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THE INVERSE SCATTERING PROBLEM FOR A SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS ON A SEMI-AXIS

Abstract

In the paper the inverse scattering problem on a semi-axis is studied, when one incident wave exists. The unique restoration of the coefficients of equations by the scattering matrix was shown when singular numbers are absent.

In the paper the problem of restoration of potential $q(x)=\left\|c_{kj}(x)\right\|_{k,j=1}^{n}$ of the system of differential equations

$$-i\frac{dy_k(x)}{dx}+\sum_{j=1}^{n}c_{kj}(x)y_j(x)=\lambda\xi_k y_k(x)\,,\ \ x\ge 0 \tag{1}$$

on a semi-axis with parameter λ by the scattering matrix is considered. It is assumed that the coefficients of the equation satisfy the conditions $(\xi_1>0>\xi_2>...>\xi_n)$ and

$$c_{kk}(x)=0,\ R_+=[0,+\infty),\ c_{kj}(x)\in L_1(R_+),\ k,j=1,2,...,n\,. \tag{2}$$

The inverse scattering problem for a system of equations (1) at $n=2$ on a semi-axis and whole axis were studied in papers [1-4], and $n\ge 3\ (\xi_1>\xi_2>...>\xi_n)$ in [5, 8]. The complete solution of general problem on the whole axis can be found in paper [9]. In papers [10, 11] the case at $n\ge 3\ (\xi_1>...>\xi_{n-1}>0>\xi_n)$ with $n-1$ incident waves and with one incident wave at $n=3\ (\xi_1>0>\xi_2>\xi_3)$ were studied. The direct scattering problem on a semi-axis is considered in [12].

The scattering problem on a semi-axis in the case, when $\xi_1>0>\xi_2>...>\xi_n$ by multiplying the both sides of the equation on (-1) is reduced to the case $\xi'_1>...>\xi'_{n-1}>0>\xi'_n$ with one given scattering problem for the case $\xi_1>...>\xi_{n-1}>0>\xi_n$ at joint consideration of $n-1$ problems with different boundary conditions and at given reflected waves was investigated.

Consider $n-1$ problems for system (1) on a semi-axis. The k-th problem is in finding of solution of system (1) at the following boundary conditions

$$y_n^k(0,\lambda)=y_k^k(0,\lambda),\ \ y_i^k(0,\lambda)=0\,,\ i\ne k,n\,, \tag{3}$$

and given asymptotic

$$y_n^k(x,\lambda)=B_n^k\exp(i\lambda\xi_n x)+o(1)\,,\ \ x\to+\infty\,,\ (\operatorname{Im}\lambda=0)$$
$$i,k=1,2,...,n-1\,. \tag{4}$$

Theorem 1. *Let the coefficients of system (1) satisfy the condition (2)* $(\xi_1>...>\xi_{n-1}>0>\xi_n)$ *and* $\operatorname{Im}\lambda=0$. *Then there exists a unique bounded solution of the scattering problem for system (1) on a semi-axis at joint consideration of problems (1), (3), (4).*

Proof. The scattering problem for the k-th problem is equivalent to the following system of integral equations

$$y_p^k(x,\lambda)=A_p^k\exp(i\lambda\xi_p x)+i\int_x^{+\infty}\sum_{j=1}^{n}c_{pj}(x')y_j^k(x',\lambda)\exp(i\lambda\xi_p(x-x'))dx'\,,$$

$$y_n^k(x,\lambda)=B_n^k\exp(i\lambda\xi_n x)+i\int_x^{+\infty}\sum_{j=1}^n c_{nj}(x')y_j^k(x')\exp(i\lambda\xi_n(x-x'))dx', \tag{5}$$

where

$$A_p^k=-i\int_0^{+\infty}\sum_{j=1}^n c_{pj}(x')y_j(x',\lambda)\exp(-i\lambda\xi_p x')dx',\ i\neq k,$$

$$A_k^k=B_n^k+i\int_0^{+\infty}\sum_{j=1}^n\left[c_{nj}(x')\exp(-i\lambda\,\xi_n x')-c_{kj}(x')\exp(-i\lambda\xi_k x')\right]y_j^k(x',\lambda)dx',$$

$$p,k=1,...,n-1.$$

The existence and uniqueness of the solution of system (5) in a class of bounded functions follow from Volterra property of system (5).

By virtue of property of condition (2) we obtain from (5)

$$y_p^k(x,\lambda)=A_p^k\exp(i\lambda\xi_p x)+o(1),\ \ p,k=1,...,n-1,$$

$$y_n^k(x,\lambda)=B_n^k\exp(i\lambda\xi_n x)+o(1),\ \ x\to+\infty. \tag{6}$$

On the basis of theorem 1 according to (6) the elements

$$A_p^k=S_{pk}B_n^k,\ \ p,k=1,2,...,n-1, \tag{7}$$

forming the matrix $S(\lambda)=\left\|S_{p_k}(\lambda)\right\|_{p,k=1}^{n-1}$ are defined.

