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ON THE EXISTENCE OF GENERALIZED SOLUTIONS OF ONE CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER ON THE WHOLE AXIS

Abstract

At the paper the sufficient conditions on the existence and uniqueness of generalized solution on whole axis are got for operator-differential equation of the fourth order, the main part of which has the multi characteristic.

Let's consider in a separable Hilbert space H the operator-differential equation

$$P\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^2 u(t)+\sum_{j=0}^{4}A_j u^{(4-j)}(t)=f(t),\quad t\in R, \qquad (1)$$

where $f(t)$ and $u(t)$ are vector-valued functions from H, and the coefficients A, A_j $\left(j=\overline{0,4}\right)$ satisfy the following conditions:

1) A is a positive definite self-adjoint operator in H;
2) the operators A_j $\left(j=\overline{0,4}\right)$ are linear in H.

It is known that the operator A generates the Hilbert scale, i.e.

$$H_\gamma=D\left(A^\gamma\right),\ (x,y)_\gamma=\left(A^\gamma x, A^\gamma y\right),\quad x,y\in D\left(A^\gamma\right),\ \gamma\ge 0,$$

moreover we suppose that $H_0=H$.

Let's denote by $L_2(R:H)$ a Hilbert space of all vector-functions $f(t)$ with the values in H, measurable quadratically-integrable in sense of Bohner and let's set that

$$\|f\|_{L_2(R:H)}=\left(\|u''\|^2_{L_2(R:H)}+\left\|A^2u\right\|^2_{L_2(R:H)}\right)^{1/2}.$$

Let's determine further a Hilbert space

$$W_2^2(R:H)=\left\{u(t)/u''(t)\in l_2(R:H),\ A^2u(t)\in l_2(R:H)\right\}$$

with the norm

$$\|u\|_{W_2^2(R:H)}=\left(\|u''\|^2_{L_2(R:H)}+\left\|A^2u\right\|^2_{L_2(R:H)}\right)^{1/2}.$$

Let's denote that here and in further the derivatives are understood in the sense of distribution theory [1]. As is known $D(R:H)$ is a set of infinite differentiable vector-functions with compact supports in $R+$ densely in the space $W_2^2(R:H)$, [1, p.28].

At the given paper we'll give the definition of a generalized solution of the equation (1) and we'll prove the theorem on the existence and uniqueness of a generalized solution of the equation (1). Let's denote that the boundary value problem on the semi-axis $R_+=(0:+\infty)$ for the equation (1) is investigated in the paper [2].

Let

$$P_0\left(\frac{d}{dt}\right)u(t)=\left(-\frac{d^2}{dt^2}+A^2\right)^2 u(t),\quad u(t)\in D(R:H),$$

$$P_1\left(\frac{d}{dt}\right)u(t)=\sum_{j=0}^{4}A_j u^{(4-j)}(t),\qquad u(t)\in D(R:H).$$

Now let's formulate the lemma which shows at which conditions on operator coefficients of the equation (1) has the meaning solution of the equation from the class $W_2^2(R:H)$.

Lemma 1. *Let the conditions 1), 2) be fulfilled, moreover the operators* $B_0 = A_0$, $B_1 = A_1 A^{-1}$, $B_2 = A^{-1} A_2 A^{-1}$, $B_3 = A^{-2} A_3 A^{-1}$ *and* $B_4 = A^{-2} A_4 A^{-2}$ *are bounded in* H. *Then the bilinear functional* $\left(P_1(d/dt)u,\Psi\right)_{L_2(R:H)}$ *determined on* $D(R:H)\oplus D(R:H)$ *is continued on the space* $W_2^2(R:H)\oplus W_2^2(R:H)$ *by continuity till the bilinear functional* $\mathrm{P}_1(u,\Psi)$ *acting by the following form*

$$\mathrm{P}_1(u,\Psi)=\sum_{j=0}^{1}(-1)^j\left(A_j u^{(2-j)},\Psi''\right)_{L_2}-\left(A_2 u',\Psi'\right)_{L_2}+\sum_{j=3}^{4}\left(A_j u^{(4-j)},\Psi\right)_{L_2}.$$

