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STRONG SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR SECOND ORDER QUASI-LINEAR ELLIPTIC EQUATIONS

Abstract

The first boundary value problem is considered for second order quasi-linear elliptic equations of non-divergent structure, whose principle part satisfies the Cordes condition. The strong (almost everywhere) solvability of this problem is established in the corresponding Sobolev space.

Let D be a bounded domain situated in n-dimensional Euclidean space $\mathbb{E}_n$ of the points $x = (x_1, ..., x_n)$, $n \geq 2$. Everywhere later not specifying this we'll assume that the boundary ∂D of the domain D belongs to the class C^2. Consider the following first boundary value problem in D

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x, u, u_x) u_{ij} + +b(x, u, u_x) = 0, \tag{1}$$

$$u|_{\partial D} = 0, \tag{2}$$

where $u_x = (u_1, ..., u_n)$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$; $i, j = 1, ..., n$; $\|a_{ij}(x, z, \vartheta)\|$ is a real symmetric matrix, whose elements are measurable in D at any fixed $z \in \mathbb{E}_1$ and $\vartheta \in \mathbb{E}_n$. Moreover

$$\mu |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, z, \vartheta) \xi_i \xi_j \leq \mu^{-1} |\xi|^2; \quad x \in D, \ z \in \mathbb{E}_1, \quad \vartheta \in \mathbb{E}_n, \quad \xi \in \mathbb{E}_n, \tag{3}$$

$$\sigma = \sup_{x \in D, z \in \mathbb{E}_1, \vartheta \in \mathbb{E}_n} \left(\sum_{i,j=1}^{n} a_{ij}^2(x, z, \vartheta) / \left[\sum_{i=1}^{n} a_{ii}(x, z, \vartheta) \right]^2 \right) < \frac{1}{n-1}. \tag{4}$$

Here $\mu \in (0, 1]$ is a constant. The condition (4) is called the Cordes condition and is understood to within equivalence and non-degenerate linear transformation in the following sense: we can cover the domain D by the finite number of subdomains $D^1, ..., D^l$ such that in each D^i we can substitute the equation (1) by equivalent to it equation $\mathcal{L}'u = 0$ and make non-degenerate linear transformation of coordinates for which leading coefficients of image of the operator $\mathcal{L}'$ satisfy the condition (4) in the image of D'; $i = 1, ..., l$. Besides we'll assume that the function $b(x, z, \vartheta)$ at any fixed $z \in \mathbb{E}_1$ and $\vartheta \in \mathbb{E}_n$ is measurable in D.

The aim of the paper is to prove the existence of strong (almost everywhere) solution of the first boundary value problem (1)-(2). Note that for the second order linear elliptic equations with continuous coefficients the analogous result is obtained in [1-2]. Relating to the equations with discontinuous coefficients whose principle part satisfies the Cordes condition, then we show in this connection, papers [3-4]. For elliptic equations whose leading coefficients are functions of VMO, the corresponding result is established in [5-6]. We also mention papers [7-9], in which the solvability of boundary value problems for non-linear second order elliptic and parabolic equations was investigated. In addition in [9] the solvability of the first bouindary value problem is proved under more rigid condition than the condition (4).

Now let's agree to some notations. We'll denote by $W_p^1(D)$ a Banach space of the functions $u(x)$ given in D with the finite norm

$$\|u\|_{W_p^1(D)} = \left(\int\limits_D \left(|u|^p + \sum_{i=1}^n |u_i|^p \right) dx \right)^{\frac{1}{p}},$$

where $p \in [1, \infty)$.

Let further $\overset{\circ}{W}{}_p^1(D)$ be the completion of $C_0^\infty(D)$ by the norm of the space $W_p^1(D)$. Denote by $W_p^2(D)$ a Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{W_p^2(D)} = \left(\int\limits_D \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p \right) dx \right)^{\frac{1}{p}},$$

and let $\dot{W}_p^2(D) = W_p^2(D) \cap \overset{\circ}{W}{}_p^1(D)$.

The function $u(x) \in \dot{W}_p^2(D)$ is called a strong solution of the first boundary value problem (1)-(2), if it satisfies the equation (1) almost everywhere in D.

Everywhere further $C(\cdots)$ denotes a positive constant C depending only on the quantities appearing in parentheses.

We mention now some facts which we use in further reasonings.

Consider the following first boundary value problem for the second order linear elliptic equation in D

$$\mathcal{M}u = \sum_{i,j=1}^n a_{ij}(x) u_{ij} + \sum_{i=1}^n b_i(x) u_i + c(x) u = f(x), \tag{5}$$

$$u|_{\partial D} = 0 \tag{6}$$

in assumption that the coefficients of the operator $\mathcal{M}$ are real, measurable in D functions, where $a_{ij}(x) = a_{ji}(x)$; $i, j = 1, ..., n$; and

$$\mu_0 |\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \le \mu_0^{-1} |\xi|^2; \quad x \in D, \quad \xi \in \mathbb{E}_n, \tag{7}$$

$$\sigma_0 = \sup_{x \in D} \left(\sum_{i,j=1}^n a_{ij}^2(x) / \left[\sum_{i=1}^n a_{ii}(x) \right]^2 \right) < \frac{1}{n-1}, \tag{8}$$

$$b_i(x) \in L_r(D); \quad i = 1, ..., n; \quad c(x) \in L_m(D). \tag{9}$$

Here $\mu_0 \in (0, 1]$ is a constant; $r = n$ if $p < 2,\ n \ge 2$ and $p = 2,\ n > 2$; $r = 2 + \nu_1$, if $p = n = 2$; $m = \max\left\{p, \frac{n}{2}\right\}$, if $p < 2,\ n \ge 2$ and $p = 2,\ n \ne 4$; $m = 2 + \nu_2$, if $p = 2$, $n = 4$ with some positive constantsν_1 and ν_2. Let $M = \sum\limits_{i=1}^n \|b_i\|_{L_r(D)} + \|c\|_{L_m(D)}$.

