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ON BASISNESS OF EIGEN- FUNCTIONS OF DISCONTINUOUS SECOND ORDER DIFFERENTIAL OPERATOR

Abstract

In paper the spectral on finite segment problem for discontinuous second order differential operator with alternating high coefficient and with discontinuous boundary conditions is considered. A class of regular boundary conditions is selected and for regular boundary problems the asymptotic of eigenvalues and Green- function has been got, the basis properties of eigen- and associated functions in the space L_p, $1<p<\infty$ *have been studied.*

Let's consider the spectral problem for the second order differential operator

$$l(y)=p_0(x)y''+p_1(x)y'+p_2(x)y=\lambda y \tag{1}$$

with the boundary conditions

$$u_\nu(y)=\sum_{s=1}^{2}\sum_{j=0}^{k_\nu}\left(\alpha_{\nu sj}y^{(j)}(x_{s-1}+0)+\beta_{\nu sj}y^{(j)}(x_s-0)\right)=0\,,\quad \nu=\overline{1,4}\,, \tag{2}$$

$-\infty<a=x_0<x_1<x_2=b<+\infty$, $0\le k_\nu\le 1$, $p_1(x)$, $p_2(x)\in L_1(a,b)$, and the function $p_0(x)$ on each interval (x_{s-1},x_s) has the form: $p_0(x)=p_{0s}(x)e^{i\varphi_s}$, $0\le\varphi_s<2\pi$, $p_{0s}(x)$ is a positive absolutely continuous on $[x_{s-1},x_s]$ function. The expression $l(y)$ and boundary conditions (2) define in space $L_p(a,b), 1<p<\infty$, linear operator L with domain $D(L)=\left\{y: y\in W_p^2(x_0,x_1)\oplus W_p^2(x_1,x_2), u_\nu(y)=0,\ \nu=\overline{1,4}\right\}$, acting by the rule $Ly=l(y)$.

In the present paper the asymptotic formulas for eigenvalues and Green's function of operator $L-\lambda I$ are obtained, basic properties of eigen- and associated functions of operator L in the space $L_p(a,b)$, $1<p<\infty$ are studied. Such questions for discontinuous differential operators were investigated in details in [1-5]. In [1-3] $p_0(x)\equiv 1$, $p_1(x)\in W_1^1(a,b)$ and in [5] $p_k(x)\in W_1^{2-k}(x_0,x_1)\oplus W_1^{2-k}(x_1,x_2)$, $k=0,1$, $p_0(x)>0$. The first order differential operator with piecewise-constant higher coefficients is considered in [4]. As distinct from the mentioned works in the present paper conditions on functions $p_0(x)$ and $p_1(x)$ were relaxed, particularly, $p_0(x)$ may be alternating function.

Let's assume that $\lambda=-\rho^2$ and denote by ω_{s1},ω_{s2} the different square roots of the complex number $-e^{i\varphi s}$, $s=1,2$. Let consider the following angles in complex ρ-plane

$$S_{s\gamma}=\left\{\rho:\frac{\pi\gamma}{2}\le\arg\rho e^{-i\frac{\varphi_s}{2}}\le\frac{\pi(\gamma+1)}{2}\right\},\quad \gamma=\overline{0,3}\,,\quad s=1,2\,.$$

As it follows from [6] it is valid the following.

Assertion 1. *On each segment* $[x_{s-1},x_s]$, $s=1,2$ *equation (1) has two linear independent solutions* $y_{s1}(x), y_{s2}(x)$ *which are regular on* $\rho\in S_{s\gamma}$ *and at sufficiently large* $|\rho|$ *have the asymptotics*

$$y_{sk}^{j}(x)=(\rho\omega_{sk})^{j}e^{\rho\omega_{sk}\alpha_s(x)}V_{sj}(x)[1+o(1)],$$

$$j=0,1\,;\; k,s=1,2\,;\; \alpha_s(x)=\int_{x_{s-1}}^{x}\frac{d\xi}{\sqrt{\rho_{0s}(\xi)}}\,,\; V_{sj}(x)=(p_{os}(x))^{-\frac{j}{2}}V_s(x),$$

$$V_s=\exp\left(-\frac{1}{2}\int_0^{\alpha_s(x)}\left(\frac{p_1(t)}{\sqrt{p_{0s}(t)}}-\frac{p_{0s}'(t)}{2p_{0s}(t)}\right)dt\right). \tag{3}$$

1. Regular boundary conditions. Suppose $\alpha_{\nu s}=\alpha_{\nu sk_\nu}V_{sk_\nu}(x_{s-1})$, $\beta_{\nu s}=$ $=\beta_{\nu sk_\nu}V_{sk_\nu}(x_s)$. Let $m=(m_1,m_2)$ be a multi-index, where m_s adopts the values $-1;-0;$ $+0;+1$. We'll determine numbers $\theta_m=\det\left(d_{\nu k}^{s}\right)$, $\nu=\overline{1,4};\,k,s=1,2$, where elements $d_{\nu k}^{s}$ are defined depending upon values of indices m_s by the following way:

$$\left.\begin{array}{l} d_{\nu 1}^{s}=\alpha_{\nu s}\omega_{s1}^{k_\nu}\,,\, d_{\nu 2}^{s}=\beta_{\nu s}\omega_{s2}^{k_\nu}\,,\ \text{if}\quad m_s=-1\,;\\ d_{\nu 1}^{s}=\beta_{\nu s}\omega_{s1}^{k_\nu}\,,\, d_{\nu 2}^{s}=\beta_{\nu s}\omega_{s2}^{k_\nu}\,,\ \text{if}\quad m_s=-0\,;\\ d_{\nu 1}^{s}=\alpha_{\nu s}\omega_{s1}^{k_\nu}\,,\, d_{\nu 2}^{s}=\alpha_{\nu s}\omega_{s2}^{k_\nu}\,,\ \text{if}\quad m_s=+0\,;\\ d_{\nu 1}^{s}=\beta_{\nu s}\omega_{s1}^{k_\nu}\,,\, d_{\nu 2}^{s}=\alpha_{\nu s}\omega_{s2}^{k_\nu}\,,\ \text{if}\quad m_s=+1\,. \end{array}\right\} \tag{4}$$

From the definition of numbers ω_{sk} it follows that $\omega_{s1}=-\omega_{s2}$, $\omega_{2k}=e^{i\frac{\varphi_2-\varphi_1}{2}}\omega_{1k}$. From here immediately follows the correctness of the following statement.

Assertion 2. *For multi-indices m, with $|m|=2$ all the numbers θ_m on absolute value are equal: $|\theta_m|=\theta$.*

Definition 1. *The boundary conditions (2) are called regular if number θ is non-zero.*

Denote by $\Delta(\rho)$ the characteristic determinant of problem (1), (2), i.e.

$$\Delta(\rho)=\det\left\|u_{\nu s}(y_{sk})\right\|_{\nu=\overline{1,4}}^{s,k=12}, \tag{5}$$

where $y_{s1}(x),\,y_{s2}(x)$ is a fundamental system of solution of equation (1) from assertion 1, and $u_{\nu s}$ are linear forms of $y^{(j)}(x_{s-1}+0)$, $y^{(j)}(x_s-0)$ defined by the equality

$$u_{\nu s}(y_{sk})=\sum_{j=0}^{k_\nu}\left(\alpha_{\nu sj}y_{sk}^{(j)}(x_{s-1}+0)+\beta_{\nu sj}y_{sk}^{(j)}(x_s-0)\right). \tag{6}$$

Definition 2. *The boundary conditions (2) are called strongly regular, if they are regular and zeros of the characteristic determinant $\Delta(\rho)$ are asymptotically simple and separated.*

The main properties of function $\Delta(\rho)$ are led in the following lemma.

Lemma 1. *Let $\Delta(\rho)$ be the characteristic determinant of regular problem (1), (2). Then:*

1^0. For any number $\varepsilon>0$ there exist such constant $m_\varepsilon>0$, depending on function $\Delta(\rho)$, that on the set obtained from complex ρ-plane by throwing out ε-neighbourhoods of zeros of $\Delta(\rho)$ the inequality

$$|\Delta(\rho)|\geq m_\varepsilon\left|\rho^{æ}\right|,\quad æ=k_1+\cdots+k_4 \tag{7}$$

holds.

2^0. Number of zeros of function $\Delta(\rho)$ in circular ring $\{f : r \le |\rho| < r+1\}$ is bounded by a constant independent on r; these zeros can be divided into 4 series with the asymptotics

$$\rho_{l,s,k} = \frac{1}{\omega_{sk}\alpha_s(x_s)}(i\pi l + O(1)),\ l = 1,2,... \tag{8}$$

if $\varphi_1 \ne \varphi_2$ and into 2 series with the asymptotics

$$\rho_{l,k} = \frac{1}{\omega_{1k}(\alpha_1(x_1) + \alpha_2(x_2))}(i\pi l + O(1)),\ l = 1,2,... \tag{9}$$

if $\varphi_1 = \varphi_2$.

Proof. Substituting asymptotic formulas (3) for $y_{sk}(x)$ into (6) and taking into account that $\alpha_s(x_s) = \int\limits_{x_{s-1}}^{x_s} \frac{dt}{\sqrt{p_{0s}(t)}} > 0$, we obtain

$$u_{\nu s}(y_{sk}) = (\rho\omega_{sk})^{k_\nu}\left([\alpha_{\nu s}] + [\beta_{\nu s}]e^{\rho\omega_{sk}\alpha_s(x_s)}\right), \tag{10}$$

Here and later on we use denotation $[A] = A + o(1)$, $|\rho| \to \infty$. Substituting (10) into (5) and taking the common multiplier in rows out the determinant sign, we obtain

$$\Delta(\rho) = \rho^{æ}\sum_{|m|=0}^{2}[\theta_m]e^{\rho\Omega_m} \overset{df}{=} \rho^{æ}\Delta_0(\rho), \tag{11}$$

