorems, which we'll need further.

Lemma 1. *Let $S = A^{1/2} K A^{-3/2}$ and $\omega_1 \notin \sigma(S)$. then*

$$P_0 u = P_0\,(d/dt) \equiv -\,(d/dt - \omega_1 A)\,(d/dt - \omega_2 A) \tag{5}$$

maps the space $W_2^2(R_+;H;K)$ on $L_2(R_+;H)$ isomorphically.

Proof. Obviously, the equation $P_0 u = 0$ has general solution from the space $W_2^2(R_+;H)$, which can be presented in the form $u_0(t) = e^{\omega_1 t A}\varphi,\; \varphi \in H_{3/2}$. Since $\omega_1 \notin \sigma(S)$, then from the condition (2) it follows that $\varphi = 0$, i.e. $u_0(t) = 0$.

From the other side, from the theorem on intermediate derivatives it follows that $||P_0 u||_{L_2(R_+;H)} \leq const\, ||u||_{W_2^2(R_+;H)}$, i.e. $P_0(d/dt)$ is a bounded operator from $W_2^2(R_+;H;K)$ to $L_2(R_+;H)$. We'll show that the equation $P_0 u = f$ has solution for all $f \in L_2(R_+;H)$ from the space $W_2^2(R_+;H;K)$.

If we denote by $\hat{f}(\xi)$ Fourier transformation of the vector-function $f(t)$, then vector-function

$$u_1(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} (-i\xi - \omega_1 A)(-i\xi - \omega_2 A)^{-1} \hat{f}(\xi) e^{i\xi t} d\xi, \quad t \in R = (-\infty; +\infty)$$

belongs to the space $W_2^2(R;H)$, and its narrowing $\widetilde{u}_1(t)$ on $[0,\infty)$ belongs to the space $W_2^2(R_+;H)$ and satisfies the equation $P_0(d/dt)u(t) = f(t)$ almost everywhere in R_+.

Then presenting the solution $u(t)$ in the form

$$u(t) = \widetilde{u}_1(t) + e^{\omega_1 t A}\varphi, \qquad \varphi \in H_{3/2},$$

from the condition (3) we can determine φ.

Theorem satetement follows from Banach theorem on the inverse operator from this lemma and from the theroem on intermediate derivatives it follows that the following number is finite

$$N(R_+;K) = \sup_{0 \neq u \in W_2^2(R_+;H;K)} \left|\left| A\frac{du}{dt} \right|\right|_{L_2(R_+;H)} ||P_0 u||^{-1}_{L_2(R_+;H)}$$

since the norms $||P_0 u||_{L_2(R_+;H)}$ and $||u||_{W_2^2(R_+;H)}$ are equivalent in the space $W_2^2(R_+;H;K)$.

It takes place the following

Theorem 1. *Number $N(R_+;K)$ is calculated by the following way*

$$N(R_+;K) = \begin{cases} (\omega_2 + |\omega_1|)^{-1}, & for \quad R(\varphi) \geq 0 \\ (\omega_2 + |\omega_1|)^{-1} \left(1 - \left| \inf\limits_{\|\varphi\|=1} \dfrac{R(\varphi)}{(\omega_2 + |\omega_1|)(\|s\varphi\|^2 + |\omega_1\omega_2|)} \right|^2 \right)^{-1/2} \end{cases},$$

for $\inf\limits_{\|\varphi\|=1} R(\varphi) < 0$, *where*

$$R(\varphi) = 4|\omega_1\omega_2| \operatorname{Re}(S\varphi,\varphi) + (\omega_1 + \omega_2)\left(\|S\varphi\|^2 - |\omega_1\omega_2|\right), \quad (\|\varphi\| = 1).$$

Proof. After simple calculations it is easy to verify that for $u \in W_2^2(R_+;H)$ and $\beta \in [0, (\omega_2 + |\omega_1|)^2)$ it takes place the following identity

$$\|P_0(d/dt)u\|^2_{L_2(R_+;H)} - \beta\|Au'\|^2_{L_2(R_+;H)} =$$

$$= \|F(d/dt;\beta;A)u\|^2_{L_2(R_+;H)} + Q(\beta;\varphi_1,\varphi_0), \tag{6}$$

where

$$F(\lambda;\beta;A) = \lambda^2 E + \sqrt{(\omega_2 + |\omega_1|)^2 - \beta}\lambda A - \omega_1\omega_2 A^2, \tag{7}$$

$$Q(\beta;\varphi_1,\varphi_0) = 4|\omega_1\omega_2| \operatorname{Re}(\varphi_1,\varphi_0) +$$