Theorem 1 [10]. *If with respect to coefficients of the operator $\mathcal{M}$ the conditions (7)-(9) are fulfilled, then there exist constants $p_1(\sigma_0, \mu_0, n) \in (1,2)$, $K(\sigma_0, \mu_0, n, M, \partial D, diamD)$ and $d(\sigma_0, \mu_0, n, M, \partial D)$, such that for $mesD \leq d$, $p \in [p_1, 2]$ and any function $u(x) \in \dot{W}_p^2(D)$ the estimation*

$$\|u\|_{W_p^2(D)} \leq K \|\mathcal{M}u\|_{L_p(D)}$$

is valid.

Here and further notation $K(\partial D)$ means that the constant $K > 0$ depends only on smoothness of the boundary ∂D.

Theorem 2 [10]. *If the conditions of the previous theorem are satisfied, then for $p \in [p_1, 2]$ and $mesD \leq d$ the first boundary value problem (5)-(6) is uniquely strongly solvable in $\dot{W}_p^2(D)$ for any $f(x) \in L_p$. At that for the solution $u(x)$ the following estimate is valid*

$$\|u\|_{W_p^2(D)} \leq K \|f\|_{L_p(D)}.$$

Note that the constant K increases according to M.

Theorem 3 (see [11]). *Let $1 < p < n$, $1 \leq q \leq \frac{np}{n-p}$. Then for any function $u(x) \in \overset{\circ}{W}{}_p^1(D)$ the estimate*

$$\|u\|_{L_q(D)} \leq C_1(p, q, n) \||u_x|\|_{L_p(D)}$$

is valid. At that embedding is compact if $q < \frac{np}{n-p}$ and bounded, if $q = \frac{np}{n-p}$.

If $p = n$, then for any $q \in [1, \infty)$ the estimate

$$\|u\|_{L_q(D)} \leq C_2(q, n) \||u_x|\|_{L_n(D)}$$

holds, moreover, the enclosure is compact.

We'll impose additional conditions on the coefficients of the operator $\mathcal{L}$: for any $x \in D$ and z^1, $z^2 \in \mathbb{E}_1$ and $\vartheta^1, \vartheta^2 \in \mathbb{E}_n$

$$\left|a_{ij}\left(x, z^1, \vartheta^1\right) - a_{ij}\left(x, z^2, \vartheta^2\right)\right| \leq$$

$$\leq H_1\left(\left|z^1 - z^2\right|^\alpha + \left|\vartheta^1 - \vartheta^2\right|^\alpha\right); i, j = 1, ..., n; \tag{10}$$

$$\left|\frac{\partial b}{\partial z}\left(x, z^1, \vartheta^1\right) - \frac{\partial b}{\partial z}\left(x, z^2, \vartheta^2\right)\right| \leq H_2\left(\left|z^1 - z^2\right|^\beta + \left|\vartheta^1 - \vartheta^2\right|^\beta\right), \tag{11}$$

$$\left|\frac{\partial b}{\partial \vartheta_i}\left(x, z^1, \vartheta^1\right) - \frac{\partial b}{\partial \vartheta_i}\left(x, z^2, \vartheta^2\right)\right| \leq H_3\left(\left|z^1 - z^2\right|^\gamma + \left|\vartheta^1 - \vartheta^2\right|^\gamma\right), \ i = 1, ..., n \tag{12}$$

$$b(x, 0, 0) \in L_2(D), \tag{13}$$

$$\left|\frac{\partial b}{\partial z}(x, z, \vartheta)\right| \leq |f_0(x)| + H_4\left(|z|^{\delta_1} + |\vartheta|^{\delta_2}\right), \tag{14}$$

$$\left|\frac{\partial b}{\partial \vartheta_i}(x,z,\vartheta)\right| \leq |f_1(x)| + H_5\left(|z|^{\delta_3} + |\vartheta|^{\delta_4}\right). \tag{15}$$

Here $H_1,...,H_5$ are non-negative constants; $\alpha \in \left(0,\alpha^0\right]$, $\alpha^0 = \frac{n(2-p_1)}{2(n-p_1)}$; $\beta \in \left(0,\beta^0\right]$, $\beta^0 = 1$ if $n=2$, $\beta^0 = \min\left\{\alpha^0 + \frac{p_1}{n-p_1}, 1\right\}$, if $n \geq 3$; $\gamma \in (0,\gamma_0]$, $\gamma^0 = 1$, if $n \leq 4$, $\gamma^0 = \min\left\{\alpha^0 + \frac{2p_1}{n-p_1}, 1\right\}$, if $n \geq 5$; $\delta_1 \in [0,\infty)$, if $n \leq 4$, $\delta_1 \in \left[0, \frac{4}{n-4}\right)$, if $n \geq 5$; $\delta_2 \in [0,\infty)$, if $n=2$; $\delta_2 \in [0,3)$, if $n=3$; $\delta_2 \in \left[0,\frac{4}{n-2}\right)$, if $n \geq 4$; $\delta_3 \in [0,\infty)$, if $n \geq 4$; $\delta_3 \in \left[0,\frac{2}{n-4}\right)$, if $n \geq 5$; $\delta_4 \in [0,\infty)$, if $n=2$, $\delta_4 \in \left[0,\frac{2}{n-2}\right)$, if $n \geq 3$; $f_0(x) \in L_m(D)$; $f_1(x) \in L_r(D)$; moreover, constants m and r were calculated for $p=2$.

Everywhere below notation $C(\mathcal{L})$ means that the positive constant C depends only on $\alpha,\beta,\gamma,\delta_1,...,\delta_4,H_1,...,H_5, \|f_0\|_{L_m(D)}, \|f_1\|_{L_r(D)}, \sigma$ and μ.

