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WEIGHTED IMBEDDING THEOREMS FOR ANISOTROPIC SOBOLEV SPACES

Abstract

In this paper the weight effects for integral operators arises on the basis of the integral representation of Il'in-Besov (see [1]) for domains Ω_k , $0 \le k \le n-1$, *which considered in [2] and weighted imbedding theorems in anisotropic Sobolev space* $W^{l_1,\dots,l_n}_{\omega_0,\omega_1,\dots,\omega_n}(\Omega_k)$ *are obtained. These weight effects for domain* Ω_{n-1} *was proved in [3].*

Let $R_n - n$- dimensional Euclidean space of points $x = (x_1,\dots,x_n), x = (x', x'')$, $x' \in R^k$, $x'' \in R^{n-k}$, $R^n_{++} = \{x : x = (x_1,\dots,x_n) \in R^n, x_i > 0, i = 1,\dots,n\}$, $a = (a_1,\dots,a_n)$, $a_i > 0$, $|a| = \sum_{i=1}^{n} a_i$, $N_0 = N \cup \{0\}$, N the set of natural number, $l, \nu \in N_0^n$ and $\rho(x) = \sum_{i=1}^{n} |x_i|^{1/a_i}$.

Assume that

$$\Omega_0 = \{x : x \in R^n, x_i^{(0)} < x_i < \infty, \ i = 1,\dots,n\},$$
$$\Omega_k = \{x : x' \in R^k, \varphi_i(x') < x_i < \infty (i = k+1,\dots,n)\}, \ k = 1,\dots,n-1, \tag{1}$$
$$\Gamma_0 = \{x : x \in R^n, x_i = x_i^{(0)}, \ i = 1,\dots,n\},$$
$$\Gamma_k = \{x : x' \in R^k, x'' = \overline{\varphi}(x')\}, \quad k = 1,\dots,n-1,$$

where the vector-function $\overline{\varphi}(x') = (\varphi_{k+1}(x'),\dots,\varphi_n(x'))$, $k = 1,\dots,n-1$ satisfies an anisotropic Hölder condition

$$\rho(\overline{\varphi}(x') - \overline{\varphi}(y')) \le M \, \rho(x' - y'), \quad \forall x', y' \in R^k,$$

$\rho(x, \Gamma_0) = \rho(x - x^{(0)})$, $\rho(x, \Gamma_k) = \inf_{y \in \Gamma_k} \rho(x - y)$, $k = 1,\dots,n-1$ and $x^{(0)} = (x_1^{(0)},\dots,x_n^{(0)})$ be a fixed point in R^n . If $x^{(0)} = (0,\dots,0)$, then $\Omega_0 = R^n_{++}$.

Let ω be a measurable, almost every positive, non-negative and locally summable in Ω_k function. Denote by $L_{p,\omega}(\Omega_k)$ the set of all measurable function $f(x)$ on Ω_k such that norm

$$\|f\|_{L_{p,\omega}(\Omega_k)} = \left(\int_{\Omega_k} |f(x)|^p \omega(x) dx \right)^{1/p}, \quad 1 \le p < \infty$$

is finite.

Let $b = (b_1,\dots,b_n)$, $c = (c_1,\dots,c_n)$, where $0 < b_i < c_i < \infty$, and $i = 1,\dots,n$. The set $R(1/a) = \{y : y_i > 0, b_i h < y_i^{1/a_i} < c_i h \ (i = 1,\dots,n), 0 < h < \infty\}$ is called $1/a$ horn (see [1]).

Lemma 1 [2]. *The domains* Ω_k , $k = 0,1,\dots,n-1$, *satisfies* $1/a$ *-horn condition, i.e. there exists* $R(1/a)$ *such that*

$$\Omega_k + R(1/a) = \Omega_k .$$

We put

$$\pi_k(x) = \rho(x'' - \overline{\varphi}(x')) = \sum_{i=k+1}^{n} \left| x_i - \varphi_i(x') \right|^{1/a_i}, \quad k = 1,...,n-1,$$

$$\pi_0(x) = \rho\left(x - x^0\right), \qquad\qquad k = 0.$$

Lemma 2 [2]. *Suppose that* Ω_k *has the form (1). Then* $\rho(x,\Gamma_k)$ *is equivalent to* $\pi_k(x)$ *for all* $x \in \Omega_k$ *more precisely,*

$$\exists C_0 > 0, \quad \forall x \in \Omega_k, \quad C_0 \pi_k(x) \le \rho(x,\Gamma\ ) \le \pi_k(x).$$

Let K_α be a given on $R_n \setminus \{0\}$ function such that supp $K_\alpha \subset R(1/a)$ and having the following properties:

a) $K_\alpha(x) = \rho(x)^{a-|a|}$ for $0 < \alpha < |a|$;

b) if $\alpha = 0$, then K_0 satisfies the following conditions

$$K_0\left(t^a x\right) = t^{-|a|} K_0(x), \quad \int_{S_k} K_0(x) \sum_{i=1}^{n} a_i x_i^2 d\sigma(x) = 0, \quad \int_0^1 \omega_{K_0}(t) \frac{dt}{t} < \infty,$$

where $\omega_{K_0}(t) = \sup\{|K_0(x) - K_0(y)|; x,y \in S_k, |x-y| \le t\}$, $S_k = S \cap \Omega_k, k = 0,1,...,n-1$, $S =$ $= \{x : \rho(x) = 1\}$ and $d\sigma$ is area element.