Matrix $S(\lambda)$ is called a scattering matrix for system (1) on a semi-axis with given reflected waves.

Note that from definition $S(\lambda)$ it follows that

$$\begin{pmatrix} A_1^1+...+A_1^{n-1}\\ A_2^1+...+A_2^{n-1}\\ ...\\ A_{n-1}^1+...+A_{n-1}^{n-1}\end{pmatrix}=S(\lambda)\begin{pmatrix}B_n^1\\ B_n^2\\ \vdots\\ B_n^{n-1}\end{pmatrix}. \tag{8}$$

The properties of $S(\lambda)$- matrix are studied with the help of integral representation of solutions.

Lemma 1 [12]. *Let coefficients of system (1) satisfy condition (2) and* $\mathrm{Im}\,\lambda=0$. *Then every bounded solution has the integral representation*

$$y_k(x)=h_k^1(x,\lambda)+\sum_{j=1}^n\int_{\xi_n x}^{\xi_1 x}A_{kj}^1(x,\tau)\exp(i\lambda\tau)d\tau\cdot y_j(0,\lambda), \tag{9}$$

$$y_k(x)=h_k^2(x,\lambda)+\int_{-\infty}^{\xi_1 x}A_{k1}^2(x,\tau)\exp(i\lambda\tau)\cdot A_1+\sum_{j=2}^n\int_{-\infty}^{\xi_2 x}A_{kj}^2(x,\tau)\exp(i\lambda\tau)d\tau\cdot y_j(0,\lambda), \tag{10}$$

$$y_k(x)=h_k^p(x,\lambda)+\sum_{j=1}^{p-2}\int_{-\infty}^{+\infty}A_{kj}^p(x,\tau)\exp(i\lambda\tau)\cdot h_j^p(0,\lambda)+\int_{-\infty}^{\xi_{p-1}x}A_{k,p-1}^p(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_{p-1}^p(0,\lambda)+$$

$$+\sum_{j=p}^n\int_{-\infty}^{\xi_p x}A_{kj}^p(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^p(0,\lambda),\ \ (3\leq p\leq n), \tag{11}$$

$$y_k(x)=h_k^{n+1}(x,\lambda)+\int_{\xi_1 x}^{+\infty}A_{k1}^{n+1}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_1^{n+1}(0,\lambda)+$$

$$+\sum_{j=2}^{n-1}\int_{-\infty}^{+\infty}A_{kj}^{n+1}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^{n+1}(0,\lambda)+\int_{-\infty}^{\xi_n x}A_{kn}^{n+1}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_n^{n+1}(0,\lambda),\quad(12)$$

$$y_k(x)=h_k^{n+p}(x,\lambda)+\sum_{j=1}^{p-1}\int_{\xi_{p-1}x}^{+\infty}A_{kj}^{n+1}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^{n+p}(0,\lambda)+$$

$$+\int_{\xi_p x}^{+\infty}A_{kp}^{n+p}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_p^{n+p}(0,\lambda)+\sum_{j=p+1}^{n}\int_{-\infty}^{+\infty}A_{kj}^{n+p}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^{n+p}(0,\lambda),$$

$$(2\le p\le n-1),\quad(13)$$

$$y_k(x)=h_k^{2n}(x,\lambda)+\sum_{j=1}^{n-1}\int_{\xi_{n-1}x}^{+\infty}A_{kj}^{2n}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_j^{2n}(0,\lambda)+$$

$$+\int_{\xi_n x}^{+\infty}A_{kn}^{2n}(x,\tau)\exp(i\lambda\tau)d\tau\cdot h_n^{2n}(0,\lambda).\quad(14)$$

The kernels of these transformations are uniquely defined by coefficients $c_{kj}(x)$ $(k,j=1,...,n)$.

Here

$$h^1(x,\lambda)=\{y_1(0)\exp(i\xi_1\lambda x),...,y_n(0)\exp(i\xi_n\lambda x)\},$$

$$h^2(x,\lambda)=\{A_1\exp(i\xi_1\lambda x),y_2(0)\exp(i\xi_2\lambda x),...,y_n(0)\exp(i\xi_n\lambda x)\},$$

$$h^k(x,\lambda)=\{A_1\exp(i\xi_1\lambda x),A_2\exp(i\xi_2\lambda x),...,A_{k-1}\exp(i\xi_{k-1}\lambda x),y_k(0)\exp(i\xi_k\lambda x),$$
$$...,y_n(0)\exp(i\lambda\xi_n x)\},\ (k=3,...,n),$$

$$h^{n+1}(x,\lambda)=\{A_1\exp(i\xi_1\lambda x),A_2\exp(i\xi_2\lambda x),...,A_{n-1}\exp(i\xi_{n-1}\lambda x),...,B_n\exp(i\xi_n\lambda x)\},$$

$$h^{n+k}(x,\lambda)=\{y_1(0)\exp(i\xi_1\lambda x),...,y_{k-1}(0)\exp(i\xi_{k-1}\lambda x),A_k\exp(i\xi_k\lambda x),$$
$$...,A_{n-1}\exp(i\lambda\xi_n x),B_n\exp(i\lambda\xi_n x)\},\ (k=2,...,n-1),$$

$$h^{2n}(x,\lambda)=\{y_1(0)\exp(i\xi_1\lambda x),...,y_{n-1}(0)\exp(i\xi_{n-1}\lambda x),B_n\exp(i\xi_n\lambda x)\}.\quad(15)$$