Proof. Allowing for $u \in D(R:H)\oplus W_2^2(R:H)$ after integrating by parts we get

$$\left(P_1\left(\frac{d}{dt}\right)u,\Psi\right)_{L_2}=\sum_{j=0}^{1}(-1)^j\left(A_j u^{(2-j)},\Psi''\right)_{L_2}-\left(A_2 u',\Psi'\right)_{L_2}+\sum_{j=3}^{4}\left(A_j u^{(4-j)},\Psi\right)_{L_2}.$$

Since $B_0 = A_0$ and $B_1 = A_1 A^{-1}$ are bounded in H, then when $j=0$ and $j=1$ we have $\left(A_j u^{(2-j)},\Psi''\right)_{L_2}=\left(A_j A^{-j} A^j u^{(2-j)},\Psi''\right)_{L_2}=\left(B_j A^j u^{(2-j)},\Psi''\right)_{L_2}$.

From the theorem on intermediate derivatives [1] it follows

$$\left|\left(A_j u^{(2-j)},\Psi''\right)_{L_2}\right|\le\left\|B_j\right\|\left\|A^j u^{(2-j)}\right\|_{L_2}\cdot\left\|\Psi''\right\|_{L_2}\le C_j\left\|B_j\right\|\cdot\|u\|_{W_2^2}\cdot\|\Psi\|_{W_2^2}. \tag{2}$$

By virtue of that the operator $B_2 = A^{-1}A_2A^{-1}$ is bounded in H, then the inequality

$$\begin{aligned}\left|\left(A_2 u',\Psi'\right)_{L_2}\right|&=\left|\left(A^{-1}A_2A^{-1}Au',A\Psi'\right)_{L_2}\right|=\left|\left(B_2Au',A\Psi'\right)_{L_2}\right|\le\\ &\le\|B_2\|\cdot\|Au'\|_{L_2}\|A\Psi'\|_{L_2}\le C_2\|B_2\|\cdot\|u\|_{W_2^2}\cdot\|\Psi\|_{W_2^2}\end{aligned} \tag{3}$$

is true.

Analogously in case $j=3$ and $j=4$

$$\begin{aligned}\left|\left(A_j u^{4-j},\Psi\right)_{L_2}\right|&=\left|\left(A^{-2}A_jA^{2-j}A^{j-2}u^{(4-j)},A^2\Psi\right)_{L_2}\right|=\left|\left(B_jA^{j-2}u^{(4-j)},A^2\Psi\right)_{L_2}\right|\le\\ &\le\left\|B_j\right\|\cdot\left\|A^{j-2}u^{(4-j)}\right\|_{L_2}\left\|A^2\Psi\right\|_{L_2}\le C_j\left\|B_j\right\|\cdot\|u\|_{W_2^2}\cdot\|\Psi\|_{W_2^2}\end{aligned} \tag{4}$$

From the inequality (2)-(4) it follows that for $u,\ \Psi \in D(R:H)$ the inequality

Lemma 2. *Let the conditions of lemma 1 hold. Then the bilinear functional* $\left(P\left(\frac{d}{dt}\right)u,\Psi\right)_{L_2}$ *determined on the space* $D(R:H)\oplus D(R:H)$ *continues by continuity till the bilinear functional*

$$\mathrm{P}(u,\Psi)=(u,\Psi)_{W_2^2}+\mathrm{P}_1(u,\Psi)+2\left(Au',A\Psi'\right)_{L_2} \tag{5}$$

acting in the space $W_2^2(R:H)\oplus W_2^2(R:H)$. *Moreover,* $\boldsymbol{P}_1(u,\Psi)$ *is determined as in lemma 1.*

The proof of the lemma follows from lemma 1 and from the equality

$$\left(P_0\left(\frac{d}{dt}\right)u,\Psi\right)_{L_2}=\left(\left(-\frac{d^2}{dt^2}+A^2\right)^2u,\Psi\right)_{L_2}=\left(u'',\Psi''\right)_{L_2}+2\left(Au',A\Psi'\right)+$$

$$+\left(A^2u,A^2\Psi\right)_{L_2}=\left(u,\Psi\right)_{W_2^2}+2\left(Au',A\Psi'\right)_{L_2},$$

which is true for the vector-function $u,\Psi\in D(R:H)$.