Consider the integral operator $K_\alpha : f \to K_\alpha f$, where

$$K_\alpha f = \int_{R^n} K_\alpha(y) f(x+y) dy, \qquad \text{supp } K_\alpha \subset R(1/a).$$

The weight ω is said to belong to $A_p(\Omega_k)$, $k = 0,1,...,n-1$, $1 < p < \infty$, if

$$\sup \left(\frac{1}{|B|} \int_{B \cap \Omega_k} \omega(x) dx \right) \left(\frac{1}{|B|} \int_{B \cap \Omega_k} \omega(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset R^n$, and $p' = \dfrac{p}{p-1}$.

Theorem 1. *Let* $0 \le \alpha < |a|, 1 < p < \dfrac{|a|}{\alpha}$, $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{\alpha}{|a|}$. *Suppose that the function* $\varphi \in A_{1+\frac{q}{p'}}(\Omega_k)$.

Then there exists a positive constant C *such that for any* $f \in L_{p,\varphi}\left(R^n\right)$ *the following inequalities*

$$\left(\int_{\Omega_k} \left| K_\alpha \left(f \cdot \varphi^{\frac{\alpha}{|a|}} \right)(x) \right|^q \varphi(x) dx \right)^{1/q} \le C \left(\int_{\Omega_k} |(f(x))|^p \varphi(x) dx \right)^{1/p}.$$

If $\alpha = 0$, *then assume that, the kernel of anisotropic singular integral operator (ASIO)* K_0 *satisfies the condition b).*

Note that, if $\Omega_k = R^n$, then for $0 < \alpha < |a|$ the theorem 1 in the isotropic case was proved in [4] and in the anisotropic case in [5], but also for $\alpha = 0$ in the isotropic case was proved in [6], and in the anisotropic case in [7]. In the case of for all domains Ω_k the prove of theorem 1 lead analogously.

Theorem 2. *Let* $1<p\le q<\infty,\ 0\le\alpha<|a|$, $\alpha=|a|\left(\frac{1}{p}-\frac{1}{q}\right)$ *and* $\varphi\in A_{1+\frac{q}{p'}}(\Omega_k)$ *the radial function depend on* $\rho(x,\Gamma_k)$ *(i.e.* $\varphi(x)=\varphi(\rho(x,\Gamma_k))$*). Suppose that* u *and* u_1 *are the positive monotone function on* $(0,\infty)$.

Suppose that the weight pair of radial function $(\omega(t),\omega_1(t))$ *satisfies the conditions 1) or 2):*

1) ω *and* $\omega_1=u_1\varphi$ *are weights defined on* $(0,\infty)$

$$\exists C>0,\ \forall t\in(0,\infty),\ \omega_1(t)^{p/q}\le C\varphi(t)^{-\frac{\alpha p}{|a|}}\omega(t),$$

where u_1 *is increasing function on* $(0,\infty)$;

2) $\omega=u\varphi$ *and* $\omega_1=u_1\varphi$ *are weights defined on* $\omega=u\varphi$ *and*

$$\sup_{t>0}\left(\int_0^{t/2}\omega_1(\tau)\tau^{|a''|-1}d\tau\right)^{p/q}\left(\int_0^{\infty}\left(\varphi(\tau)^{-\frac{\alpha p}{|a|}}\omega(\tau)\right)^{1-p'}\tau^{-1-|a''|\,p'/q}d\tau\right)^{p-1}<\infty,$$

where u *and* u_1 *are decreasing function on* $(0,\infty)$ *and* $\varphi(\rho(x,\Gamma_k))$ *is equivalently to* $\varphi(\pi_k(x))$.

Then if $1<p<q<\infty$, *then the operator* $f\to K_\alpha f$ *gives a bounded mapping from* $L_{p,\omega}(\Omega_k)$ *to* $L_{q,\omega_1}(\Omega_k)$, *i.e.*

$$\left(\int_{\Omega_k}\left|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)(x)\right|^q\omega_1(\rho(x,\Gamma_k))dx\right)^{1/q}\le C\left(\int_{\Omega_k}|f(x)|^p\omega(\rho(x,\Gamma_k))dx\right)^{1/p}. \quad (2)$$

In the case $\alpha=0$, *assume that, the kernel of ASIO* K_0 *satisfies the b). In this case the operator* K_0 *acts boundedly from* $L_{p,\omega}(\Omega_k)$ *to* $L_{p,\omega_1}(\Omega_k)$.

Proof of theorem 2. Let $f\in L_{p,\omega(\rho(x,\Gamma_k))}(\Omega_k)$ and suppose that the weight pair $(\omega(\rho(x,\Gamma_k)),\omega_1(x,\Gamma_k))$ satisfies condition 1) of theorem 2.