We'll write equation (1) in the equivalent form. We have

$$b(x,u,u_x) = b(x,0,u_x) + \int_0^u \frac{\partial b}{\partial z}(x,z,u_x)\,dz =$$

$$= b(x,0,u_x) + u\int_0^1 \frac{\partial b(x,u\tau,u_x)}{\partial(u\tau)}d\tau. \tag{16}$$

Further we obtain

$$b(x,0,u_x) = b(x,0,u_1,...,u_n) =$$

$$= b(x,0,0,u_2,...,u_n) + \int_0^{u_1} \frac{\partial b}{\partial \vartheta_1}(x,0,\vartheta_1,u_2,...,u_n)\times$$

$$\times d\vartheta_1 = b(x,0,0,u_2,...,u_n) + u_1\int_0^1 \frac{\partial b}{\partial(u_1\tau)}(x,0,u_1\tau,u_2,...,u_n)\,d\tau. \tag{17}$$

Acting analogously, from (16)-(17) we conclude

$$b(x,u,u_x) = b(x,0,0) + \sum_{i=1}^n B_i(x,u,u_x)\,u_i + C(x,u,u_x)\,u,$$

where

$$B_i(x,u,u_x) = \int_0^1 \frac{\partial b}{\partial(u_i\tau)}(x,0,...,0,u_i\tau,u_{i+1},...,u_n)\,d\tau; \quad i=1,...,n;$$

$$C(x,u,u_x) = \int_0^1 \frac{\partial b}{\partial(u\tau)}(x,u\tau,u_1,...,u_n)\,d\tau.$$

Thus, we can write the first boundary value problem in the equivalent form

$$\mathcal{L}^0 u = \sum_{i,j=1}^{n} a_{ij}(x,u,u_x)\, u_{ij} + \sum_{i=1}^{n} B_i(x,u,u_x)\, u_i + C(x,u,u_x)\, u = F(x)\,, \tag{18}$$

$$u|_{\partial D} = 0, \tag{19}$$

where $F(x) = -b(x,0,0)$.

We shall denote the following functional set $\Big\{ u(x) : u(x) \in \overset{\circ}{W}{}_q^1(D)\,, \|u\|_{W_2^2(D)} \le N \Big\}$ by $\mathcal{A}$, where $q = \frac{p_1 n}{n-p_1}$, and constant N will be chosen later.

It's easy to see that the set $\mathcal{A}$ is a convex compact in $\overset{\circ}{W}{}_q^1(D)$. In fact, let $u^1, u^2 \in \mathcal{A},\ \lambda \in [0,1]\,,\ u = \lambda u^1 + (1-\lambda)\, u^2$. It's clear that $u \in \overset{\circ}{W}{}_q^1(D)$, moreover, $\|u\|_{W_2^2(D)} \le \lambda \left\|u^1\right\|_{W_2^2(D)} + (1-\lambda)\left\|u^2\right\|_{W_2^2(D)} \le N$, i.e. $u \in \mathcal{A}$.

The compactness of $\mathcal{A}$ follows from theorem 3, since by virtue of increase of function $\frac{p_1 n}{n-p_1}$ with respect to p_1, the inequality $\frac{p_1 n}{n-p_1} < \frac{2n}{n-2}$ holds if $n \ge 3$. If $n = 2$, then it's sufficient to apply the second part of the mentioned theorem.

Consider now the auxiliary problem for the linear elliptic equation

$$\mathcal{L}_w u = \sum_{i,j=1}^{n} e_{ij}(x)\, u_{ij} + \sum_{i=1}^{n} g_i(x)\, u_i + h(x)\, u = F(x)\,, \tag{20}$$

$$u|_{\partial D} = 0, \tag{21}$$

where $e_{ij}(x) = a_{ij}(x, w(x), w_x(x))\,,\ g_i(x) = B_i(x, w(x), w_x(x))\,,\ h(x) = C(x, w(x), w_x(x))\,;\ i,j = 1,...,n$, and $w(x) \in \mathcal{A}$.

Lemma 1. *If $mesD \le d$ and relative to the coefficients of the operator $\mathcal{L}$ conditions (3)-(4), (13)-(15) are fulfilled, then the first boundary value problem (20)-(21) is uniquely strongly solvable in $\dot{W}_2^2(D)$ for any $w(x) \in \mathcal{A}$.*

Proof. At first consider the case $n \ge 3,\ \delta_4 > \frac{1}{n-1}$. For $i = 1,...,n$ we have

$$\|g_i\|_{L_n(D)} \le \|f_1\|_{L_n(D)} + H_5 \left\| |w|^{\delta_3} \right\|_{L_n(D)} + H_5 \left\| |w_x|^{\delta_4} \right\|_{L_n(D)}. \tag{22}$$

Let's paraphrase now the statement of theorem 3. If $q > \frac{n}{n-1}$, then for any function $u(x) \in \overset{\circ}{W}{}_p^1(D)$ the first estimate of the mentioned theorem is valid if $p \ge \frac{qn}{n+q}$. Therefore

$$\left\| |w_x|^{\delta_4} \right\|_{L_n(D)} = \||w_x|\|^{\delta_4}_{L_{n\delta_4}(D)} \le C_3(\delta_4, n)\, \|w\|^{\delta_4}_{W^2_{\frac{\delta_4 n}{1+\delta_4}}(D)}\,, \tag{23}$$

since if $q = n\delta_4$, then $p \ge \frac{a^2 \delta_4 n^2}{n + n\delta_4} = \frac{\delta_4 n}{1+\delta_4}$. On the other hand by virtue of increasing of funciton $\frac{\delta_4 n}{1+\delta_4}$ with respect to δ_4 we have

$$\frac{\delta_4 n}{1+\delta_4} < \frac{2n/(n-2)}{1 + 2/n-2} = 2.$$

Consequently there exists $\varepsilon_1(\delta_4, n)$ such that $\frac{\delta_4 n}{1+\delta_4} = 2 - \varepsilon_1$. Thus, from (23) we obtain

$$\left\| |w_x|^{\delta_4} \right\|_{L_n(D)} \leq C_3 \|w\|^{\delta_4}_{W^2_{2-\varepsilon_1}(D)} . \tag{24}$$