Lemma 2. *Let* $B_n^k\exp(i\lambda\xi_n x)$ $(k=1,...,n-1)$ *be given reflected waves. Then* $y_k^k(0,\lambda)$ $(k=1,...,n-1)$ *are defined by the following formulas*

$$y_k^k(0,\lambda)=y_n^k(0,\lambda)=\left(1-A_{nk+}^{2n}(\lambda)\right)^{-1}\left(1+A_{nn+}^{2n}(\lambda)\right)B_n^k\equiv\left(1+A_{k+}(\lambda)\right)B_n^k,$$
$$(k=1,2,...,n-1).\quad(16)$$

Proof. From (14) taking into account (3) we obtain

$$y_n^k(0,\lambda)=B_n^k+\int_0^{+\infty}A_{nk}^{2n}(0,\tau)\exp(i\lambda\tau)d\tau\cdot y_k^k(0,\lambda)+\int_0^{+\infty}A_{nn}^{2n}(0,\tau)\exp(i\lambda\tau)d\tau\cdot B_n^k.$$

From here

$$y_k^k(0,\lambda)=\left(1-A_{nk+}^{2n}(\lambda)\right)^{-1}\left(1+A_{nn+}^{2n}(\lambda)\right)B_n^k,\quad k=1,2,...,n-1,$$

where

$$A_{ni+}^{2n}(\lambda)=\int_0^{+\infty}A_{ni}^{2n}(0,\tau)\exp(i\lambda\tau)d\tau,\quad i=1,2,..,n.$$

Lemma 3. *For the* k *-th* $(k=1,...,n-1)$ *problem, the following equalities hold*

$$y_1^1(0,\lambda)=\left(1-A_{1n-}^2(\lambda)\right)\left(1+A_{11-}^2(\lambda)\right)A_1^1\equiv\left(1+A_{1-}(\lambda)\right)A_1^1,$$

$$A_1^k = -\left(1+A_{11+}^2(\lambda)\right)^{-1}\left(A_{1k-}^2(\lambda)+A_{1n-}^2(\lambda)\right)y_k^k(0,\lambda)=G_{2-}^k(\lambda)y_k^k(0,\lambda),\ k=2,...,n-1. \quad (17)$$

Proof. From (10) it follows that

$$y_1^k(0,\lambda)=A_1^k+\int_{-\infty}^0 A_{11}^2(0,\tau)\exp(i\lambda\tau)d\tau\cdot A_1^k+\int_{-\infty}^0\sum_{j=2}^n A_{1j}^2(0,\tau)\exp(i\lambda\tau)d\tau\cdot y_j^k(0,\lambda)$$

or

$$y_1^k(0,\lambda)=\left(1+A_{11-}^2(\lambda)\right)A_1^k+\sum_{j=2}^n A_{1j-}^2(\lambda)\cdot y_j^k(0,\lambda).$$

From here subject to (3) we have

$$\left(1-A_{1n-}^2(\lambda)\right)y_1^1(0,\lambda)=\left(1+A_{11-}^2(\lambda)\right)A_1^1,$$

$$\left(1+A_{11-}^2(\lambda)\right)A_1^k+\left(A_{1k-}^2(\lambda)+A_{1n-}^2(\lambda)\right)y_k^k(0,\lambda)=0.$$

Consequently, we obtain relation (17).

Lemma 4. *Let* $B_n^1=B_n^2=...=B_n^{n-1}\equiv B_n$. *Then functions* $y_i^i(0,\lambda)\ (i=1,2,...,n-1)$ *are defined by the following formula*

$$y_j^j(0,\lambda)=\left(1-A_{n,j+}^{2n-1}(\lambda)+A_{n-1,j+}^{2n-1}(\lambda)\right)^{-1}\left(1+A_{n-1,n-1+}^{2n-1}(\lambda)-A_{n,n-1+}^{2n-1}(\lambda)\right)\left(A_{n-1}^{n-1}-A_{n-1}^j\right),$$

$$y_{n-1}^{n-1}(0,\lambda)=\Big[1+A_{n-1,n-1+}^{2n-1}(\lambda)-A_{n-1,j+}^{2n-1}(\lambda)\left(1-A_{n,j+}^{2n-1}(\lambda)+A_{n-1,j+}^{2n-1}(\lambda)\right)^{-1}\times$$

$$\times\left(1+A_{n-1,n-1+}^{2n-1}-A_{n,n-1+}^{2n-1}(\lambda)\right)\Big]\left(A_{n-1}^{n-1}-A_{n-1}^j\right),\quad j=1,2,...,n-2. \quad (18)$$