Without restriction of generality we may assume that the function u has the form $u_1(t)=u_1(0)+\int_0^t\psi(t)d\tau$, where $u_1(0)=\lim_{t\to 0}u_1(t)$ and ψ is non-negative on $(0,\infty)$ function. In fact there exists a sequence of absolutely continuous functions u_n such that $u_n(t)\le u(t)$ and $\lim_{n\to\infty}u_n(t)=u(t)$ for any $t\in(0,\infty)$. For such functions we may take $u_n=$ $=u(+0)+n\int_0^t\left[u(\tau)-u\left(\tau-\frac{1}{n}\right)\right]d\tau$.

Estimate the left-hand side of inequality (2):

$$\left\|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)\right\|_{L_{q,\omega_1}(\Omega k)}\le\left(u_1(0)\int_{\Omega_k}\left|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)(x)\right|^q\varphi(\rho(x,\Gamma_k))dx\right)^{1/q}+$$

$$+\left(\int_{\Omega_k}\left|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)(x)\right|^q\left(\int_0^{\rho(x,\Gamma_k)}\psi(\tau)d\tau\right)\varphi(\rho(x,\Gamma_k))dx\right)^{1/q}=A_1+A_2\,.$$

If $u_1(0)=0$, then $A_1=0$. However if $u_1(0)\neq 0$, then by theorem 1 and also by condition theorem 2 we have

$$A_1\le C_1u_1^{\frac{1}{q}}(0)\left(\int_{\Omega_k}|f(x)|^p\varphi(x)dx\right)^{1/p}\le C_1\left(\int_{\Omega_k}|f(x)|_B^p\varphi(x)u_1^{\frac{p}{q}}(\rho(x,\Gamma k))dx\right)^{1/p}\le$$

$$\le C_2\left(\int_{\Omega_k}|f(x)|^p\omega(\rho(x,\Gamma k))dx\right)^{1/p}.$$

Estimate A_2. It is not hard to prove that $\pi_k(y)>\pi_k(x)$ for $x\in\Omega_k$ and $y\in R(1/a)$, $k=0,1,,...,n-1$. If use the fact that $K_\alpha(x)$ is zero outside the horn $R(1/a)$, then in view of the condition 1), theorem 1 and lemma 2, we obtain

$$A_2=\left(\int_0^\infty\psi(\tau)d\tau\int_{\rho(x,\Gamma_k)>\tau}\left|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)(x)\right|^p dx\right)^{1/q}=$$

$$=\left(\int_0^\infty\psi(\tau)d\tau\int_{\pi_k(x)>\tau/c}\left|\int_{\pi_k(y)>\pi_k(x)}K_\alpha(y-x)f(y)\varphi^{\frac{\alpha}{|a|}}(y)\right|^q dx\right)^{1/q}=$$

$$=\left(\int_0^\infty\psi(\tau)d\tau\int_{\pi_k(x)>\tau/c}\left|\int_{\pi_k(y)>\tau/c}K_\alpha(y-x)f(y)\varphi^{\frac{\alpha}{|a|}}(y)\right|^q dx\right)^{1/q}\le$$

$$\le C\left[\int_0^\infty\psi(\tau)d\tau\left(\int_{\pi_k(x)>\tau/c}|f(x)|^p\varphi(x)dx\right)^{q/p}\right]^{1/q}\le C\left[\int_{\Omega_k}|f(x)|^p\varphi(x)\left(\int_0^{c\pi_k(x)}\psi(\tau)d\tau\right)^{p/q}dx\right]^{1/p}\le$$

$$\le C\left[\int_{\Omega_k}|f(x)|^p\varphi(x)u_1(c\pi_k(x))^{p/q}dx\right]^{1/p}\le C\left[\int_{\Omega_k}|f(x)|^p\varphi(x)u_1(\rho(x,\Gamma_k))^{p/q}dx\right]^{1/p}\le$$

$$\le C\left(\int_{\Omega_k}|f(x)|^p\omega(\rho(x,\Gamma_k))dx\right)^{1/p}.$$

The first part of theorem 2 is proved.

We now consider the case weight pair (ω,ω_1) satisfies the condition 2).

Let u_1 be an arbitrary positive decreasing on $(0,\infty)$ and let us extend the function u_1 on the left of zero by $u_1(\infty)$. Consider the sequence of functions

$$\vartheta_n(t)=u_1(\infty)+\int_t^\infty\psi_n(\tau)d\tau\,,$$

where $\psi_n(\tau)=n[u_1(\tau)-u_1(\tau+1/n)]$. Since u_1 decreases we have $\psi n(\tau)\geq 0$. On the other hand,

$$\vartheta_n(t)=n\int_t^{t+1/n}u_1(\tau)d\tau,$$

and thus $\vartheta_n(t)\leq u_1(t)$ and $\lim_{n\to\infty}\vartheta_n(t)=u_1(t)$ a.e. $t>0$. Consequently

$$\lim_{n\to\infty}\vartheta_n(\rho(x,\Gamma_k))=u_1(\rho(x,\Gamma_k))$$

for almost $x\in\Omega_k$ (see [3, 10]).