If $n = 2$ the as it follows from (23), the estimation

$$\left\| |w_x|^{\delta_4} \right\|_{L_{2+\nu_1}(D)} \leq C'_3 (\delta_4) \|w\|^{\delta_4}_{W^2_{2-\varepsilon'_1}(D)} ; \quad \varepsilon'_1 = \varepsilon'_1 (\delta_4) . \tag{24'}$$

is valid at $\frac{(2+\nu_1)\delta_4}{1+\left(1+\frac{\nu_1}{2}\right)\delta_4} < 2$, i.e. for any $\delta_4 < \infty$. Thus the inequalities (24) and (24') are valid at any $n \geq 2$, if only $\delta_4 > \frac{1}{n-1}$. Applying the Hölder inequality and restricted for the simplicity to the case $n \geq 3$ we have

$$\|w\|^{\delta_4}_{W^2_{2-\varepsilon_1}(D)} \leq C_4 (\delta_4, n) \left[\left(\int_D |w|^{2-\varepsilon_1} dx \right)^{\frac{\delta_4}{2-\varepsilon_1}} + \sum_{i=1}^n \left(\int_D |w_i|^{2-\varepsilon_1} dx \right)^{\frac{\delta_4}{2-\varepsilon_1}} + \right.$$

$$\left. + \sum_{i,j=1}^n \left(\int_D |w_{ij}|^{2-\varepsilon_1} dx \right)^{\frac{\delta_4}{2-\varepsilon_1}} \right] \leq C_4 \left\{ \left[\left(\int_D w^2 dx \right)^{\frac{2-\varepsilon_1}{2}} (mesD)^{\frac{\varepsilon_1}{2}} \right]^{\frac{\delta_4}{2-\varepsilon_1}} + \right.$$

$$+ \sum_{i=1}^n \left[\left(\int_D w_i^2 dx \right)^{\frac{2-\varepsilon_1}{2}} (mesD)^{\frac{\varepsilon_1}{2}} \right]^{\frac{\delta_4}{2-\varepsilon_1}} + \sum_{i,j=1}^n \left[\left(\int_D w_{ij}^2 dx \right)^{\frac{2-\varepsilon_1}{2}} \times \right.$$

$$\left. \left. \times (mesD)^{\frac{\varepsilon_1}{2}} \right]^{\frac{\delta_4}{2-\varepsilon_1}} \right\} \leq C_5 (\delta_4, n) \|w\|^{\delta_4}_{W^2_2(D)} (mesD)^{\frac{\varepsilon_1 \delta_4}{2(2-\varepsilon_1)}} \leq C_5 N^{\delta_4} d^{\delta^0_4}, \tag{25}$$

where $\delta^0_4 = \delta^0_4 (\delta_4, n)$.

Let now $\delta_4 \leq \frac{1}{n-1}$. Without losing generality we can assume that $\delta_4 > 0$. Then

$$\left\| |w_x|^{\delta_4} \right\|_{L_n(D)} \leq C_6 (\delta_4, n) \sum_{i=1}^n \left(\int_D |w|^{\delta_4 n} dx \right)^{\frac{1}{n}} . \tag{26}$$

Denote by $k > 1$ any number for which $\frac{1}{n-1} < k\delta_4 < \frac{2}{n-2}$ (if $n = 2$, then $k > 1$ is arbitrary). Applying the Hölder inequality, we obtain the following estimate from (26)

$$\sum_{i=1}^n \left(\int_D |w_i|^{\delta_4 n} dx \right)^{\frac{1}{n}} \leq \sum_{i=1}^n \left[\left(\int_D |w_i|^{k\delta_4 n} dx \right)^{\frac{1}{k}} (mesD)^{\frac{k-1}{k}} \right]^{\frac{1}{n}} \leq$$

$$\leq C_7\left(\delta_4,n\right)(mesD)^{\frac{k-1}{kn}}\left\||w_x|\right\|^{\delta_4}_{L_{k\delta_4 n(D)}}\leq C_8\left(\delta_4,n\right)d^{\frac{k-1}{kn}}\left\|w\right\|^{\delta_4}_{W^2_{2-\varepsilon_2}(D)}, \tag{27}$$

where $\varepsilon_2=\varepsilon_2\left(\delta_4,n\right)$. It follows from (26)-(27) that

$$\left\||w_x|^{\delta_4}\right\|_{L_n(D)}\leq C_9\left(\delta_4,n\right)N^{\delta_4}d^{\delta_4^0+\frac{k-1}{kn}}. \tag{28}$$

Thus, assuming for definiteness that $d\leq 1$, we conclude from (24), (24'), (25) and (28) that for $n\geq 2$ the estimation

$$\left\||w_x|^{\delta_4}\right\|_{L_r(D)}\leq C_{10}\left(\delta_4,n\right)N^{\delta_4}d^{\delta_4^0}. \tag{29}$$

is valid.

We obtain absolutely analogously for $n\geq 5$ and $\delta_3>\frac{1}{n-1}$

$$\left\||w_x|^{\delta_3}\right\|_{L_n(D)}=\left\|w\right\|^{\delta_3}_{L_n\delta_3(D)}\leq C_{11}\left(\delta_3,n\right)\left\||w_x|\right\|^{\delta_3}_{L_{\frac{n\delta_3}{1+\delta_3}}(D)}. \tag{30}$$