$$+\left(\sqrt{(\omega_2+|\omega_1|)^2-\beta}+(\omega_1+\omega_2)\right)\|\varphi_1\|^2+$$

$$+|\omega_1\omega_2|\left(\sqrt{(\omega_2+|\omega_1|)^2-\beta}-(\omega_1+\omega_2)\right)\|\varphi_0\|^2,$$

and $\varphi_0=A^{3/2}u(0)\,,\quad \varphi_1=A^{1/2}u'(0)\,.$

From here we can see, that for $u\in W_2^2(R_+;H;0;1)$ $(u(0)=0,\ u'(0)=0)$ from (6) it follows that $\|F(d/dt;\beta;A)\,u\|^2_{L_2(R_+;H)}=\|P_0(d/dt)\,u\|^2_{L_2(R_+;H)}-\beta\|Au'\|^2_{L_2(R_+;H)}$.

Passing to the limit for $\beta\to(\omega_2+|\omega_1|)^2$ we obtain that $N(R_+;0;1)$ the norm of the operator $A\dfrac{d}{dt}:W_2^2(R_+;H;0;1)\to L_2(R_+;H)$ has the estimation $N(R_+;0;1)\le$ $\le(\omega_2+|\omega_1|)^{-1}$. Applying the methods of work [2] we can prove that $N(R_+;0;1)=$ $=(\omega_2+|\omega_1|)^{-1}$. From the other side, from (6) it follows that for any $u\in W_2^2(R_+;H;K)$ and $\beta\in[0,(\omega_2+|\omega_1|)^2)$ it takes place the equality

$$\|P_0(d/dt)\,u\|^2_{L_2}-\beta\|A\frac{du}{dt}\|^2_{L_2}=\|F(d/dt;\beta;A)\,u\|^2_{L_2}+Q(\beta;S\varphi_0,\varphi_0)\,,\tag{8}$$

where

$$Q(\beta;S\varphi_0,\varphi_0)=4|\omega_1\omega_2|\operatorname{Re}(S\varphi_0,\varphi_0)+$$

$$+\left(\sqrt{(\omega_2+|\omega_1|)^2-\beta}+(\omega_1+\omega_2)\right)\|S\varphi_0\|^2+$$

$$+|\omega_1\omega_2|\left(\sqrt{(\omega_2+|\omega_1|)^2-\beta}-(\omega_1+\omega_2)\right)\|\varphi_0\|^2\quad\left(\varphi_0=A^{3/2}u(0)\right).\tag{9}$$

Since $W_2^2(R_+;H;K)\supset W_2^2(R;0;1)$, then $N(R_+;K)\ge N(R_+;0;1)=(\omega_2+|\omega_1|)^{-1}$. From the equality (9) it follows that $N(R_+;K)=(\omega_2+|\omega_1|)^{-1}$ if and only if $Q(\beta;S\varphi_0,\varphi_0)>0$ for all $\beta\in[0,(\omega_2+|\omega_1|)^2)$. Further, from the equality (7) it follows that Cauchy problem $F(d/dt;\beta;A)\,u=0$, $A^{-1/2}u'(0)=K\varphi_0$, $A^{3/2}u(0)=$ $=\varphi_0$, $\forall\varphi_0\in H$ has unique solution from the space $W_2^2(R_+;H)$. From here it follows that $N(R_+;K)=N(R_+;0;1)=(\omega_2+|\omega_1|)^{-1}$ if and only if $Q(0;S\varphi_0,\varphi_0)=$ $=R(\varphi_0)\ge0$, $\varphi_0\in H$, $\|\varphi_0\|=1$.

Now we suppose that $\inf\limits_{\|\varphi\|=1}R(\varphi)<0$. In this case $N(R_+;K)>(\omega_2+|\omega_1|)^{-1}$ and because of it $N^{-2}(R_+;K)\in(0,(\omega_2+|\omega_1|)^2)$. Then for $\beta\in\left(0,N^{-2}(R_+;K)\right)$ from the equality (6) it follows that

$$\|F(d/dt;\beta;A)\,u\|^2_{L_2(R_+;H)}+Q(\beta;S\varphi_0,\varphi_0)=$$

$$=\|P_0(d/dt)\,u\|^2_{L_2(R_+;H)}-\beta\|Au'\|^2_{L_2(R_+;H)}\ge$$

$$=\|P_0(d/dt)\,u\|^2_{L_2(R_+;H)}+$$

$$+\left(1-\beta\sup_{u\in W_2^2(R_+;H;K)}\|A\frac{d}{dt}u\|^2_{L_2}\|P_0(d/dt)\,u\|^{-2}_{L_2(R_+;H)}\right)=$$

$$= \|P_0 (d/dt) u\|^2_{L_2(R_+;H)} + \left(1 - \beta N^{-2} (R_+; K)\right) > 0.$$

Thus, for $u \in W_2^2 (R_+; H; K)$ and $\beta \in \left(0, N^{-2} (R_+; K)\right)$ the equality

$$\|F (d/dt; \beta; A) u\|^2_{L_2} + Q (\beta; S\varphi_0, \varphi_0) > 0 \tag{10}$$

takes place.