Definition. *The vector-function* $u(t)\in W_2^2(R:H)$ *is called the generalized solution of the equation (1) if for any vector-function* $\Psi(t)\in W_2^2(R:H)$ *it holds the equality*

$$\mathrm{P}(u,\Psi)=(f,\Psi)_{L_2},$$

where $\mathrm{P}(u,\Psi)$ *is determined from the equality (5).*

It holds the following

Theorem. *Let* A *be a positive determined self-adjoint operator in* H *the operators* $B_0=A_0$, $B_1=A_1A^{-1}$, $B_2=A^{-1}A_2A^{-1}$, $B_3=A^{-2}A_3A^{-1}$, $B_4=A^{-2}A_2A^{-2}$ *are bounded in* H *and it holds the inequality*

$$\sum_{j=0}^{4}\gamma_j\left\|B_j\right\|<1, \tag{6}$$

where $\gamma_0=\gamma_1=1,\ \gamma_1=\gamma_3=\frac{1}{2},\ \gamma_2=\frac{1}{4}$. *Then the equation (1) has the unique generalized solution, moreover*

$$\|u\|_{W_2^2(R:H)}\le const\|f\|_{L_2(R:H)}. \tag{7}$$

Proof. It is evident that for any $\Psi\in D(R:H)$ it holds the inequality

$$\left|\left(P_0\left(\frac{d}{dt}\right)\Psi,\Psi\right)_{L_2}\right|=\|\Psi\|_{W_2^2}^2+2\|A\Psi'\|_{L_2}^2=|||\Psi|||^2, \tag{8}$$

consequently for any $\Psi\in W_2^2(R:H)$ the inequality

$$\left|\left(P\left(\frac{d}{dt}\right)\Psi,\Psi\right)_{L_2}\right|\ge|||\Psi|||^2-\left|\left(P_1\left(\frac{d}{dt}\right)\Psi,\Psi\right)_{L_2}\right| \tag{9}$$

is true.

It is evident that

$$\left|P_1\left(\left(\frac{d}{dt}\right)\Psi,\Psi\right)_{L_2}\right|\le\sum_{j=0}^{1}\left|\left(A_j\Psi^{(2-j)},\Psi''\right)_{L_2}\right|+\left|\left(A_2\Psi',\Psi'\right)_{L_2}\right|+\sum_{j=3}^{4}\left|\left(A_j\Psi^{(4-j)},\Psi\right)_{L_2}\right|. \tag{10}$$

On the other hand when $\Psi\in D(R:H)$

$$\left|\left(A_2\Psi',\Psi'\right)_{L_2}\right|=\left|\left(B_2A\Psi',A\Psi'\right)_{L_2}\right|\le\|B_2\|\cdot\|A\Psi'\|_{L_2}^2. \tag{11}$$

Allowing for

$$\|A\Psi'\|_{L_2}^2=\int_{-\infty}^{+\infty}\left(A\Psi',A\Psi'\right)dt=-\int_{-\infty}^{+\infty}\left(A^2\Psi,\Psi''\right)dt\le\left\|A^2\Psi\right\|_{L_2}\|\Psi''\|_{L_2}^2\le\frac{1}{2}\|\Psi''\|_{L_2}^2,$$

i.e.

$$2\|A\Psi'\|_{L_2}^2\le\|\Psi\|_{W_2^2}^2=\|\Psi\|_{W_2^2}^2+2\|A\Psi'\|_{L_2}^2-2\|A\Psi'\|_{L_2}^2$$

or

$$4\|A\Psi'\|_{L_2}^2 \le \|\Psi\|_{W_2^2}^2 + 2\|A\Psi'\|_{L_2}^2,$$

i.e.