Proof. By virtue of condition (3) from representation (13), at $k=n-1$ by subtraction of the j-th problem from $n-1$-st one $(j=1,...,n-2)$

$$y_{n-1}^{n-1}(0,\lambda)=A_{n-1}^{n-1}-A_{n-1}^j-\int_0^{+\infty}A_{n-1,j-}^{2n-1}(0,\tau)\exp(i\lambda\tau)d\tau\cdot y_j^j(0,\lambda)+\int_0^{+\infty}A_{n,n-1}^{2n-1}(0,\tau)\exp(i\lambda\tau)d\tau\times$$

$$\times\left(A_{n-1}^{n-1}-A_{n-1}^j\right),$$

$$y_n^{n-1}(0,\lambda)-y_n^j(0,\lambda)=y_{n-1}^{n-1}(0)-y_j^j(0,\lambda)=\int_0^{+\infty}A_{n,j}^{2n-1}(0,\tau)\exp(i\lambda\tau)d\tau\cdot y_j^j(0,\lambda)+$$

$$+\int_0^{+\infty}A_{n,n-1}^{2n-1}(0,\tau)\exp(i\lambda\tau)d\tau\left(A_{n-1}^{n-1}-A_{n-1}^j\right)$$

or

$$y_{n-1}^{n-1}(0,\lambda)+A_{n-1,j+}^{2n-1}(\lambda)\cdot y_j^j(0,\lambda)=\left(1+A_{n-1,n-1+}^{2n-1}(\lambda)\right)\left(A_{n-1}^{n-1}-A_{n-1}^j\right),$$

$$y_{n-1}^{n-1}(0,\lambda)-\left(1-A_{n,j+}^{2n-1}(\lambda)\right)\cdot y_j^j(0,\lambda)=A_{n,n-1+}^{2n-1}(\lambda)\left(A_{n-1}^{n-1}-A_{n-1}^j\right).$$

Solution of this system gives (18).

Lemma 5. *Let* $A_1^1+...+A_1^{n-1}=0,...,A_{k-1}^1+...+A_{k-1}^{n-1}=0\ (2\le k\le n-1)$. *Then the following equality holds*

$$y_p^p(0,\lambda)=\left(1-A_{pn-}^{k+1}(\lambda)\right)^{-1}A_{p,k-}^{k+1}(\lambda)\left(A_k^1+...+A_k^{n-1}\right)=B_{p-}^k(\lambda)\left(A_k^1+...+A_k^{n-1}\right),$$

$$(p=1,2,...,k-1),$$

$$y_k^k(0,\lambda)=\left(1-A_{kn-}^{k+1}(\lambda)\right)^{-1}\left(1+A_{kk-}^{k+1}(\lambda)\right)\left(A_k^1+...+A_k^{n-1}\right)=\left(1+A_{k-}(\lambda)\right)\left(A_k^1+...+A_k^{n-1}\right),$$

$$(p=2,3,...,n-1). \quad (19)$$

Proof. From (11) we obtain

$$y_i(0,\lambda)=A_i+\int_{-\infty}^{+\infty}\sum_{j=1}^{k-2}A_{ij}^k(0,\tau)\exp(i\lambda\tau)d\tau\cdot A_j+\int_{-\infty}^{0}A_{i,k-1}^k(0,\tau)A_{k-1}+$$
$$+\int_{-\infty}^{0}\sum_{j=k}^{n}A_{ij}^k(0,\tau)\exp(i\lambda\tau)d\tau\cdot y_j(0,\lambda)\quad (i=1,2,...k-1).$$

From here for the p-th problem we have

$$y_i^p(0,\lambda)=A_i^p+\sum_{j=1}^{k-2}A_{ij}^k(\lambda)A_j^p+A_{i,k-1-}^k(\lambda)\cdot A_{k-1}^p+\sum_{j=k}^{n}A_{ij-}^k(\lambda)\cdot y_j^p(0,\lambda),$$
$$(i,p=1,...,k-1).$$

Summing from $p=1$ to $p=k-1$ and using (3) we obtain

$$y_p^p(0,\lambda)=A_{p,k-1-}^k(\lambda)\left(A_{k-1}^1+...+A_{k-1}^{n-1}\right)+A_{p,n-}^k(\lambda)y_p^p(0,\lambda),\ (p=1,...,k-2),$$
$$y_{k-1}^{k-1}(0,\lambda)=\left(1+A_{k-1,k-1-}^k(\lambda)\right)\left(A_{k-1}^1+...+A_{k-1}^{n-1}\right)+A_{k-1,n-}^k(\lambda)y_{k-1}^{k-1}(0,\lambda).$$

Substituting instead of $k\to k+1$ and solving with respect to $y_p^p(0,\lambda)\,(p=1,...,k-1)$, $y_k^k(0,\lambda)\,(k=2,3,...n-1)$ we arrive at equality (19).