We showed earlier that if $\delta_4<\frac{2}{n-2}$, then the inequality (24) is satisfied. In our case substituting δ_4 by $\frac{\delta_3}{1+\delta_3}$ and taking into account that the condition $\frac{\delta_3}{1+\delta_3}<\frac{2}{n-2}$ is equivalent to the condition $\delta_3<\frac{2}{n-4}$, we conclude on the validity of the estimatiuon

$$\left\||w|^{\delta_3}\right\|_{L_{n(D)}}\leq C_{12}\left(\delta_3,n\right)\left\|w\right\|^{\delta_3}_{W^2_{2-\varepsilon_3}(D)},\quad (31) \tag{31}$$

where $\varepsilon_3=\varepsilon_3\left(\delta_3,n\right)$. If $n=2$, then as it follows from the second part of theorem 32, the inequality of the form (31) is valid for any $\delta_3<\infty$. Let now $n=3$. Then letting in the mentioned theorem $q=\frac{3\delta_3}{1+\delta_3}$, we conclude that the inequality of the form (31) is fulfilled , if $\frac{3q}{3+q}<2$. But the last condition is equivalent to the condition $\frac{3\delta_3}{1+2\delta_3}<2$, which is valid for any $\delta_3<\infty$. Let finally $n=4$. Letting $q=\frac{4\delta_3}{1+\delta_3}$ in theorem 3, we are convinced that the inequality of the form (31) will hold, if $\frac{4q}{4+q}<2$. The last condition is equivalent to the condition$\frac{4\delta_3}{1+2\delta_3}<2$ valid for any $\delta_3<\infty$. Thus, allowing for (30) and (31) we conclude that if $\delta_3>\frac{1}{n-1}$, then for any $n\geq 2$ the estimation

$$\left\||w_x|^{\delta_3}\right\|_{L_r(D)}\leq C_{13}\left(\delta_3,n\right)N^{\delta_3}d^{\delta_3^0}. \tag{32}$$

where $\delta_3^0=\delta_3^0\left(\delta_3,n\right)$ is valid.

Using the some concepts as at the proof of the inequality (18) now we are convinced in the validity of the estimation (32) and for $\delta_3\leq\frac{1}{n-1}$. Thus, allowing for (29) and (32) in (22) we obtain

$$\sum_{i=1}^{n}\left\|g_i\right\|_{L_r(D)}\leq\left\|f_1\right\|_{L_r(D)}+C_{14}\left(\mathcal{L},n\right)\left(N^{\delta_3}d^{\delta_3^0}+N^{\delta_4}d^{\delta_4^0}\right). \tag{33}$$

Analogously we have

$$\left\|h\right\|_{L_m(D)}\leq\left\|f_0\right\|_{L_m(D)}+H_4\left\||w|^{\delta_1}\right\|_{L_m(D)}+H_4\left\||w_x|^{\delta_2}\right\|_{L_m(D)}. \tag{34}$$

Let at first $n \geq 3$ and $\delta_2 > \frac{n}{m(n-1)}$. According to theorem 3 we obtain

$$\left\| |w_x|^{\delta_2} \right\|_{L_m(D)} = \| |w_x| \|^{\delta_2}_{L_m \delta_2 (D)} \leq C_{15} (\delta_2, m, n) \| w \|^{\delta_2}_{W^2_{\frac{m\delta_2 n}{m\delta_2 + n}}(D)} . \tag{35}$$

By virtue of increasing of the function $\frac{m\delta_2 n}{m\delta_2+n}$ with respect to δ_2 we conclude that for $n \geq 4$

$$\frac{m\delta_2 n}{m\delta_2 + n} < \frac{mn\frac{4}{n-2}}{\frac{4m}{n-2} + n} = \frac{4mn}{4m + n^2 - 2n}.$$

Hence it follows that if $n \geq 5$ (i.e. $m = \frac{n}{2}$), then $\frac{4mn}{4m+n^2-2n} = 2$. If $n = 4$ (i.e. $m = 2 + \nu_2$), then there exists $\varepsilon_4 (\delta_2)$ such that $\delta_2 < 2 - \varepsilon_4$. Then

$$\frac{m\delta_2 n}{m\delta_2 + n} < \frac{4 (2 + \nu_2) (2 - \varepsilon_4)}{4 + (2 + \nu_2) (2 - \varepsilon_4)} = \frac{4 \left(4 - \varepsilon_4^2\right)}{4 + 4 - \varepsilon_4^2} = \frac{4 - \varepsilon_4^2}{2 - \frac{\varepsilon_4^2}{4}} < 2,$$

if we choose $\nu_2 = \varepsilon_4$. Let $n = 3$ (i.e. $m = 2$). Then $\frac{m\delta_2 n}{m\delta_2+n} < \frac{18}{9} = 2$. Thus, if $n \geq 3$, then we obtain from (35)

$$\left\| |w_x|^{\delta_2} \right\|_{L_m(D)} \leq C_{16} (\delta_2, n) \| w \|^{\delta_2}_{W^2_{2-\varepsilon_5}(D)} , \tag{36}$$

where $\varepsilon_5 = \varepsilon_5 (\delta_2, n)$.

Let now $n = 2$ (i.e. $m = 2$). Then for the validity of the equality (36) it's sufficient that $\frac{4\delta_2}{2\delta_2+2} < 2$. The last condition as it's easy to see, is satisfied for $\delta_2 < \infty$. Thus the inequality (36) is valid for $n \geq 2$, if only $\delta_2 > \frac{n}{m(n-1)}$. But on the other hand applying the Hölder inequality we have

$$\| w \|^{\delta_2}_{W^2_{2-\varepsilon_5}(D)} \leq C_{17} (\delta_2, n) \left\{ \left[\left(\int_D w^2 dx \right)^{\frac{2-\varepsilon_5}{2}} (mesD)^{\frac{\varepsilon_5}{2}} \right]^{\frac{\delta_2}{2-\varepsilon_5}} + \right.$$

$$+ \sum_{i=1}^{n} \left[\left(\int_D w_i^2 dx \right)^{\frac{2-\varepsilon_5}{2}} (mesD)^{\frac{\varepsilon_5}{2}} \right]^{\frac{\delta_2}{2-\varepsilon_5}} +$$

$$\left. + \sum_{i,j=1}^{n} \left[\left(\int_D w_{ij}^2 dx \right)^{\frac{2-\varepsilon_5}{2}} (mesD)^{\frac{\varepsilon_5}{2}} \right]^{\frac{\delta_2}{2-\varepsilon_5}} \right\} \leq$$

$$\leq C_{18} (\delta_2, n) \| w \|^{\delta_2}_{W^2_2(D)} (mesD)^{\frac{\varepsilon_5 \delta_2}{2(2-\varepsilon_5)}} \leq C_{18} N^{\delta_2} d^{\delta_2^0}, \tag{37}$$

where $\delta_2^0 = \delta_2^0 (\delta_2, n)$. We obtain from (36)-(37)

$$\left\| |w_x|^{\delta_2} \right\|_{L_m(D)} \leq C_{19} (\delta_2, n) N^{\delta_2} d^{\delta_2^0}. \tag{38}$$

If we reason as at the proof of the inequality (28) we come to conclusion that the estimation (38) is valid for $\delta_2 \leq \frac{n}{m(n-1)}$.