where $æ = k_1 + k_2 + k_3 + k_4$, $m = (m_1, m_2)$, $|m| = |m_1| + |m_2|$, m_1, m_2 possess the values $\pm 1, \pm 0$ $(|\pm 1| = 1, |\pm 0| = 0)$; numbers θ_m are defined by (4); $W_m = \Omega_{m_1} + \Omega_{m_2}$, moreover $\Omega_{m_s} = 0$, if $m_s = \pm 0$, $\Omega_{m_s} = \omega_{s1}\alpha_s(x_s)$, if $m_s = 1$, $\Omega_{m_s} = \omega_{s2}\alpha_s(x_s)$ if $m_s = -1$. Let numbers ω_{s1}, ω_{s2} be enumerated such that (see [7]) for all $\rho \in S_{s\gamma}$ the inequality $\operatorname{Re}\rho\omega_{s1} \le \operatorname{Re}\rho\omega_{s2}$ holds. Consider half-string $\Pi_{sk}(h) = \left\{\rho : \rho = \zeta\exp\left(\frac{i}{2}(\varphi_s(k-1)\pi)\right),\ \operatorname{Re}\zeta \ge 0, |\operatorname{Im}\zeta| \le h\right\}$, $k,s = 1,2$. Note, that at ρ $\delta < |\rho| \le R_0$, where R_0 is sufficiently large number, inequality (7) is obvious. Therefor, at sufficiently large $|\rho|$, having asymptotic representation (11) and taking into account that θ_m for $|m| = 2$ are different from zero, number h may be chosen so large that on boundaries of strings $\Pi_{sk}(h)$ the following inequality will be satisfied

$$0 < c_1 < |\Delta_0(\rho)|\,e^{-H(\rho)} < c_2 < \infty,$$

where $H(\rho)$ is a Minkowsky function [8] of rectangular D with vertices at points ω_{sk}, $s,k = 1,2$. It means, that function $\Delta_0(\rho)$ belongs to the class S_D of integer functions of exponential type and assertion and the fist part of assertion 2 follow from assertion 1^0 theorem 2,2 of paper [8].

Let $\varphi_1 \ne \varphi_2$ and $\rho \in \Pi_{1k}(h)$. In this case $\operatorname{Re}\rho\omega_{21} < 0, \operatorname{Re}\rho\omega_{22} > 0$ and from (11) it follows that

$$\Delta_0(\rho) = e^{\rho\omega_{22}\alpha_2(x_2)}\left(\left[\theta_{(+1,-1)}\right]e^{\rho\omega_{11}\alpha_1(x_1)} + \left[\theta_{(-0,-1)} + \theta_{(+0,-1)}\right] + \left[\theta_{(-1,-1)}\right]e^{\rho\omega_{12}\alpha_1(x_1)}\right).$$

At $\rho \in \Pi_{2k}(h)$ we have $\operatorname{Re}\rho\omega_{11} < 0$, $\operatorname{Re}\rho\omega_{12} > 0$ and from (11) we obtain

$$\Delta_0(\rho)=e^{\rho\omega_{12}\alpha_1(x_1)}\left(\left[\theta_{(-1,+1)}\right]e^{\rho\omega_{21}\alpha_2(x_2)}+\left[\theta_{(-1,-0)}+\theta_{(-1,+0)}\right]+\left[\theta_{(-1,-1)}\right]e^{\rho\omega_{22}\alpha_2(x_2)}\right).$$

Now by repeating analogous reasoning from [7, p.80-83] we obtain formulas (8).

In the case $\varphi_1=\varphi_2$ numbers ω_{1k} and ω_{2k} coincide. In accordance with this $\Pi_{1k}(h)$ and $\Pi_{2k}(h)$ also coincide. Therefore, from (11) taking into account, that $\omega_{11}=-\omega_{21}$, we obtain

$$\Delta_0(\rho)=e^{-\omega_{11}(\alpha_1(x_1)+\alpha_2(x_2))}\left(\left[\theta_{(+1,+1)}\right]e^{2\rho\omega_{11}(\alpha_1(x_1)+\alpha_2(x_2))}+\left[\theta_{(+1,-1)}\right]e^{2\rho\omega_{11}\alpha_1(x_1)}+\left[\theta_{(+1,+0)}+\theta_{(+1,-0)}\right]\times\right.$$
$$\times e^{\rho\omega_{11}(2\alpha_1(x_1)+\alpha_2(x_2))}+\left[\theta_{(+0,+0)}+\theta_{(-0,-0)}\right]e^{\rho\omega_{11}(\alpha_1(x_1)+\alpha_2(x_2))}+\left[\theta_{(-0,+1)}+\theta_{(+0,+1)}\right]e^{\rho\omega_{11}(\alpha_1(x_1)+2\alpha_2(x_2))}+$$
$$\left.+\left[\theta_{(-0,-1)}+\theta_{(+0,-1)}\right]e^{\rho\omega_{11}\alpha_1(x_1)}+\left[\theta_{(-1,-0)}+\theta_{(-1,+0)}\right]e^{\rho\omega_{11}\alpha_2(x_2)}+\left(\theta_{(-1,+1)}\right)e^{2\rho\omega_{11}\alpha_2(x_2)}+\left[\theta_{(-1,-1)}\right]\right).$$

In this case the validity of formula (9) follows from lemma 1 of paper [9, p.113].

An important part later on plays the following lemma.