Lemma 6. *Let* $A_i^k=A_i^p$, $k\neq p,\ i,k,p=1,...,n-1$. *Then* $B_n^k=B_n^p$.

Proof. From representations (12) we obtain

$$y_i^k(0,\lambda)-y_i^p(0,\lambda)=A_i^k-A_i^p+A_{i1+}^{n+1}(\lambda)\left(A_1^k-A_1^p\right)+$$
$$+\sum_{j=2}^{n-1}A_{ij}^{n+1}(\lambda)\left(A_j^k-A_j^p\right)+A_{in-}^{n+1}(\lambda)\left(B^k-B^p\right),\ i=1,2,...,n-1, \tag{20}$$
$$y_n^k(0,\lambda)-y_n^p(0,\lambda)=B^k-B^p+A_{n1+}^{n+1}(\lambda)\left(A_1^k-A_1^p\right)+$$
$$+\sum_{j=2}^{n-1}A_{nj}^{n+1}(\lambda)\left(A_j^k-A_j^p\right)+A_{nn-}^{n+1}(\lambda)\left(B^k-B^p\right). \tag{21}$$

By virtue of (3) and by the condition of the lemma we have

$$y_k^k(0,\lambda)=A_{kn-}^{n+1}(\lambda)\left(B^k-B^p\right),$$
$$-y_p^p(0,\lambda)=A_{pn-}^{n+1}(\lambda)\left(B^k-B^p\right),$$
$$y_k^k(0,\lambda)-y_p^p(0,\lambda)=\left(1+A_{nn-}^{n+1}(\lambda)\right)\left(B^k-B^p\right).$$

From here

$$\left(1+A_{nn-}^{n+1}(\lambda)-A_{pn-}^{n+1}(\lambda)-A_{kn-}^{n+1}(\lambda)\right)\left(B^k-B^p\right)=0$$

or

$$B^k=B^p.$$

The lemma is proved.

Remark. In lemmas 2-6 we assume that functions
$1-A_{nk+}^{2n}(\lambda)\,(k=1,...,n-1)$, $1+A_{nn+}^{2n}(\lambda)$, $1+A_{11-}^{2}(\lambda)$, $1-A_{1n-}^{2}(\lambda)$, $1-A_{n,j+}^{2n-1}(\lambda)-A_{n-j,j+}^{2n-1}$ $(j=1,...,n-2)$, $1+A_{n-1,n-1+}^{2n-1}(\lambda)-A_{n,n-1+}^{2n-1}(\lambda)$, $1+A_{n-1,n-1+}^{2n-1}(\lambda)-A_{n-1,j+}^{2n-1}(\lambda)\left(1-A_{n,j+}^{2n-1}(\lambda)+\right.$ $\left.+A_{n-1,j+}^{2n-1}(\lambda)\right)^{-1}\left(1+A_{n-1,n-1+}^{2n-1}(\lambda)-A_{n,n-1+}^{2n-1}(\lambda)\right)$, $1-A_{in-}^{k+1}(\lambda)\,(i=1,...,k;\ k=2,...,n-1)$,
$1+A_{kk-}^{k+1}(\lambda)$
have no zeros. Zeros of these functions we call singular numbers.

Theorem 2. *Let the coefficients of the system of equations (1) satisfy the condition (2)* $(\xi_1 > ... > \xi_{n-1} > 0 > \xi_n)$, *and singular numbers are absent. Then* $S(\lambda)$ *has an inverse operator* $S^{-1}(\lambda) = \left\|\gamma_{ij}\right\|_{i,j=1}^{n-1}$. *In addition* $S(\lambda)$ *and* $S^{-1}(\lambda)$ *have the following form:*

$$S(\lambda) = I + \int_{-\infty}^{+\infty} F(\tau) e^{i\lambda\tau} d\tau, \tag{22}$$

$$S^{-1}(\lambda) = I + \int_{-\infty}^{+\infty} J(\tau) e^{i\lambda\tau} d\tau, \tag{23}$$

where I *is a unit matrix,* $F, J \in L_1(R)$. *Matrix functions* $\Delta_k(\lambda) = \left\|S_{ij}(\lambda)\right\|_{i,j=1}^{k}$ $(k = 1,...,n-1)$ *are also invertible. Moreover, functions* $(\gamma_{11}(\lambda) + ... + \gamma_{n-1,1}(\lambda))^{-1}$, $\left(\Delta_k^{-1}(\lambda)\right)_{kk}$ *admit the factorization*

$$(\gamma_{11}(\lambda) + ... + \gamma_{n-1,1}(\lambda))^{-1} = (1 + R_+(\lambda))^{-1}(1 + R_-(\lambda)), \tag{24}$$