Let now $n \geq 5$ and $\delta_1 > \frac{n}{m(n-1)}$. Then according to theorem 3

$$\left\| |w|^{\delta_1} \right\|_{L_m(D)} = \|w\|^{\delta_1}_{L_{m\delta_1}(D)} \leq C_{20}(\delta_1, m, n) \| |w_x| \|^{\delta_1}_{L_{\frac{m\delta_1 n}{m\delta_1+n}}(D)}.$$

We showed earlier that if $\delta_2 < \frac{4}{n-2}$, then the inequality (36) is valid. In our case substituting δ_2 by $\frac{\delta_1 n}{m\delta_1+n}$ and taking into account that the condition $\frac{\delta_1 n}{m\delta_1+n} < \frac{4}{n-2}$ is equaivalent to the condition $\delta_1 < \frac{4}{n-4}$, we conclude on the validity of the estimation

$$\left\| |w|^{\delta_1} \right\|_{L_m(D)} \leq C_{21}(\delta_1, n) \|w\|^{\delta_1}_{W^2_{2-\varepsilon_6}(D)}, \tag{39}$$

where $\varepsilon_6 = \varepsilon_6(\delta_1, n)$.

Let $n = 2$ (i.e. $m = 2$). Then setting $q = \frac{m\delta_1 n}{\mu\delta_1+n} = \frac{2\delta_1}{1+\delta_1}$, we obtain from theorem 3 that the inequality of the form (39) is valid, if $\frac{2q}{2+q} < 2$. The last condition is equivalent to the condition $\frac{\delta_1}{1+2\delta_1} < 1$,which is satisfied for any $\delta_1 < \infty$. Let $n = 3$ (i.e.$m = 2$).Then setting $q = \frac{m\delta_1 n}{m\delta_1+n} = \frac{6\delta_1}{2\delta_1+3}$ and using theorem 3 we conclude that for the validity of the estimation of the form (35) it's sufficient that $\frac{3q}{3+q} < 2$.The last equality is equivalent to the condition $\frac{6\delta_1}{4\delta_1+3} < 2$,which is valid for any $\delta_1 < \infty$. Let finally $n = 4$ (i.e.$m = 2+\nu_2$).Then assuming $q = \frac{m\delta_1 n}{m\delta_1+n} = \frac{4(2+\nu_2)\delta_1}{(2+\nu_2)\delta_1+4}$,we obtain from theorem 3 that the inequality of theform (9) holds, if $\frac{4q}{4+q} < 2$. The last condition is equivalent to the condition $\frac{(2+\nu_1)\delta_1}{(2+\nu_1)\delta_1+2} < 1$satisfied for any $\delta_1 < \infty$.Thus, the inequality (39) is valid for any $n \geq 2$, if $\delta_1 > \frac{n}{m(n-1)}$. Hence it follows that

$$\left\| |w|^{\delta_1} \right\|_{L_m(D)} \leq C_{22}(\delta_1, n) N^{\delta_1} d^{\delta_1^0}. \tag{40}$$

where $\delta_1^0 = \delta_1^0(\delta_1, n)$.

In order to be convinced in the validity of the inequality (40) for $\delta_1 \leq \frac{n}{m(n-1)}$, it's sufficient to apply the reasonings used at the proof of the estimation (28). Allowing (37) and (40) in (34) we conclude

$$\|h\|_{L_m(D)} \leq \|f_0\|_{L_m(D)} + C_{23}(\mathcal{L}, n)\left(N^{\delta_1} d^{\delta_1^0} + N^{\delta_2} d^{\delta_2^0}\right). \tag{41}$$

Now it follows from (33) and (34) that

$$M_0 = \sum_{i=1}^{n} \|g_i\|_{L_r(D)} + \|h\|_{L_m(D)} \leq \|f_0\|_{L_m(D)} + \|f_1\|_{L_r(D)} +$$

$$+C_{24}(\mathcal{L}, n)\left(N^{\delta_1} d^{\delta_1^0} + N^{\delta_2} d^{\delta_2^0} + N^{\delta_3} d^{\delta_3^0} + N^{\delta_4} d^{\delta_4^0}\right). \tag{42}$$

To complete the proof it's sufficient to apply theorem 2. The lemma is proved.

Remark. It follows from the statement of lemma 1 that the operator $\mathcal{H}$ from $\mathcal{A}$ in $\dot{W}^2_2(D)$ which associates the strong solution $u(x)$ of the first boundary value problem (20)-(21) to each function $w(x) \in \mathcal{A}$ is defined. In addition, if $u = \mathcal{H}w$, then

$$\|u\|_{W_2^2(D)} \le K(M_0, \sigma, \mu, n, \partial D, diamD) \|F\|_{L_2(D)}. \tag{43}$$

Lemma 2. *Let the conditions of the previous lemma be satisfied with respect to the coefficients of the operator $\mathcal{L}$. Then there exist the constants $N(\mathcal{L}, n, \partial D, diamD)$ and $d_0(\mathcal{L}, n, \partial D)$ such that if $mesD \le d_0$ then the operator $\mathcal{H}$ maps the set $\mathcal{A}$ into itself.*

Proof. To shorten the notation we'll denote the constant K of theorems 1, 2 and the inequality (43) by $K(M_0)$ omitting its dependence on the other parameters. It's obvious that for the validity of the statement of lemma it's sufficient that the condition

$$K(M_0) \|F\|_{L_2(D)} \le N \tag{44}$$

was satisfied.