Lemma 2. *For Green's function of the operator $L+\rho^2 I$ generated by regular boundary value problem (1), (2) the following asymptotic representation is valid*

$$2\rho G(x,\xi,-\rho^2)=\mathsf{P}_{0s\tau}(x,\xi,\rho)+\sum_{k,j=1}^{2}\mathsf{A}_{s\tau kj}(x,\xi,\rho)u_{sk}(x,\rho)\vartheta_{\tau j}(\xi,\rho),\qquad(12)$$

where $x\in[x_{s-1},x_s]$, $\xi\in[x_{\tau-1},x_\tau]$, $\rho\in S_{s\gamma}\cap S_{\tau\gamma'}$, $s,\tau=1,2$; $\gamma,\gamma'=\overline{1,4}$

$$\left.\begin{aligned}&u_{s1}(x,\rho)=\exp(\rho\omega_{s1}\alpha_s(x)),\ u_{s2}(x,\rho)=\exp(\rho\omega_{s2}(\alpha_s(x)-a_s(x_s))),\\&\vartheta_{\tau1}(\xi,\rho)=\exp(\rho\omega_{\tau1}(\alpha_\tau(x_\tau)-\alpha_\tau(\xi))),\ \vartheta_{\tau2}(\xi,\rho)=\exp(-\rho\omega_{\tau2}\alpha_\tau(\xi));\end{aligned}\right\}\qquad(13)$$

$\mathsf{P}_{0s\tau}(x,\xi,\rho)\equiv0$ *at* $s\neq\tau$ *and* $\mathsf{P}_{0ss}(x,\xi,\rho)$ *is a regular in the sector* $\mathsf{S}_{s\gamma}$ *function*

$$\mathsf{A}_{s\tau kj}(x,\xi,\rho)=\frac{V_{0s}(x)}{V_{0\tau}(\xi)}a_{s\tau kj}(\rho)[1+\varphi_{sk}(x,\rho)][1+\psi_{\tau j}(\xi,\rho)],\qquad(14)$$

where $a_{s\tau kj}(\rho)$ is a meromorphic function with the poles in zeros of $\Delta_0(\rho)$, at that if from the sector $S_{s\gamma}\cap S_{\tau\gamma'}$ we throw out circles of the same radius $\varepsilon>0$ and with the centers at zeros of $\Delta_0(\rho)$, then in remained part the following estimation

$$\left|a_{s\tau kj}(\rho)\right|\le C(\varepsilon),\qquad(15)$$

and functions $\varphi_{sk}(x,\rho)$ and $\psi_{\tau j}(\xi,\rho)$ tend to zero as $|\rho|\to\infty$ uniformly on $x\in[x_{s-1},x_s]$ and $\xi\in[x_{\tau-1},x_\tau]$.

Proof. Let ρ belong to the fixed sector $S_{s\gamma}\cap S_{\tau\gamma'}$ (assume, that intersection is non-empty) and numbers ω_{sk}, $s,k=1,2$ are enumerated such that $\operatorname{Re}\rho\omega_{s1}\le\operatorname{Re}\rho\omega_{s2}$. As was shown in [5] at $x\in[x_{s-1},x_s]$, $\xi\in[x_{\tau-1},x_\tau]$ for Green's function the following formula is valid

$$G(x,\xi,-\rho^2)=\frac{\mathsf{H}_{s\tau}(x,\xi,\rho)}{\Delta(\rho)}=\frac{1}{\Delta(\rho)}\det\left\|H^0_{s\tau}\,H^1_s\,H^2_s\right\|,$$

where $\Delta(\rho)$ is a characteristic determinant, defined by (5); $H^0_{s\tau}$ and H^τ_s are block-matrices of dimensions 1×5 and 2×5 respectively, defined by the equalities

$$H^0_{s\tau}=\begin{pmatrix} g_{s\tau} \\ u_{1\tau}(g_\tau) \\ u_{2\tau}(g_\tau) \\ u_{3\tau}(g_\tau) \\ u_{4\tau}(g_\tau) \end{pmatrix}, \quad H^\tau_s=\begin{pmatrix} y^\tau_{s1} & y^\tau_{s2} \\ u^\tau_{11} & u^\tau_{12} \\ u^\tau_{21} & u^\tau_{22} \\ u^\tau_{31} & u^\tau_{32} \\ u^\tau_{41} & u^\tau_{42} \end{pmatrix}, \quad s,\tau=1,2,$$

where $g_{s\tau}=\delta_{s\tau}\,g_\tau$, $y^\tau_{sk}=\delta_{s\tau}y_{sk}$, $\delta_{s\tau}$ is Cronecker's symbol,

$$g_\tau(x,\xi,\rho)=\begin{cases} \frac{1}{2}y_{\tau 1}(x)z_{\tau 1}(\xi), & \text{if} \quad x_\tau \ge x > \xi \ge x_{\tau-1}, \\ -\frac{1}{2}y_{\tau 1}(x)z_{\tau 1}(\xi), & \text{if} \quad x_{\tau-1} \le x \le \xi \le x_\tau; \end{cases}$$

$$z_{\tau 1}(\xi)=\frac{y_{\tau 2}(\xi)}{P_{0\tau}(\xi)W_\tau(\xi)}, \qquad z_{\tau 2}(\xi)=\frac{y_{\tau 1}(\xi)}{P_{0\tau}(\xi)W_\tau(\xi)}; \tag{16}$$

$$W_\tau(\xi)=y_{\tau 1}(\xi)y'_{\tau 2}(\xi)-y_{\tau 2}(\xi)y'_{\tau 1}(\xi). \tag{17}$$