If in the inequality (10) we take as $u (t)$v the solution of Cauchy problem $F (d/dt; \beta; A) u = 0$, $u (0) = A^{-1/2}\varphi$, $u' (0) = KA^{-3/2}\varphi$, $\varphi \in H$, then we obtain that for all $\varphi \in H$ $\quad Q (\beta; S\varphi, \varphi)$, i.e. for $\|\varphi\| = 1$, $\varphi \in H$ it takes place $m (\beta) \equiv \min\limits_{\|\varphi\|=1} Q (\beta; S\varphi, \varphi) > 0$.

From the definition of the number $N (R_+; K)$ it follows that for

$$\beta \in \left(N^{-2} (R_+; K),\ (\omega_2 + |\omega_1|)^2\right)$$

there is vector-function $v (t, \beta) \in W_2^2 (R_+; H; K)$ such that

$$\|P_0 (d/dt) v\|^2_{L_2(R_+;H)} \le \beta \|Av'\|^2_{L_2(R_+;H)}.$$

Then from the equality (6) for the vector-function $v (t, \beta)$ and

$$\beta \in \left(N^{-2} (R_+; K), (\omega_2 + |\omega_1|)^2\right)$$

it follows that $Q \left(\beta; S\varphi_\beta; \varphi_\beta\right) < 0$. Consequently, function $m (\beta) = \min\limits_{\|\varphi\|=1} Q (\beta; S\varphi; \varphi)$ changes its sign for $\beta = N^{-2} (R_+; K)$. Consequently, $m \left(N^{-2} (R_+; K)\right) = 0$, i.e. $\min\limits_{\|\varphi\|=1} Q \left(N^{-2} (R_+; K) ; S\varphi; \varphi\right) = 0$. It follows from here that

$$N (R_+; K) = (\omega_2 + |\omega_1|)^{-1} \left[1 - \left|\inf_{\|\varphi\|=1} \frac{R (\varphi)}{(\omega_2 + |\omega_1|) (\|S\varphi\|^2 + |\omega_2\omega_1|)}\right|^2\right]^{-1/2}.$$

Theorem is proved.

Theorem 2. *Let A be a selfadjoint positive operator, $\omega_1 \notin \sigma (S)$, $B = A_1 A^{-1}$ is bounded in H, and $\|B_1\| < N^{-1} (R_+; K)$, where $N (R_+; K)$ is defined from theorem 1. then problem (2), (3) is reqularly sovable.*

Proof. After substitution $y (t) = u (t) - e^{\omega_1 tA} A^{-3/2} (\omega_1 E - S)^{-1} A^{1/2}\varphi$ we can reduce the problem (2), (3) to the problem

$$P (d/dt) y (t) = g (t) \tag{11}$$

$$y' (0) - Ky (0) = 0 \tag{12}$$

where $g (t) = B_1 e^{\omega_1 tA}\psi \in L_2 (R_+; H)$, $\psi = \omega_1 A^{-1} (\omega_1 E - S)^{-1} A^{1/2}\varphi$, $y (t) \in \in W_2^2 (R_+; H)$. Obviously, $\|g (t)\|_{L_2(R_+;H)} \le const\|\varphi\|_{1/2}$. This boundary-value problem can be written in the form of equation $(P_0 + P_1) y = g$, $y \in W_2^2 (R_+; H; K)$, $g \in L_2 (R_+; H)$, $P_1 y = A_1 y'$

As according to lemma 1 P_0 has bounded inverse, then after substitution $y = P_0^{-1}v$ we obtain the following equation in the space $L_2(R_+; H)$:

$$\left(E + P_1 P_0^{-1}\right) v = g\ , \quad v, g \in L_2(R_+; H)\,. \tag{13}$$

As according to theorem 1

$$\|P_1 P_0^{-1} v\|_{L_2} = \|P_1 y\|_{L_2} = \|A_1 y'\|_{L_2} \le \|B_1\|\ \|Ay'\|_{L_2} \le$$

$$\le \|B_1\| N(R_+; K)\, \|P_0 y\|_{L_2} = \|B_1\| N(R_+; K)\, \|v\|_{L_2},$$

then $\|P_1 P_0^{-1}\|_{L_2 \to L_2} < 1$, and thus of it $y(t) = \left(P_0^{-1}\left(E + P_1 P_0^{-1}\right)^{-1} g\right)(t)$.