Let's fix $N = K(1) \|F\|_{L_2(D)}$. Then the estimation (44) holds if $K(M_0) \le K(1)$, i.e. $M_0 \le 1$. Thus, according to (42) it's sufficient to choose d_0 from the condition

$$\|f_0\|_{L_m(D)} + \|f_1\|_{L_r(D)} + C_{24}\left(N^{\delta_1} d^{\delta_1^0} + N^{\delta_2} d^{\delta_2^0} + N^{\delta_3} d^{\delta_3^0} + N^{\delta_4} d^{\delta_4^0}\right) \le 1,$$

which must be fulfilled at $d \le d_0$.

Let d' be such that

$$\|f_0\|_{L_m(D)} + \|f_1\|_{L_r(D)} \le \frac{1}{2} \text{ when } mesD \le d',$$

and d'' be so small that

$$N^{\delta_1} d^{\delta_1^0} + N^{\delta_2} d^{\delta_2^0} + N^{\delta_3} d^{\delta_3^0} + N^{\delta_4} d^{\delta_4^0} \le \frac{1}{2C_{24}} \text{ when } d \le d''.$$

Now it's sufficient to choose $d_0 = \min\{d, d', d''\}$ and the lemma is proved.

Lemma 3. *If with respect to the coefficients of the operator $\mathcal{L}$ the conditions (3)-(4), (10)-(15) are satisfied and $mesD \le d_0$, then the operator $\mathcal{H}$ is continuous.*

Proof. Let $w^1 \in \mathcal{A}$, $w^2 \in \mathcal{A}$, $u^1 = \mathcal{H}w^1$, $u^2 = \mathcal{H}w^2$, $q = \frac{np_1}{n-p_1}$. Let's fix arbitrary $\varepsilon > 0$. Assume that

$$\left\|w^1 - w^2\right\|_{W_q^1} < \lambda, \tag{45}$$

where the positive constant λ will be chosen later. We have

$$\mathcal{L}_{w^1}\left(u^1 - u^2\right) = \mathcal{L}_{w^1}u^1 - \mathcal{L}_{w^1}u^2 = F(x) - \left(\mathcal{L}_{w^1} - \mathcal{L}_{w^2}\right)u^2 - \mathcal{L}_{w^2}u^2 =$$

$$= -\left(\mathcal{L}_{w^1} - \mathcal{L}_{w^2}\right)u^2 = P(x).$$

But on the other hand

$$|P(x)| \le \sum_{i,j=1}^n \left|a_{ij}\left(x, w^1, w_x^1\right) - a_{ij}\left(x, w^2, w_x^2\right)\right| \left|u_{ij}^2\right| +$$

$$+\sum_{i=1}^n \left|B_i\left(x, w^1, w_x^1\right) - B_i\left(x, w^2, w_x^2\right)\right| \left|u_i^2\right| +$$

$$+\left|C\left(x,w^1,w_x^1\right)-C\left(x,w^2,w_x^2\right)\right|\left|u^2\right|\leq$$

$$\leq H_1\left(\left|w^1-w^2\right|^{\alpha}+\left|w_x^1-w_x^2\right|^{\alpha}\right)\sum_{i,j=1}^{n}\left|u_{ij}^2\right|+$$

$$+H_2\left(\left|w^1-w^2\right|^{\beta}+\left|w_x^1-w_x^2\right|^{\beta}\right)\sum_{i,j=1}^{n}\left|u_{i}^2\right|+$$

$$+H_3\left(\left|w^1-w^2\right|^{\gamma}+\left|w_x^1-w_x^2\right|^{\gamma}\right)\left|u^2\right|. \tag{46}$$

Using theorems 3 and 2 and also the estimation (46), we further obtain

$$\left\|u^1-u^2\right\|_{W_q^1(D)}\leq C_{25}\left(\mathcal{L},n\right)\left\|u^1-u^2\right\|_{W_{p_1}^2(D)}\leq C_{26}\left(\mathcal{L},n,\partial D,diamD\right)\times$$

$$\times\left\|P\right\|_{L_{p_1}(D)}\leq C_{27}\left(\mathcal{L},n,\partial D,diamD\right)\left\{\left[\int\limits_D\left(\left|w^1-w^2\right|^{\alpha}+\left|w_x^1-w_x^2\right|^{\alpha}\right)^{p_1}\times\right.\right.$$

$$\left.\times\left(\sum_{i,j=1}^{n}\left|u_{ij}^2\right|\right)^{p_1}dx\right]^{\frac{1}{p_1}}+\left[\int\limits_D\left(\left|w^1-w^2\right|^{\beta}+\left|w_x^1-w_x^2\right|^{\beta}\right)^{p_1}\left(\sum_{i=1}^{n}\left|u_{i}^2\right|^{p_1}\right)dx\right]^{\frac{1}{p_1}}+$$

$$+\left[\int\limits_D\left(\left|w^1-w^2\right|^{\gamma}+\left|w_x^1-w_x^2\right|^{\gamma}\right)^{p_1}\left|u^2\right|^{p_1}dx\right]^{\frac{1}{p_1}}\leq C_{27}\left(J_1+J_2+J_3\right). \tag{47}$$

Applying the Hölder inequality we conclude

$$J_1\leq C_{28}\left(\mathcal{L},n,\partial D,diamD\right)\left[\left(\int\limits_D\sum_{i,j=1}^{n}\left(u_{ij}^2\right)^2dx\right)^{\frac{p_1}{2}}\times\left(\int\limits_D\left|w^1-w^2\right|^{\frac{2\alpha p_1}{2-p_1}}+\right.\right.$$

$$\left.\left.+\left|w_x^1-w_x^2\right|^{\frac{2\alpha p_1}{2-p_1}}\right)dx\right)^{\frac{2-p_1}{2}}\right]^{\frac{1}{p_1}}\leq C_{29}\left(\mathcal{L},n,\partial D,diamD\right)\times$$

$$\times\left(\int\limits_D\sum_{i,j=1}^{n}\left(u_{ij}^2\right)^2dx\right)^{\frac{1}{2}}\left\|w^1-w^2\right\|^{\alpha}_{W^{\alpha}_{\frac{2\alpha p_1}{2-p_1}}(D)}. \tag{48}$$