$$\|A\Psi'\|_{L_2}^2 \le \frac{1}{4}\left(\|\Psi\|_{W_2^2}^2 + 2\|A\Psi'\|_{L_2}^2\right) = \frac{1}{4}|||\Psi|||^2, \tag{12}$$

allowing for the inequality (12) in inequality (11) we find

$$\left|(A_2\Psi', \Psi')_{L_2}\right| = \frac{1}{4}\|B_2\| \cdot |||\Psi|||^2, \tag{13}$$

when $j = 0$ we have

$$\left|(A_0\Psi'', \Psi'')_{L_2}\right| \le \|B_0\| \cdot \|\Psi''\|_{L_2}^2 \le \|B_0\|\left(\|\Psi\|_{W_2^2}^2 + 2\|A\Psi'\|_{L_2}^2\right) = \|B_0\| |||\Psi|||^2, \tag{14}$$

and when $j = 1$ analogously we get

$$\left|(A_2\Psi', \Psi'')_{L_2}\right| \le \left|(B_1 A\Psi', \Psi'')_{L_2}\right| \le \|B_1\| \cdot \|A\Psi'\|_{L_2} \|\Psi''\|_{L_2} \le \frac{1}{2}\|B_1\| \times$$

$$\times \left(\|A\Psi'\|_{L_2}^2 + \|\Psi''\|_{L_2}^2\right) \le \frac{1}{2}\|B_1\| |||\Psi|||^2, \tag{15}$$

when $j = 3$ from Cauchy inequality and from the definition $|||\Psi|||$ it follows:

$$\left|(A_3\Psi', \Psi)_{L_2}\right| \le \left|(B_3 A\Psi', A^2\Psi)_{L_2}\right| \le \|B_3\| \cdot \|A\Psi'\|_{L_2} \|A^2\Psi\|_{L_2} \le \|B_3\| \times$$

$$\times \frac{1}{2}\left(\|A\Psi'\|_{L_2}^2 + \|\Psi''\|_{L_2}^2\right) \le \frac{1}{2}\|B_3\| |||\Psi|||^2, \tag{16}$$

when $j = 4$ the inequality

$$\left|(A_4\Psi, \Psi)_{L_2}\right| = \left|(B_4 A^2\Psi, A^2\Psi)_{L_2}\right| \le \|B_4\| \cdot \|A^2\Psi\|_{L_2}^2 \le \|B_4\| |||\Psi|||^2. \tag{17}$$

is true.

Allowing for the inequality (13)-(17) in the inequality (10) we get

$$\left|\left(P_1\left(\frac{d}{dt}\right)\Psi, \Psi\right)_{L_2}\right| \le |\mathrm{P}(\Psi, \Psi)| \le \sum_{j=0}^{4} \gamma_j \|B_j\| |||\Psi|||^2,$$

where $\gamma_0 = \gamma_4 = 1$, $\gamma_1 = \gamma_3 = \frac{1}{2}$, and $\gamma = \frac{1}{4}$. Thus from the equality (9) it follows that

$$|\mathrm{P}(\Psi, \Psi)| \ge |||\Psi|||^2 - \left(\sum_{j=0}^{4} \gamma_j \|B_j\|\right)|||\Psi|||^2 = \left(1 - \sum_{j=0}^{4} \gamma_j \|B_j\|\right)|||\Psi|||^2 \ge const|||\Psi|||. \tag{18}$$

On the other hand from the determination of the generalized solution it follows that

$$\mathrm{P}(u, \Psi) = (u, \Psi)_{W_2^2} + 2(Au', A\Psi')_{L_2} + \mathrm{P}_1(u, \Psi) = (f, \Psi), \tag{19}$$

and the right part of this formula determines the continuous functional in the space $W_2^2(R:H)$, and the left part satisfies the conditions of Lax-Milgram theorem [3], since the inequality (8) is true. Therefore there exists the unique vector-function $u(t) \in W_2^2(R:H)$ satisfying the equality (19). Since from the inequality (18) for $\Psi = u$ it follows that

$$|\mathrm{P}(u,u)| = \left|(f, u)_{L_2}\right| \ge const|||u|||^2 \ge const\|u\|_{W_2^2}^2$$

consequently

$$\|u\|_{W_2^2}^2 \le const\|f\|_{L_2}.$$

The theorem is proved.

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