$$\left(\Delta_k^{-1}(\lambda)\right)_{kk} = (1 + A_{k+}(\lambda))^{-1}(1 + A_{k-}(\lambda)), \quad k = 1,...,n-1 \tag{25}$$

and functions $S_{n-1,n-1}(\lambda) - S_{n-1,k}(\lambda)$ $(k = 1,2,...n-2)$, $S_{1k}(\lambda)$ $(k = 2,...,n-1)$, $\gamma_{k,n-1}(\lambda)$ $(k = 1,..., n-2)$ *have the following form*

$$S_{n-1,n-1}(\lambda) - S_{n-1,k}(\lambda) = 1 + Q_{k+}(\lambda), \quad k = 1,...,n-2, \tag{26}$$

$$S_{1k}(\lambda) = G_{2-}^{k}(\lambda)(1 + A_{k+}(\lambda)) \ (k = 2,...,n-1), \tag{27}$$

$$\gamma_{k,n-1}(\lambda) = (1 + A_{k+}(\lambda))^{-1} B_{k-}(\lambda), \quad k = 1,...,n-2 \tag{28}$$

here

$R_+(\lambda) = A_{11+}^{n+1}(\lambda) + ... + A_{n-1,1+}^{n+1}(\lambda) - A_{n1+}^{n+1}(\lambda)$,

$R_-(\lambda) = A_{nn-}^{n+1}(\lambda) - ... - A_{1n-}^{n+1}(\lambda) - ... - A_{n-1,n-}^{n+1}(\lambda)$,

$A_{k+}(\lambda) = \left(1 - A_{nk+}^{2n}(\lambda)\right)^{-1}\left(1 + A_{nn+}^{2n}(\lambda)\right) - 1$, $A_{k-}(\lambda) = \left(1 - A_{kn-}^{k+1}(\lambda)\right)^{-1}\left(1 + A_{kk-}^{k+1}(\lambda)\right) - 1$, $(k = 1,...,n-1)$,

$Q_{k+}(\lambda) = \left(1 - A_{n,k+}^{2n-1}(\lambda) + A_{n-1,+}^{2n-1}(\lambda)\right)^{1}\left(1 + A_{n-1,n-1+}^{2n-1}(\lambda) - A_{n,n-1+}^{2n-1}(\lambda)\right)$, $(k = 1,...,n-2)$,

$G_{2-}^{k}(\lambda) = -\left(1 + A_{11-}^{2}(\lambda)\right)^{-1}\left(A_{1k-}^{2}(\lambda) + A_{1n-}^{2}(\lambda)\right)$, $(k = 2,...,n-1)$,

$B_{k-}(\lambda) = \left(1 - A_{kn-}^{n}(\lambda)\right)^{-1} A_{k,n-1-}^{n}(\lambda)$, $(k = 1,...,n-2)$.

Proof. 1. Let singular numbers be absent. We'll show that the inverse operator $S^{-1}(\lambda)$ exists. If $A_1^1 + ... + A_1^{n-1} = 0,..., A_{n-1}^1 + ... + A_{n-1}^{n-1} = 0$, then by lemma 5 it follows that $y_1^1(0,\lambda) = ... = y_{n-1}^{n-1}(0,\lambda) = 0$, and from (16) we obtain

$$(1 + A_{k+}(\lambda))B_n^k = 0, \ k = 1,...,n-1$$

or

$$B_n^1 = B_n^2 = ... = B_n^{n-1} = 0,$$

i.e. $S(\lambda)$ has an inverse operator $S^{-1}(\lambda) = \left\|\gamma_{ij}(\lambda)\right\|_{i,j=1}^{n-1}$.

2. We'll show that (22) and (23) are fulfilled. From the representation (12) at $x = 0$ we have

$$y_k(0,\lambda)=A_k+A_{k1+}^{n+1}(\lambda)\cdot A_1+\sum_{j=2}^{n-1}A_{kj}^{n+1}(\lambda)\cdot A_j+A_{kn-}^{n+1}(\lambda)B_n,\ k=1,2,...,n-1,$$

$$y_n(0,\lambda)=B_n+A_{n1+}^{n+1}(\lambda)\cdot A_1+\sum_{j=2}^{n-1}A_{nj}^{n+1}(\lambda)\cdot A_j+A_{nn-}^{n+1}(\lambda)B_n. \qquad (29)$$

From here applying lemma 2 for the first problem we obtain

$$\begin{pmatrix}\left(1+A_{nn-}^{n+1}(\lambda)-A_{1n-}^{n+1}(\lambda)\right)B_n^1\\ 0\\ \vdots\\ 0\end{pmatrix}=\begin{pmatrix}1+A_{11+}^{n+1}(\lambda)-A_{n1+}^{n+1}(\lambda) & A_{12}^{n+1}(\lambda)-A_{n2}^{n+1}(\lambda) & \dots & A_{1,n-1}^{n+1}(\lambda)-A_{n,n-1}^{n+1}(\lambda)\\ A_{21+}^{n+1}(\lambda) & 1+A_{22}^{n+1}(\lambda) & \dots & A_{2,n-1}^{n+1}(\lambda)\\ \vdots & \vdots & \dots & \vdots\\ A_{n-1,1+}^{n+1} & A_{n-1,2}^{n+1}(\lambda) & \dots & 1+A_{n-1,n-1}^{n+1}\end{pmatrix}\begin{pmatrix}A_1^1\\ A_2^1\\ \vdots\\ A_{n-1}^1\end{pmatrix}.$$