Without restriction of generality we may assume that the function u_1 has the form

$$u_1(t)=u_1(\infty)+\int_t^{\infty}\psi(\tau)d\tau,$$

where $u_1(\infty)=\lim_{t\to+\infty}u_1(t)$ and ψ is positive on $(0,\infty)$ function.

In [8, lemma 3] (for $1<p=q<\infty$ see[9, lemma 2.5]), if $b\geq 1$, then there exists a positive constant c such that for an arbitrary $t>0$ the inequality

$$u_1^{\frac{p}{q}}\left(\frac{t}{\beta}\right)\leq cu(t) \tag{3}$$

holds, where c depend only on $|a|$ and β.

We have

$$\left\|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)\right\|_{L_{q,\omega_1(\rho(x,\Gamma_k))}(\Omega_k)}\leq\left(\int_{\Omega_k}\left|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)(x)\right|^q u_1(\infty)\varphi(x)dx\right)^{\frac{1}{q}}+$$

$$+\left(\int_{\Omega_k}\left|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)(x)\right|^q\varphi(x)\left(\int_{\varphi(x,\Gamma_k)}^{\infty}\psi(t)dt\right)dx\right)^{\frac{1}{q}}=B_1+B_2.$$

If $u_1(\infty)=0$, then $B_1=0$. However if $u(\infty)>0$, then by theorem 1 we obtain

$$B_1\leq C\,u_1^{\frac{1}{q}}(\infty)\left(\int_{\Omega_k}|f(x)|^p\varphi(x)dx\right)^{\frac{1}{p}}\leq C\left(\int_{\Omega_k}|f(x)|^p\varphi(x)u_1^{\frac{p}{q}}(\rho(x,\Gamma_k))dx\right)^{\frac{1}{p}}.$$

Using the inequality (3) we get

$$B_1\leq C\left(\int_{\Omega_k}|f(x)|^p\omega(\rho(x,\Gamma_k))dx\right)^{\frac{1}{p}}.$$

Now estimate B_2

$$B_2=\left(\int_0^{\infty}\psi(\tau)\left(\int_{\rho(x,\Gamma_k)<\tau}\left|K_\alpha\left(f\cdot\varphi^{\frac{\alpha}{|a|}}\right)(x)\right|^q\varphi(x)dx\right)d\tau\right)^{\frac{1}{q}}\leq$$

$$\leq \left(\int_0^\infty \psi(\tau) \left(\int_{\rho(x,\Gamma_k)<\tau} \varphi(x) \left| \int_{\rho(x,\Gamma_k)<\frac{2}{C_0}\tau} K_\alpha(y-x) f(y) \varphi^{\frac{\alpha}{|a|}}(y) dy \right|^q dx \right) d\tau \right)^{\frac{1}{q}} +$$

$$+ \left(\int_0^\infty \psi(\tau) \left(\int_{\rho(x,\Gamma_k)<\tau} \varphi(x) \left| \int_{\rho(x,\Gamma_k)\geq\frac{2}{C_0}\tau} K_\alpha(y-x) f(y) \varphi^{\frac{\alpha}{|a|}}(y) dy \right|^q dx \right) d\tau \right)^{\frac{1}{q}} = B_{21} + B_{22}.$$

By theorem 1 and by the generalized Minkowski inequality we get

$$B_{21} \leq C \left(\int_0^\infty \psi(\tau) \left(\int_{\rho(x,\Gamma_k)<\tau} \int_{\rho(x,\Gamma_k)<\frac{2}{C_0}\tau} |f(x)|^p \varphi(x) dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \leq$$

$$\leq C \left(\int_{\Omega_k} |f(x)|^p \varphi(x) \left(\int_{\frac{C_0 \rho(x,\Gamma_k)}{2}} \psi(\tau) d\tau \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Using again the inequality (3) we get

$$B_{21} \leq C \left(\int_{\Omega_k} |f(x)|_B^p \omega(\rho(x,\Gamma_k)) dx \right)^{\frac{1}{p}}.$$

Now estimate B_{22}. Since $\varphi(\rho(x,\Gamma_k)) \sim \varphi(\pi_k(x))$, then

$$B_{22} \leq C \left[\int_0^\infty \psi(\tau) d\tau \int_{\rho(x,\Gamma_k)<\tau} \varphi(x) \left(\int_{\rho(y,\Gamma_k)\geq 2\tau/c_0} \frac{|f(y)| \varphi^{\frac{\alpha}{|a|}}(y)}{\rho(x-y)^{|a|-\alpha}} dy \right)^q dx \right]^{1/q} =$$

$$= \left(\int_0^\infty \psi(\tau) A^q(\tau) d\tau \right)^{1/q}.$$

Further by virtue of lemma 2 we have

$$A^q(\tau) \leq \int_{\pi_k(x)<\tau/c} \varphi(\pi_k(x)) \left(\int_{\pi_k(y)>2\tau} \frac{|f(y)| \varphi^{\frac{\alpha}{|a|}}(\pi_k(y))}{\rho(x-y)^{|a|-\alpha}} dy \right)^q dx =$$

$$= \int_{\rho(x''-\overline{\varphi}(x'))<\tau/c} \varphi\left(c\rho(x''-\overline{\varphi}(x')) dx'' \int_{R^k} dx' \left(\int_{R^k} dy' \times \right.\right.$$

$$\times \int_{\rho(y''-\overline{\varphi}(y'))>2\tau} \frac{\|f(y)\|_B \varphi^{\frac{\alpha}{|a|}}(\rho(y''-\overline{\varphi}(y')))}{\rho(x-y)^{|a|-\alpha}} \Bigg)^q .$$