It follows from here that

$$\|u(t)\|_{W_2^2} = \|y(t)\|_{W_2^2} + const\|\varphi\|_{1/2} \le const\|\varphi\|_{1/2},$$

Theorem is proved.

Theorem 3. *Let A be a positively-defined selfadjoint operator, $\omega_1 \notin \sigma(S)$, $\left(S = A^{1/2} K A^{-3/2}\right)$, $\|B_1\| < N^{-1}(R_+; K)$ and one of the following conditions is satisfied:*

a) $A^{-1} \in \sigma_p \quad (0 < p \le 1)$, b) $A^{-1} \in \sigma_p \quad (0 < p < \infty)$, $B_1 \in \sigma_\infty(H)$

Then the system of derivatives chains $\{\psi_{p,n}\}$ is complete in $H_{1/2}$.

Proof. If the system $\{\psi_{p,n}\}$ is not complete in $H_{1/2}$, then there is the vector $0 \ne \psi \in H_{1/2}$ such that $\left(\psi, \psi_{p,n}\right)_{1/2} = 0$, $n = 1, 2, ...$ Then from the decomposition of the main part of resolvent $P^{-1}(\lambda)$ in the neighbourhood of eigen values (see, for example [3]) it follows that vector-function $\left(A^{1/2}\left(\overline{\lambda}E - K\right) P^{-1}\left(\overline{\lambda}\right)\right)^* A^{1/2}\psi$ is holomorphic in the left semi-plane. From theorem 2 it follows that if theorems condition is satisfied, then the problem (2), (3) is regularly solvable and its solution $u(t)$ can be presented in the form

$$u(t) = \frac{1}{2\pi i}\int_{-\infty}^{i\infty} \hat{u}(\lambda)\, e^{\lambda t} d\lambda,$$

where $\hat{u}(\lambda) = P^{-1}(\lambda)\left((\lambda E + A_1) u(0) + u'(0)\right)$. As for $\lambda = i\xi$, $\xi \in R$ the inequality

$$\|\lambda A_1 P_0^{-1}(\lambda)\| \le \|B_1\| \sup_{\mu_n \in \sigma(A)} \left|\xi \mu_n (i\xi - \omega_1 \mu_n)^{-1} (i\xi - \omega_2 \mu_n)^{-1}\right| \le$$

$$\le \|B_1\| \left(\omega_2 + |\omega_1|\right)^{-1} < 1$$

takes place, then operator bunch $P(\lambda) = \left(E + \lambda A_1 P_0^{-1}(\lambda)\right) P_0(\lambda)$ is invertible on the imaginary axis and the estimation $\|P^{-1}(\lambda)\| \le c(1 + |\lambda|)^{-2}$ takes place for it. That's why integrating contour $(-i\infty, i\infty)$ can be substituted for the contour $\Gamma_{\pm\theta}\{\lambda | \lambda = r \exp \pm i(\theta + \pi/2)),\ 0 < r < \infty\}$ for sufficiently small $\theta > 0$. Further, from the results of the works [3], [4] it follows that $A^2 P^{-1}(\lambda)$ can be presented in the form of two whole functions of the order p and minimal type for the order p. Then for $t > 0$

$$\left(u'(t) - Ku(t), \varphi\right)_{1/2} =$$

$$= \frac{1}{2\pi i} \int\limits_{\Gamma_{\pm\theta}} \left(\left(\lambda E + A_1\right) u\left(0\right) + u'\left(0\right), \left(A^{1/2}\left(\lambda E - K\right) P^{-1}\left(\lambda\right)\right)^* A^{1/2}\varphi \right) e^{\lambda t} d\lambda$$

Further, applying Fragmen-Lindelef theorem, we obtain that the function, standing before the member $e^{\lambda t}$ under the integral is polynomial. Thus of it for $t > 0$

$$\left(u'\left(t\right) - Ku\left(t\right), \varphi\right)_{1/2} = 0.$$

Passing to limit for $t \to 0$ we obtain that $(\psi, \psi)_{1/2} = 0$, i.e. $\psi = 0$. Theorem is proved.

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Sabir S. Mirzoyev
Institute of Mathematics & Mechanics of NAS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20 (off.)

Khanum V. Yagubova
Baku State University.
23, Z.I.Khalilov str., 370148.

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