Analogously the inevitability of the following matrix is proved

$$\begin{pmatrix}1+A_{11+}^{n+1}(\lambda)-A_{n1+}^{n+1}(\lambda) & A_{12}^{n+1}(\lambda)-A_{n2}^{n+1}(\lambda) & \dots & A_{1,n-1}^{n+1}(\lambda)-A_{n,n-1}^{n+1}(\lambda)\\ A_{21+}^{n+1}(\lambda) & 1+A_{22}^{n+1}(\lambda) & \dots & A_{2,n-1}^{n+1}(\lambda)\\ \dots & \dots & \dots & \dots\\ A_{n-1,1+}^{n+1}(\lambda) & A_{n-1,2}^{n+1}(\lambda) & \dots & 1+A_{n-1,n-1}^{n+1}(\lambda)\end{pmatrix}.$$

Hence,

$$\begin{pmatrix}S_{11}(\lambda)\\ S_{21}(\lambda)\\ \vdots\\ S_{n-1,1}(\lambda)\end{pmatrix}=\begin{pmatrix}1+A_{11+}^{n+1}(\lambda)-A_{n1+}^{n+1}(\lambda) & A_{12}^{n+1}(\lambda)-A_{n2}^{n+1}(\lambda) & \dots & A_{1,n-1}^{n+1}(\lambda)-A_{n,n-1}^{n+1}(\lambda)\\ A_{21+}^{n+1}(\lambda) & 1+A_{22}^{n+1}(\lambda) & \dots & A_{2,n-1}^{n+1}(\lambda)\\ \dots & \dots & \dots & \dots\\ A_{n-1,1+}^{n+1}(\lambda) & A_{n-1,2}^{n+1}(\lambda) & \dots & 1+A_{n-1,n-1}^{n+1}(\lambda)\end{pmatrix}^{-1}\times$$

$$\times\begin{pmatrix}1+A_{nn-}^{n+1}(\lambda)-A_{1n-}^{n+1}(\lambda)\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

Similarly using the second, ...,$(n-1)$-th problem we find

$$\begin{pmatrix}S_{12}(\lambda)\\ S_{22}(\lambda)\\ \vdots\\ S_{n-1,2}(\lambda)\end{pmatrix}=\begin{pmatrix}1+A_{11+}^{n+1}(\lambda) & A_{12}^{n+1}(\lambda) & \dots & A_{1,n-1}^{n+1}(\lambda)\\ A_{21+}^{n+1}(\lambda)-A_{n1+}^{n+1}(\lambda) & 1+A_{22}^{n+1}(\lambda)-A_{n2}^{n+1}(\lambda) & \dots & A_{2,n-1}^{n+1}(\lambda)-A_{n,n-1}^{n+1}(\lambda)\\ \dots & \dots & \dots & \dots\\ A_{n-1,1+}^{n+1}(\lambda) & A_{n-1,2}^{n+1}(\lambda) & \dots & 1+A_{n-1,n-1}^{n+1}(\lambda)\end{pmatrix}^{-1}\times$$

$$\times\begin{pmatrix}0\\1+A_{nn-}^{n+1}(\lambda)-A_{2n-}^{n+1}(\lambda)\\ \vdots\\0\end{pmatrix},$$

....

$$\begin{pmatrix}S_{1,n-1}(\lambda)\\S_{2,n-1}(\lambda)\\ \vdots\\S_{n-1,n-1}(\lambda)\end{pmatrix}=\begin{pmatrix}1+A_{11+}^{n+1}(\lambda) & A_{12}^{n+1}(\lambda) & \dots & A_{1,n-1}^{n+1}(\lambda)\\ A_{21+}^{n+1}(\lambda) & 1+A_{22}^{n+1}(\lambda) & \dots & A_{2,n-1}^{n+1}(\lambda)\\ \dots & \dots & \dots & \dots\\ A_{n-1,1+}^{n+1}(\lambda)-A_{n1+}^{n+1}(\lambda) & A_{n-1,2}^{n+1}(\lambda)-A_{n2}^{n+1}(\lambda) & \dots & 1+A_{n-1,n-1}^{n+1}(\lambda)-A_{n,n-1}^{n-1}(\lambda)\end{pmatrix}^{-1}\times$$

$$\times\begin{pmatrix}0\\0\\ \vdots\\1+A_{nn-}^{n+1}(\lambda)-A_{n-1,n-}^{n+1}(\lambda)\end{pmatrix}.$$

Thus, $S(\lambda)$ has the form of (22). From Wiener theorem we also obtain (23).