By elementary transformations ([7], p.93-96) of elements of the first column of determinant $\mathsf{H}_{s\tau}(x,\xi,\rho)$ we reduce to the following form

$$P_{s\tau}=\frac{1}{2\rho}\begin{pmatrix} \mathsf{P}_{0s\tau} \\ \rho^{k_1}\mathsf{P}_{1\tau} \\ \rho^{k_2}\mathsf{P}_{2\tau} \\ \rho^{k_3}\mathsf{P}_{3\tau} \\ \rho^{k_4}\mathsf{P}_{4\tau} \end{pmatrix},$$

where $\mathsf{P}_{0s\tau}=\delta_{s\tau}\mathsf{P}_{0\tau\tau}$,

$$\frac{1}{2\rho}\mathsf{P}_{0\tau\tau}=\begin{cases} y_{t1}(x)z_{\tau 1}(\xi), & \text{if} \quad x_\tau \ge x > \xi \ge x_{\tau-1}, \\ -y_{\tau 2}(x)z_{\tau 2}(\xi), & \text{if} \quad x_{\tau-1} \le x \le \xi \le x_\tau; \end{cases} \tag{18}$$

$$\rho^{k_\nu}\mathsf{P}_{\nu\tau}=\frac{1}{2\rho}\left(e^{-\rho\omega_{t2}\alpha_\tau(\xi)}\left[\alpha_{\nu\tau}\omega_{\tau 2}\right]-e^{\rho\omega_{\tau 1}(\alpha_\tau(x_\tau)-\alpha_\tau(\xi))}\left[\beta_{\nu\tau}\omega_{\tau 1}\right]\right). \tag{19}$$

On the other hand, from (16), (17) and (3) it follows that

$$\tau_{\tau k}(\xi)=\frac{1}{\rho V_{0\tau}(\xi)}e^{-\rho\omega_{\tau k}\alpha_\tau(\xi)}\left[-\omega_{\tau k}\right]. \tag{20}$$

Taking into account (18)-(20) and estimation (7), open the determinant $\mathsf{H}_{s\tau}(x,\xi,\rho)$ and divide the obtained addends by $\Delta(\rho)$ we obtain the statements of the lemma.

2. Basisness in space $L_p(a,b)$, $1<p<\infty$. We remind, that the system $\{e_k\}_1^\infty$ of elements of Banach space X is called a basis with brackets in this space if there exists the sequence of indices $0=m_0<m_1<m_2<\ldots$ such that each element $x\in X$ can be expanded in series

$$x=\sum_{k=0}^{\infty}\sum_{j=m_k+1}^{m_{k+1}}c_j e_j$$

convergent by norm of X. If $m_k = k$ then the system $\{e_k\}_1^\infty$ is called simply a basis in X.

Let's formulate the main result of the work.

Theorem. *Eigen- and associated functions of regular problem (1), (2) form a basis with brackets in space* $L_p(a,b)$, $1 < p < \infty$ *and an ordinary basis in this space if the boundary conditions are strongly regular.*

Proof. At first consider the case $\varphi_1 \neq \varphi_2$. Without loss of generality it can be assumed that $\varphi_1 = 0$ (in opposite case by introducing new spectral parameter $\widetilde{\rho} = \rho e^{-i\frac{\varphi_1}{2}}$ this can be archieved). In this case half-strings $\Pi_{1,1}(h)$ and $\Pi_{1,2}(h)$ are arranged along the positive and negative real axes respectively. If $0 < \varphi_2 \le \pi$, then strings $\Pi_{1,1}(h)$ and $\Pi_{2,1}(h)$ completely lie in half-plane $\widetilde{S}$ obtained from the half-plane $S = S_{1,0} \cup S_{1,3}$ by shifting for some number z $\left(\widetilde{S} = S + z\right)$. If $\pi < \varphi_2 < 2\pi$, then $\Pi_{2,2}(h) \subset S_{1,3}$ and in this case strings $\Pi_{1,1}(h)$ and $\Pi_{2,2}(h)$ are completely in half-plane $\widetilde{S}$. For definiteness consider case $\varphi_2 \le \pi$. The case $\pi < \varphi_2 < 2\pi$ is considered analogously. Let $\rho_{1,1,k} = \rho_k^{(1)}$ and $\rho_{2,1,k} = \rho_k^{(2)}$ be zeros of the function $\Delta_0(\rho)$ lying in poles $\Pi_{1,1}(h)$ and $\Pi_{2,1}(h)$ respectively. Write around each point $\rho_k^{(s)}$ circles $O_{sk}(\varepsilon) = \left\{\rho : \left|\rho - \rho_k^{(s)}\right| < e\right\}$, $s = 1,2$ and form the domain $G_s(\varepsilon) = \bigcup\limits_{k=1}^{\infty} O_{sk}(\varepsilon)$. From the properties of function $D(\rho)$ in lemma 1 it follows that at small $\varepsilon > 0$ $G_s(\varepsilon) = \bigcup\limits_{N=1}^{\infty} G_{sN}(\varepsilon)$, where $G_{sN}(\varepsilon)$ are simply connected components containing no more than N_0 points $\rho_k^{(s)}$, at that number N_0 is independent of N. From asymptotic formulas (8) it follows that we can show a system of contours Γ_N, having the following properties:

1) $\Pi_{1,1}(h)$ is a part of circle of the radius R_N located in the sector $S_{1,0} \cup S_{1,3}$;
2) radii R_N tend to infinity and $R_N - R_{N-1} = O(1)$ as $N \to \infty$;
3) between the neighbouring contours Γ_N and Γ_{N+1} there is only one domain $G_{1N}(\varepsilon)$ and $G_{2N}(\varepsilon)$;
4) at mapping $\lambda = -\rho^2$ images of each of domains $G_{1N}(\varepsilon)$ and $G_{2N}(\varepsilon)$ are found for sufficiently large N at distance $\ge \delta$ from each other and from circles- images of contours Γ_N.