But on the other hand

$$\frac{2\alpha p_1}{2-p_1}\leq\frac{2\alpha^0p_1}{2-p_1}=\frac{2p_1\frac{n(2-p_1)}{2(n-p_1)}}{2-p_1}=\frac{np_1}{n-p_1}=q.$$

We therefore derive from (48) and (45)

$$J_1\leq C_{30}\left(\mathcal{L},n,\partial D,diamD\right)\left\|u^2\right\|_{W_2^2(D)}\left\|w^1-w^2\right\|^{\alpha}_{W_q^1(D)}<C_{30}\lambda^{\alpha}\left\|u^2\right\|_{W_2^2(D)}. \tag{49}$$

If we reason analogously, we obtain for $n \geq 3$

$$J_2 \leq C_{31}(\mathcal{L}, n, \partial D, diamD)\left(\int_D \sum_{i=1}^{n} \left|u_i^2\right|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{2n}} \left\|w^1 - w^2\right\|^{\beta}_{W^1_{\frac{2n\beta p_1}{n(2-p_1)+2p_1}}(D)}. \quad (50)$$

Taking into account that

$$\frac{2n\beta p_1}{n(2-p_1)+2p_1} \leq \frac{2n\beta^0 p_1}{n(2-p_1)+2p_1} \leq \frac{2np_1 \frac{n(2-p_1)+2p_1}{2(n-p_1)}}{n(2-p_1)+2p_1} = \frac{np_1}{n-p_1} = q$$

and using theorem 3 and the inequality (45) we conclude from (50)

$$J_2 \leq C_{32}(\mathcal{L}, n, \partial D, diamD)\left\|u^2\right\|_{W_2^2(D)} \left\|w^1 - w^2\right\|^{\beta}_{W_q^1(D)} < C_{32}\lambda^{\beta}\left\|u^2\right\|_{W_2^2(D)}. \quad (51)$$

If $n = 2$ then as it follows from the second part of theorem 3, the estimation (51) is valid for any $\beta \leq 1$.

Let $n \geq 5$. Then again applying the Holder inequality we obtain

$$J_3 \leq C_{33}(\mathcal{L}, n, \partial D, diamD)\left(\int_D \left|u^2\right|^{\frac{2n}{n-4}} dx\right)^{\frac{n-4}{2n}} \left\|w^1 - w^2\right\|^{\gamma}_{W^1_{\frac{2n\gamma p_1}{2n-p_1(n-4)}}(D)}. \quad (52)$$

It follows from theorem 3 that

$$\left\|u^2\right\|_{L_{\frac{2n}{n-4}}(D)} \leq C_{34}(n)\left\|\left|u_x^2\right|\right\|_{L_{\frac{2n}{n-2}}(D)} \leq C_{35}(n)\left\|u^2\right\|_{W_2^2(D)}.$$

Besides

$$\frac{2n\gamma p_1}{2n-p_1(n-4)} \leq \frac{2n\gamma^0 p_1}{2n-p_1(n-4)} \leq \frac{2np_1\frac{n(2-p_1)+4p_1}{2(n-p_1)}}{n(2-p_1)+4p_1} = \frac{np_1}{n-p_1} = q.$$

Therefore from (52) subject to (45) we derive

$$J_3 < C_{36}(\mathcal{L}, n, \partial D, diamD)\lambda^{\gamma}\left\|u^2\right\|_{W_2^2(D)}. \quad (53)$$

It's proved with the help of the second part of theorem 3 by simply form that the estimation (53) at $n \leq 4$ is valid for any $\gamma \leq 1$.

Thus, allowing for (49), (51) and (53) in (47) we finally obtain

$$\left\|u^1 - u^2\right\|_{W_q^1(D)} \leq C_{37}(\mathcal{L}, n, \partial D, diamD)\left(\lambda^{\alpha} + \lambda^{\beta} + \lambda^{\gamma}\right)\left\|u^2\right\|_{W_2^2(D)} \leq$$

$$\leq C_{38}(\mathcal{L}, n, \partial D, diamD)\left(\lambda^{\alpha} + \lambda^{\beta} + \lambda^{\gamma}\right)\|F\|_{L_2(D)}. \quad (54)$$

We choose λ so small that

$$\left(\lambda^{\alpha} + \lambda^{\beta} + \lambda^{\gamma}\right)\|F\|_{L_2(D)} < \frac{\varepsilon}{C_{38}}.$$

Then it follows from (54) that $\left\|u^1 - u^2\right\|_{W_q^1(D)} < \varepsilon$. The lemma is proved.

Theorem 4. *Let the conditions (3)-(4), (10)-(15) are satisfied with respect to the coefficients of the operator $\mathcal{L}$ and $mesD \leq d_0$. Then the first boundary value problem (1)-(2) has a strong solution $u(x) \in \dot{W}_2^2(D)$. In addition*

$$\|u\|_{W_2^2(D)} \leq C_{39}(\mathcal{L}, n, \partial D, diamD) \|b(x,0,0)\|_{L_2(D)}. \tag{55}$$

Proof. We'll use the following Schauder theorem: the continuous mapping of convex compact into itself contains a fixed point. According to this theorem there exists a function $u(x) \in \mathcal{A}$ such that $u = \mathcal{H}u$. But when $w = u$ the first boundary problem (20)-(21) coincides with the problem (18)-(19) and by the same token with the initial problem (1)-(2). The existence of a strong solution of the problem (1)-(2) is proved. Now the estimation (55) immediately follows from theorem 2.

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