It is easy to show that

$$\rho(\eta'-\xi',\eta''+\overline{\varphi}(\eta')-\xi''-\overline{\varphi}(\xi')) \geq C\rho(\eta-\xi). \qquad (4)$$

Indeed,

$$\rho(\eta-\xi)=\rho(\eta'-\xi')+\rho(\eta''-\xi''-\overline{\varphi}(\eta')+\overline{\varphi}(\xi')+\overline{\varphi}(\xi')+\overline{\varphi}(\eta')-\overline{\varphi}(\xi')) \leq$$

$$\leq \rho(\eta'-\xi')+2^{\frac{1}{a''_{\min}}}(\rho(\eta''-\xi''-\overline{\varphi}(\eta')+\overline{\varphi}(\xi'))+\rho(\overline{\varphi}(\eta')-\overline{\varphi}(\xi'))) \leq$$

$$\leq \rho(\eta'-\xi')+2^{\frac{1}{a''_{\min}}}\rho(\eta''-\xi''-\overline{\varphi}(\eta')+\overline{\varphi}(\xi'))+2^{\frac{1}{a''_{\min}}}M\rho(\eta'-\xi') \leq$$

$$\leq \left(1+2^{\frac{1}{a''_{\min}}}M\right)[\rho(\eta'-\xi')+\rho(\eta''-\xi''-\overline{\varphi}(\eta')+\overline{\varphi}(\xi'))]=$$

$$=\left(1+2^{\frac{1}{a''_{\min}}}M\right)\rho(\eta'-\xi',\ \eta''-\xi''-\overline{\varphi}(\eta')+\overline{\varphi}(\xi')),$$

where $a''_{\min}=\min\limits_{k+1\leq i\leq n} a_i$.

Making the substitution $\eta'=y'$, $\xi'=x'$, and $\xi''=x''-\varphi(x')$, $\eta''=y''-\varphi(y')$ and applying the generalized Minkowski inequality we get

$$A(\tau)\leq C\left[\int_{\rho(\xi'')<\tau/c}\varphi(\rho(\xi''))d\eta''\int_{R^k}d\xi'\left(\int_{\rho(\eta'')>2\tau}d\eta''\times\right.\right.$$

$$\left.\left.\times\int_{R^k}\frac{|f(\eta',\eta''+\overline{\varphi}(\eta'))|\varphi^{\frac{\alpha}{|a|}}(\rho(\eta''))}{\rho(\xi'-\eta',\ \xi''-\overline{\varphi}(\xi')-\eta''+\overline{\varphi}(\eta'))^{\alpha-|a|}}\right)^q\right]^{1/q} \leq$$

$$\leq C\left[\int_{\rho(\xi'')<\tau}\varphi(\rho(\xi''))\left(\int_{\rho(\eta'')>2\tau}\varphi^{\frac{\alpha}{|a|}}(\rho(\eta''))B(\xi'',\eta'')d\eta''\right)^q d\xi''\right]^{1/q},$$

where

$$B(\xi'',\eta'')=\left[\int_{R^k}\left(\int_{R^k}|f(\eta',\eta''+\overline{\varphi}(\eta'))|\rho(\eta-\xi)^{\alpha-|a|}d\eta'\right)^q d\xi'\right]^{1/q}.$$

Applying the Young's inequality with $\frac{1}{q}=\frac{1}{p}+\frac{1}{r}-1$ we get

$$B(\xi'',\eta'')\leq\left(\int_{R^k}|f(\eta',\eta''+\overline{\varphi}(\eta'))|^p d\eta'\right)^{1/p}\left(\int_{R^k}\rho(\eta',\eta''-\xi'')^{(\alpha-|a|)r}d\eta'\right)^{1/r} =$$

$$= f_1(\eta'')\left(\int_{R^k}\rho(\eta',\eta''-\xi'')^{-|a|}d\eta'\right)^{1/r} = Cf_1(\eta'')\times$$

$$\times\left(\int_{R^k}(\rho(\eta')+\rho(\eta''-\xi''))^{-|a|}d\eta'\right)^{1/r} \le C\, f_1(\eta'')\rho(\eta''-\xi'')^{-\frac{|a''|}{r}}.$$