Let $R(\lambda) = (L - \lambda I)^{-1}$ be a resolvent of the operator L. Denote

$$E_{2k-1} = -\frac{1}{2\pi i} \int\limits_{\partial \widetilde{G}_{1k}} R(\lambda) d\lambda, \qquad E_{2k} = -\frac{1}{2\pi i} \int\limits_{\partial \widetilde{G}_{2k}} R(\lambda) d\lambda$$

and for $f(x) \in L_p(a,b)$, $\sigma_N(f) = \sum\limits_{k=1}^{N} E_k f$, where $\widetilde{G}_{sk}$ is an image of domain $G_{sk}(\varepsilon)$ at mapping $\lambda = -\rho^2$. With the help of contours Γ_N we construct a new system of contours by the following way: we drow from the origin two zero-free rays of $\Delta_0(\rho)$ and forming angles ψ_1 and ψ_2 with positive real axis respectively, such that $0<\psi_1<\frac{\varphi_2}{2}<\psi_2<\frac{\pi}{2}$. The cross points of rays with contours Γ_N denote by a_N and b_N, and parts Γ_N, on which it is divided by points a_N, b_N by $\gamma_N^{(2)}$, $\gamma_N^{(3)}$ (at counter clockwise). Suppose $\Gamma'_{2N-1}=\gamma_N^{(1)}\cup[a_N,a_{N-1}]\cup\gamma_{N-1}^{(2)}\cup[b_{N-1},b_N]\cup\gamma_N^{(3)}$, $\Gamma'_{2N}=\Gamma_N$. Then

$$\sigma_N(f)=\sum_{k=1}^{N}E_k f=-\frac{1}{2\pi i}\int_{\widetilde{\Gamma}'_N}R(\lambda)fd\lambda=-\frac{1}{\pi i}\int_{\Gamma'_N}\rho\int_a^b G(x,\xi,-\rho^2)f(\xi)d\xi\,d\rho,$$

where $\widetilde{\Gamma}'_N$ is an image of Γ'_N at mapping $\lambda=-\rho^2$.

Let $x\in[x_{s-1},x_s]$. Then using representation (12) and taking into account that $\mathrm{P}_{0s\tau}(x,\xi,\rho)$ is a regular function on λ-plane, we obtain

$$\sigma_N(f)(x)=\frac{i}{\pi}\sum_{k,j=1}^{2}\int_{\Gamma'_N}\sum_{\tau=1}^{2}\int_{x_{\tau-1}}^{x_\tau}\mathrm{A}_{s\tau kj}(x,\xi,\rho)u_{sk}(x,\rho)\vartheta_{\tau j}(\xi,\rho)f(\xi)d\xi\,d\rho. \tag{21}$$

We'll show that partial sums $\sigma_N(f)$ are uniformly bounded by norm of $L_p(a,b)$, i.e. there exists a positive constant $C>0$ independent of N and f such that

$$\|\sigma_N(f)\|_p\le C\|f\|_p,\quad N=1,2,\dots. \tag{22}$$

We introduce the following denotations

$$\delta_{s\tau kj}(x,\xi)=\begin{cases}\omega_{s1}\alpha_s(x)+\omega_{\tau1}(\alpha_\tau(x_\tau)-\alpha_\tau(\xi)), & \text{if } k=j=1;\\ \omega_{s1}\alpha_s(x)-\omega_{\tau2}\alpha_\tau(\xi), & \text{if } k=1,\ j=2;\\ \omega_{s2}(\alpha_s(x)-\alpha_s(x_s))+\omega_{\tau1}(\alpha_\tau(x_\tau)-\alpha_\tau(\xi)), & \text{if } k=2,\ j=1;\\ \omega_{s2}(\alpha_s(x)-\alpha_s(x_s))-\omega_{\tau2}\alpha_\tau(\xi), & \text{if } k=j=2\end{cases} \tag{23}$$

and also

$$\alpha_{s\tau kj}(x,\xi)=\begin{cases}\alpha_s(x)+\alpha_\tau(x_\tau)-\alpha_\tau(\xi), & \text{if } k=j=1;\\ \alpha_s(x)+\alpha_\tau(\xi), & \text{if } k=1,\ j=2;\\ \alpha_s(x_s)-\alpha_s(x)+\alpha_\tau(x_\tau)-\alpha_\tau(\xi), & \text{if } k=2,\ j=1;\\ \alpha_s(x_s)-\alpha_s(x)+\alpha_\tau(\xi), & \text{if } k=j=2.\end{cases} \tag{24}$$

Taking into consideration denotations (23), (24) in (21) and changing the order of integration, we obtain

$$\sigma_N(f)(x)=\frac{i}{2\pi}\sum_{k,j,\tau=1}^{2}\int_{x_{\tau-1}}^{x_\tau}f(\xi)\int_{\Gamma'_N}\mathrm{A}_{s\tau kj}(x,\xi,\rho)e^{\rho\delta_{s\tau kj}(x,\xi)}d\rho\,d\xi. \tag{25}$$