For $\rho(\xi'')<\tau$ and $\rho(\eta'')>2\tau$ it follows $\rho(\xi''-\eta'')>\frac{1}{2}\rho(\eta'')$. Therefore we have

$$A(\tau)\le$$

$$\le C\left[\int_{\rho(\xi'')<\tau/c}\varphi(\rho(\xi''))\left(\int_{\rho(\eta'')>2\tau}\varphi^{\frac{\alpha}{|a|}}(\rho(\eta''))\rho(\xi''-\eta'')^{-|a''|/r} f_1(\eta'')d\eta''\right)^q d\xi''\right]^{1/q} \le$$

$$\le C\left[\left(\int_{\rho(\xi'')<\tau/c}\varphi(\rho(\xi''))d\xi''\right)\left(\int_{\rho(\eta'')>2\tau}\varphi^{\frac{\alpha}{|a|}}(\rho(\eta''))\rho(\eta'')^{-|a''|/r} f_1(\eta'')d\eta''\right)^q\right]^{1/q} \le$$

$$\le C\left(\int_0^{\tau/c}\varphi(t)t^{|a''|-1}dt\right)\int_{2\tau}^{\infty}s^{-|a''|/r+|a''|-1}\varphi^{\frac{\alpha}{|a|}}(s)\left(\int_{S_{++}^{n-h}}f_1\left(t^{a''}\zeta''\right)d\sigma(\zeta'')\right)ds,$$

where $S_{++}^{n-k-1}=\{x'':x''\in R_{++}^{n-k};\rho(x'')=1\}$.

Consequently,

$$B_{22}\le C\left[\int_0^{\infty}\psi(\tau/2)\left(\int_0^{\tau/2c}\varphi(t)t^{|a''|-1}dt\right)\left(\int_{\tau}^{\infty}t^{-|a''|/r+|a''|-1}\varphi^{\frac{\alpha}{|a|}}(s)\times\right.\right.$$

$$\left.\left.\times\left(\int_{S_{++}^{n-k-1}}f_1\left(t^{a''}\zeta''\right)d\sigma(\zeta'')\right)dt\right)^q d\tau\right]^{1/q}.$$

Besides, we have

$$\int_0^t\psi\left(\frac{\tau}{2}\right)\left(\int_0^{\tau/2c}\varphi(s)s^{|a''|-1}ds\right)d\tau\le\int_0^{t/2}\psi(t)\left(\int_0^{\tau/c}\varphi(s)s^{|a''|-1}ds\right)d\tau=$$

$$=\int_0^{t/2}\varphi(s)s^{|a''|-1}\left(\int_{cs}^{t/2}\psi(\tau)d\tau\right)\le\int_0^{t/2}u_1(s)\varphi(s)s^{|a''|-1}ds=\int_0^{t/2}\omega_1(s)s^{|a''|-1}ds.$$

Therefore

$$\left(\int_0^t\psi\left(\frac{\tau}{2}\right)\left(\int_0^{\tau/2c}\varphi(s)s^{|a''|-1}ds\right)d\tau\right)^{p/q}\left(\int_{t/c}^{\infty}\left(\varphi(\tau)^{-\frac{\alpha p}{|a|}}\omega(\tau)\right)^{1-p'}\tau^{-1-|a''|p'/q}d\tau\right)^{p-1}\le$$

$$=\left(\int_0^{t/2}\omega_1(\tau)\tau^{|a''|-1}d\tau\right)^{p/q}\left(\int_{t/c}^{i}\left(\varphi(\tau)^{-\frac{\alpha p}{|a|}}\omega(\tau)\right)^{1-p'}\tau^{-1-|a''|p'/q}d\tau\right)^{p-1}<\infty.$$

Take into account the last inequality and theorem 1.7 from [3], we have

$$\left[\int_0^\infty \psi(\tau/2)\left(\int_0^{\tau/2c} \varphi(t)t^{|a''|-1}dt\right)\left(\int_\tau^\infty \delta^{-|a''|/r+|a''|-1}\varphi^{\frac{\alpha}{|a|}}(\delta)\times\right.\right.$$

$$\left.\left.\times\left(\int_{S_{++}^{n-k-1}} f_1\left(\delta^{a''}\zeta''\right)d\sigma(\zeta'')\right)d\delta\right)^q d\tau\right]^{1/q} \le$$

$$\le C\left(\int_0^\infty t^{-\frac{|a''|p}{r}+(|a''|-1)p}\left(\int_{S_{++}^{n-k-1}} f_1\left(t^{a''}\zeta''\right)d\sigma(\zeta'')\right)^p \omega(t)t^{-\frac{|a''|p}{r}+(|a''|-1)(p-1)}dt\right)^{1/p} =$$

$$= C\left(\int_0^\infty t^{|a''|-1}\left(\int_{S_{++}^{n-k-1}} f_1\left(\delta^{a''}\zeta''\right)d\sigma(\zeta'')\right)^p \omega(t)dt\right)^{1/p} \le C\left(\int_{R_{++}^{n-k}} f_1(\eta'')^p\,\omega(\rho(\eta''))d\eta''\right)^{1/p} =$$

$$= \left(\int_{R_{++}^{n-k}}\int_{R^k} \left|f\left(\eta',\eta''+\overline{\varphi}(\eta')\right)\right|_B^p \omega(\rho(\eta''))d\eta''d\eta'\right)^{1/p} = C\left(\int_{\Omega_k} |f(y)|^p\,\omega(\pi_k(y))dy\right)^{1/p} \le$$

$$\le \left(\int_{\Omega_k} |f(y)|^p\,\omega(\rho(y,\Gamma_k))dy\right)^{1/p}.$$

The theorem is proved.