According to lemma 2 functions $\mathrm{A}_{s\tau kj}(x,\xi,\rho)$ are uniformly bounded by $x\in[x_{s-1},x_s]$, $\xi\in[x_{\tau-1},x_\tau]$ and $\rho\in\Gamma'_N$. Suppose

$$c_1=\max_{x\in[x_{s-1},x_s],\ \xi\in[x_{\tau-1},x_\tau],\ \rho\in\Gamma'_N}\left|\mathrm{A}_{s\tau kj}(x,\xi,\rho)\right|.$$

Then, estimating module (25), we obtain

$$|\sigma_N(f)(x)|=c_2\sum_{k,j,\tau=1}^{2}\int_{x_{\tau-1}}^{x_\tau}|f(\xi)|\int_{\Gamma'_N}e^{\mathrm{Re}(\rho\delta_{s\tau kj}(x,\xi))}|d\rho|\,d\xi.$$

We separately estimate $\left|\sigma_{2N}(f)(x)\right|$ and $\left|\sigma_{2N-1}(f)(x)\right|$. Since $\Gamma'_{2N}=\Gamma_N$ then assuming $\rho=R_N e^{i\varphi}$ in this case we obtain

$$\left|\sigma_{2N}(f)(x)\right|=c_2 R_N \sum_{k,j,\tau=1}^{2}\int_{x_{\tau-1}}^{x_\tau}\left|f(\xi)\right|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{R_N \operatorname{Re}\left(e^{i\varphi}\delta_{s\tau kj}(x,\xi)\right)}d\varphi\, d\xi\,. \tag{26}$$

We revise the values of ω_{sk} in all interval of integration in (26):

if $\varphi\in\left[-\frac{\pi}{2},-\frac{\pi}{2}+\frac{\varphi_2}{2}\right]$, then $\rho\in S_{1,3}\cap S_{2,2}$ and $\omega_{11}=-\omega_{12}=-i$, $\omega_{21}=-\omega_{22}=-i\,e^{-i\frac{\varphi_2}{2}}$;

if $\varphi\in\left[-\frac{\pi}{2}+\frac{\varphi_2}{2};0\right]$, then $\rho\in S_{1,3}\cap S_{2,3}$ and $\omega_{11}=-\omega_{12}=-i$, $\omega_{21}=-\omega_{22}=-i\,e^{-i\frac{\varphi_2}{2}}$;

if $\varphi\in\left[0,\frac{\varphi_2}{2}\right]$, then $\rho\in S_{1,0}\cap S_{2,3}$ and $\omega_{11}=-\omega_{12}=i$, $\omega_{21}=-\omega_{22}=-i\,e^{-i\frac{\varphi_2}{2}}$;

if $\varphi\in\left[\frac{\varphi_2}{2},\frac{\pi}{2}\right]$, then $\rho\in S_{1,0}\cap S_{2,0}$ and $\omega_{11}=-\omega_{12}=i$, $\omega_{21}=-\omega_{22}=i\,e^{-i\frac{\varphi_2}{2}}$.

Allowing for the last relation (23) and estimating integrals in (26) in each integration interval separately we arrive at the following estimation:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{R_N \operatorname{Re}\left(e^{i\varphi}\delta_{s\tau kj}(x,\xi)\right)}d\varphi\le\frac{c_3}{R_N\alpha_{s\tau kj}(x,\xi)}\,. \tag{27}$$

From (26) and (27) it follows that

$$\int_{x_{s-1}}^{x_s}\left|\sigma_{2N}(f)(x)\right|^p dx\le c_4\left(\sum_{\tau=1}^{2}\int_{x_{\tau-1}}^{x_\tau}\frac{\left|f(\xi)\right|}{\alpha_{s\tau kj}(x,\xi)}d\xi\right)^p dx\,. \tag{28}$$

Making change of variable $\eta=\alpha_s(x)$, if $k=1$, $\eta=\alpha_s(x_s)-\alpha_s(x)$ if $k=2$ and also $\zeta=\alpha_\tau(\xi)$, if $j=2$, $\zeta=\alpha_\tau(x_\tau)-\alpha_\tau(\xi)$, if $j=1$ and by the next application of Riesz theorem [10] on boundedness of Hilbert transformation, we obtain

$$\int_{x_{s-1}}^{x_s}\left(\int_{x_{\tau-1}}^{x_\tau}\frac{\left|f(\xi)\right|}{\alpha_{s\tau kj}(x,\xi)}d\xi\right)^p dx\le c_5\int_0^{\alpha_s(x_s)}\left(\int_0^{\alpha_\tau(x_\tau)}\frac{\left|\tilde f(\zeta)\right|}{\eta+\zeta}d\zeta\right)^p d\eta\le c_6\int_0^{\alpha_s(x_s)}\left|\tilde f(\eta)\right|^p d\eta\le$$

$$\le c_7\int_{x_{s-1}}^{x_s}\left|f(x)\right|^p dx\,. \tag{29}$$

From (28) and (29) the estimation (22) follows for even partial sums. In order to estimate the norm $\sigma_{2N-1}(f)(x)$ we remember, that contours Γ'_{2N-1} consist of arcs $\gamma_N^{(1)}$, $\gamma_{N-1}^{(2)}$ and $\gamma_{N-1}^{(3)}$ on which estimation (22) is already proved. Therefore, we'll prove estimation (22) for segments $\left[a_N,a_{N-1}\right]$ (for segments $\left[b_{N-1},b_N\right]$ it's proved analogously). Note, that $a_N=R_N e^{i\psi_1}$. Then making change of variable $\rho=r\,e^{i\psi_1}$ and taking into account, that $\operatorname{Re}\left(\rho\delta_{s\tau kj}(x,\xi)\right)\le 0$, we obtain

$$\left|\int_{a_N}^{a_{N-1}} e^{\rho\delta_{s\tau\, kj}(x,\xi)}d\rho\right| \le \int_{R_{N-1}}^{R_N} e^{\mathrm{Re}\left(\rho\delta_{s\tau\, kj}(x,\xi)\right)}d\,r \le c_8\left(R_N - R_{N-1}\right) \le c_9 .$$

Subject to the latter

$$\int_{x_{s-1}}^{x_s} \left|\sigma_{2N-}(f)(x)\right|^p dx \le c_{10} \sum_{k,j,\tau=1}^{2} \int_{x_{s-1}}^{x_s} \left(\int_{x_{\tau-1}}^{x_\tau} \left|f(\xi)\right| \int_{\Gamma'_{2N-1}} e^{\mathrm{Re}\left(\rho\delta_{s\tau\, kj}(x,\xi)\right)} |d\rho| d\xi \right)^p dx =$$

$$= c_{10} \sum_{k,j,\tau=1}^{2} \int_{x_{s-1}}^{x_s} \left(\int_{x_{\tau-1}}^{x_\tau} \left|f(\xi)\right| \left(\int_{\gamma_N^{(1)}} + \int_{a_N}^{a_{N-1}} + \int_{\gamma_{N-1}^{(2)}} + \int_{b_{N-1}}^{b_N} + \int_{\gamma_N^{(3)}} \right) e^{\mathrm{Re}\left(\rho\delta_{s\tau\, kj}(x,\xi)\right)} |d\rho| d\xi \right)^p dx \le c_{11} \|f\|_p^p .$$

Thus estimation (22) is established for all N.

In order to complete the proof of the first part of the theorem note that according to Lorch theorem [11] there exists a projection operator E such that at $N \to \infty$ sequence $\sigma_N(\cdot)$ strongly converges to E, i.e. for any function $f \in L_p(a,b)$

$$\lim_{N\to\infty} \sigma_N(f) = E\,f . \tag{30}$$

On the other hand, having estimation (12) for the kernel of the resolvent operator $R(\lambda)$ by a standard method (see, e.g. [12], p.445) we can show that the system $\{Z_k\}$ of eigen- and associated functions of adjoint operator L^* forms the complete system in the space $L_q(a,b)$, $q^{-1} + p^{-1} = 1$. Since the weak convergence follows from strong convergence from (30) we have

$$\lim_{N\to\infty} \left(\sigma_N(f), Z_k\right) = \left(E\,f, Z_k\right), \qquad k = 1,2,... .$$

But from biorthogonality of $\{Z_k\}$ to the system $\{y_k\}$ of EAF of operator L it follows, that at sufficiently large N

$$\left(f, Z_k\right) = \left(\sigma_N(f), Z_k\right).$$

From here, passing to limit at $N \to \infty$, we obtain

$$\left(f, Z_k\right) = \left(Ef, Z_k\right), \qquad k = 1,2,... .$$

Then from completeness of the system $\{Z_k\}$ it follows that $E\,f = f$, i.e.

$$\lim_{N\to\infty} \sigma_N(f) = f , \qquad f \in L_p(a,b).$$

In order to prove the second part of the theorem note that in case of strongly regular problems domain $G_s(\varepsilon)$ consists of mutually disjoint circles $O_{sN}(\varepsilon)$ (at sufficiently small $\varepsilon > 0$). Therefore, between contours Γ'_N and Γ'_{N+1} several circles $O_{sN}(\varepsilon)$ can lie and their at that number is bounded above. Proceeding as above a new system of contours Γ''_N can be formed, such that between two neighbouring contours Γ''_N and Γ''_{N+1} only one circle $O_{sN}(\varepsilon)$ is found, and on this contours estimation (22) also will be valid.

In the case $\varphi_1 = \varphi_2$ numbers $\omega_{2k} = \omega_{1k}$, half-strips $\Pi_{1,k}$ and $\Pi_{2,k}$ also coincide. Subject to this, proof of the theorem for this case is carried out without any changes.

Remark 1. In case $\varphi_1 \ne \varphi_2$ regular boundary conditions will be strongly regular, if $\left(\theta_{(-0,-1)} + \theta_{(+0,-1)}\right)^2 - 4\theta_{(+1,-1)} \cdot \theta_{(-1,-1)} \ne 0$ and $\left(\theta_{(-1,-0)} + \theta_{(-1,+0)}\right)^2 - 4\theta_{(-1,+1)} \cdot \theta_{(-1,-1)} \ne 0$.

Remark 2. The analogous results are correct also for boundary value problems with any finite number of discontinuity points inside the interval.

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