From this theorem the following corollaries imply.

Corollary 1. *Let* $1<p<q<\infty$ *and* $\alpha=|a|\left(\frac{1}{p}-\frac{1}{q}\right)$. *Then the operator* $f\to K_\alpha f$ *acts boundedly from* $L_{p,\rho(x,\Gamma_k)^\beta}(\Omega_k)$ *to* $L_{q,\rho(x,\Gamma_k)^{q\beta/p}}(\Omega_k)$ *for any* $\beta>0$ *and* $k=0,1,2,...,n-1$.

In the case $1<p=q<\infty$, *the ASIO* $f\to K_0 f$ *acts boundedly from* $L_{p,\rho(x,\Gamma_k)^\beta}(\Omega_k)$ *to* $L_{p,\rho(x,\Gamma_k)^\beta}(\Omega_k)$.

Corollary 2. *Let* $1<p<q<\infty$ *and* $\alpha=|a|\left(\frac{1}{p}-\frac{1}{q}\right)$. *Then the operator* $f\to K_\alpha f$ *acts boundedly from* $L_{p,\exp\left(\rho(x,\Gamma_k)^\beta\right)}(\Omega_k)$ *to* $L_{q,\exp\left(\rho(x,\Gamma_k)^\beta\right)}(\Omega_k)$ *for any* $\beta>0$ *and* $k=0,1,2,...,n-1$.

In the case $1<p=q<\infty$, *the ASIO* $f\to K_0 f$ *acts boundedly from* $L_{p,\exp\left(\rho(x,\Gamma_k)^\beta\right)}(\Omega_k)$ *to* $L_{p,\exp\left(\rho(x,\Gamma_k)^\beta\right)}(\Omega_k)$.

Corollary 3. *Let* $1<p<q<\infty$ *and* $\alpha=|a|\left(\frac{1}{p}-\frac{1}{q}\right)$. *Then for any increasing radial function* $\omega(x)$ *the operator* $f\to K_\alpha f$ *acts boundedly from* $L_{p,\omega(\rho(x,\Gamma_k))}(\Omega_k)$ *to* $L_{q,\omega(\rho(x,\Gamma_k))^{q/p}}(\Omega_k)$.

In the case $1<p=q<\infty$ *the ASIO* $f\to K_0 f$ *acts boundedly from* $L_{p,\omega(\rho(x,\Gamma_k))}(\Omega_k)$ *to* $L_{p,\omega(\rho(x,\Gamma_k))}(\Omega_k)$.

The weight anisotropic space Sobolev $W^{l_1,\dots,l_n}_{p,\omega_0,\omega_1,\dots,\omega_n}(\Omega_k)$ is defined as the collection of all function $f(x)\in L_1^{loc}(\Omega_k),\ x\in R^n$, having the generalized derivatives $D_i^{l_i}f$ with the finite norm

$$\|f\|_{W^{l_1,\dots,l_n}_{p,\omega_0,\omega_1,\dots,\omega_n}(\Omega_k)}=\|f\|_{L_{p,\omega_0}(\Omega_k)}+\sum_{i=1}^{n}\left\|D_i^{l_i}f\right\|_{L_{p,\omega_i}(\Omega_k)},$$

where l_i non-negative integers and $1\le p<\infty$.

We give an integral representation of Il'in-Besov in terms of generalized derivatives of function in $R(l)$ (see [1]):

$$f(x)=f_{h^a}(x)+\sum_{i=1}^{n}\int_0^h \upsilon^{-|a|}d\upsilon\int_{R^n}D_i^{l_i}f(x+y)\Phi_i\left(yh^{-a}\right)dy,\ \ x\in\Omega_k,$$

where $a_i=1/l_i$, $i=1,\dots,n$ and $f_{h^a}(x)=h^{-|a|}\int_{R^n}\Phi_0\left(yh^{-a}\right)f(x+y)dy$ is the average of f and $\int_{R^n}\Phi_0(x)dx=1$. The smooth compactly supported kernels $\Phi_i\in C_0^\infty\left(R^n\right)$ are concentrated in an arbitrary previously specified cube in the first coordinate angle and are such that

$$\int_{R^n}\Phi_i(x)dx=0\ ,\ \ i=1,\dots,n.$$

By virtue of this integral representation we prove the following imbedding theorems.

Theorem 3. *Let* $a=1/l$, $1<p\le q<\infty$, $æ=(\nu+1/p-1/q,1/l)\le 1$ *and* $æ=(\nu,1/l)=1$, *where* $\nu=(\nu_1,\dots,\nu_n)$, *and* ν_i *are non-negative integer number. Suppose that the weight pairs* (ω,ω_j) $j=0,1,\dots,n$, *satisfy the conditions of theorem 2.*

Then the continuous imbedding

$$D^\nu W^{l_1,\dots,l_n}_{p,\omega_0(\rho(x,\Gamma_k)),\dots,\omega_n(\rho(x,\Gamma_k))}(\Omega_k)\subset L_{q,\omega(\rho(x,\Gamma_k))}(\Omega_k)$$

is valid.

Further, the inequality

$$\left\|D^\nu f\right\|_{L_{q,\omega}(\Omega_k)}\le C\|f\|_{W^{l_1,\dots,l_n}_{p,\omega_0,\dots,\omega_n}(\Omega_k)}$$

holds, with a constant C *is independent of* f.

Proof of theorem 3. Applying the differentiation operation D^ν to equality

$$f_{\varepsilon^\lambda}(x)=f_{h^\lambda}(x)+\sum_{i=1}^{n}\lambda_i\int_\varepsilon^h \vartheta^{|\lambda|}d\vartheta\int_{R^n}L_i\left(\vartheta^{-\lambda}y\right)D_i^{l_i}f(x+y)dy$$

and theorem 2, we get

$$\left\|\int_\varepsilon^h \vartheta^{|\lambda|-(\nu,\lambda)}d\vartheta\int_{\Omega_k}L_i^{(k)}\left(\vartheta^{-\lambda}y\right)D_i^{l_i}f(x+y)dy\right\|_{L_{p,\omega}(\Omega_k)}\le C\left\|D_i^{l_i}f\right\|_{L_{p,\omega_1}(\Omega_k)}.$$

Besides,

$$\left\|D^\nu f_{h^\lambda}\right\|_{L_{p,\omega}(\Omega_k)}\le C\|f\|_{L_{p,\omega_0}(\Omega_k)}.$$

Thus, combining the estimates we obtain

$$\left\| D^{\nu} f_{\varepsilon^{\lambda}} \right\|_{L_{p,\omega}(\Omega_k)} \le C \| f \|_{W^{l_1,\dots,l_n}_{p\omega_0,\omega_1}(\Omega_k)}.$$

To conclude the proof of the theorem two facts are established: first, it is proved that $D^{\nu} f_{\varepsilon^{\lambda}}$ converges to some element of $L_{p,\omega}(\Omega_k)$ for $\varepsilon \to 0$, second, it is proved that this limit element is the generalized derivative $D^{\nu} f$ of the function f to which the $f_{\varepsilon^{\lambda}}$ converge for $\varepsilon \to 0$.

For the proved of converges $D^{\nu} f_{\varepsilon^{\lambda}}$ to some element of $L_{p,\omega}(\Omega_k)$ for $\varepsilon \to 0$, it is proved that the sequence $\{D^{\nu} f_{\varepsilon^{\lambda}}\}$ is fundamental at norm $L_{p,\omega}(\Omega_k)$.

We have

$$\left\| D^{\nu} f_{\varepsilon^{\lambda}} - D^{\nu} f_{\eta^{\lambda}} \right\|_{L_{p,\omega}(\Omega_k)} \le C \sum_{i=1}^{n} \int_{\varepsilon}^{\eta} \upsilon^{-æ} d\upsilon \| M_i \|_{L_{1,\omega}(\Omega_k)} \left\| D_i^{l_i} f \right\|_{L_{p,\omega}(\Omega_k)} \le$$

$$\le C \eta^{1-æ} \left\| D_i^{l_i} f \right\|_{L_{p,\omega_i}(\Omega_k)},$$

where $0 < \varepsilon < \eta$.

Then by theorem Lebesgue we conclude that the sequence $\{D^{\nu} f_{\varepsilon^{\lambda}}\}$ is Cauchy sequence.

Hence in view of the fact that the space $L_{p,\omega}(\Omega_k)$ is complete, then $D^{\nu} f_{\varepsilon^{\lambda}}$ converges to some element g of $L_{p,\omega}(\Omega_k)$ for $\varepsilon \to 0$. By the definition of generalized derivative in the sense of Sobolev at each a fixed ε for arbitrary function $\psi \in C_0^{\infty}(\Omega_k)$ the equality

$$\int_{R^n} D^{\nu} \psi \, f_{\varepsilon^{\lambda}} dx = (-1)^{|\nu|} \int_{R^n} \psi \, D^{\nu} f_{\varepsilon^{\lambda}} dx$$

holds.

Taking into account, that $f \in L_1^{loc}(\Omega_k)$ and mean $f_{\varepsilon^{\lambda}} \to f$ in $L_1^{loc}(\Omega_k)$ and passing to the limit for $\varepsilon \to 0$ we give:

$$\int_{R^n} D^{\nu} \psi \, f(x) dx = (-1)^{|\nu|} \int_{R^n} \psi(x) g(x) dx$$

from that imply the limit element g of the sequence $\{D^{\nu} f_{\varepsilon^{\lambda}}\}$ is generalized derivative $D^{\nu} f$ function f.

The theorem is proved.

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