3. We'll prove formulas (24). Using formulas (29) we use

$$y_k^1(0,\lambda)+\dots+y_k^{n-1}(0,\lambda)=A_k^1+\dots+A_k^{n-1}+A_{k1+}^{n+1}(\lambda)\left(A_1^1+\dots+A_1^{n-1}\right)+$$
$$+\sum_{j=2}^{n-1}A_{kj}^{n+1}\left(A_j^1+\dots+A_j^{n-1}\right)+A_{kn-}^{n+1}(\lambda)\left(B_n^1+\dots+B_n^{n-1}\right),\ k=1,2,\dots,n-1 \tag{30}$$

and

$$y_n^1(0,\lambda)+\dots+y_n^{n-1}(0,\lambda)=B_n^1+\dots+B_n^{n-1}+A_{n1+}^{n+1}(\lambda)\left(A_1^1+\dots+A_1^{n-1}\right)+$$
$$+\sum_{j=2}^{n-1}A_{nj}^{n+1}(\lambda)\left(A_j^1+\dots+A_j^{n-1}\right)+A_{nn-}^{n+1}(\lambda)\left(B_n^1+\dots+B_n^{n-1}\right). \tag{31}$$

Let $A_2^1+\dots+A_2^{n-1}=0,\dots,A_{n-1}^1+\dots+A_{n-1}^{n-1}=0$. Summing (30) we obtain

$$y_1^1(0,\lambda)+y_2^2(0,\lambda)+\dots+y_{n-1}^{n-1}(0,\lambda)=A_1^1+\dots+A_1^{n-1}+\left(A_{11+}^{n+1}(\lambda)+\dots+A_{n-1,1+}^{n+1}(\lambda)\right)\times$$
$$\times\left(A_1^1+\dots+A_1^{n-1}\right)+\left(A_{1n-}^{n+1}(\lambda)+\dots+A_{n-1,n-}^{n+1}(\lambda)\right)\left(B_n^1+\dots+B_n^{n-1}\right),$$
$$y_1^1(0,\lambda)+y_2^2(0,\lambda)+\dots+y_{n-1}^{n-1}(0,\lambda)=y_n^1(0,\lambda)+y_n^2(0,\lambda)+\dots+y_n^{n-1}(0,\lambda)=$$
$$=B_n^1+\dots+B_n^{n-1}+A_{n1+}^{n+1}(\lambda)\left(A_1^1+\dots+A_1^{n-1}\right)+A_{nn-}^{n+1}(\lambda)\left(B_n^1+\dots+B_n^{n-1}\right).$$

Thus, the following equality will be fulfilled

$$\left(1+A_{nn-}^{n+1}(\lambda)-A_{1n-}^{n+1}(\lambda)-\dots-A_{n-1,n-}^{n+1}(\lambda)\right)\left(B_n^1+\dots+B_n^{n-1}\right)=$$
$$=\left(1+A_{11+}^{n+1}(\lambda)+\dots+A_{n-1,1+}^{n+1}(\lambda)-A_{n1+}^{n+1}(\lambda)\right)\left(A_1^1+\dots+A_1^{n-1}\right). \tag{32}$$

Using (8) we obtain

$$\begin{pmatrix}S_{11}(\lambda) & S_{12}(\lambda)-S_{11}(\lambda) & \dots & S_{1,n-1}(\lambda)-S_{11}(\lambda)\\ S_{21}(\lambda) & S_{22}(\lambda)-S_{21}(\lambda) & \dots & S_{2,n-1}(\lambda)-S_{21}(\lambda)\\ \dots & \dots & \dots & \dots\\ S_{n-1,1}(\lambda) & S_{n-1,2}(\lambda)-S_{n-1,1}(\lambda) & \dots & S_{n-1,n-1}(\lambda)-S_{n-1,1}(\lambda)\end{pmatrix}\begin{pmatrix}B_n^1+B_n^2+\dots+B_n^{n-1}\\B_n^2\\ \vdots\\B_n^{n-1}\end{pmatrix}=$$

$$=\begin{pmatrix} A_1^1+A_1^2+...+A_1^{n-1} \\ A_2^1+A_2^2+...+A_2^{n-1} \\ ... \\ A_{n-1}^1+A_{n-1}^2+...+A_{n-1}^{n-1} \end{pmatrix}. \qquad (33)$$

Further the following equality holds

$$\begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda)-S_{11}(\lambda) & ... & S_{1,n-1}(\lambda)-S_{11}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda)-S_{21}(\lambda) & ... & S_{2,n-1}(\lambda)-S_{21}(\lambda) \\ ... & ... & ... & ... \\ S_{n-1,1}(\lambda) & S_{n-1,2}(\lambda)-S_{n-1,1}(\lambda) & ... & S_{n-1,n-1}(\lambda)-S_{n-1,1}(\lambda) \end{pmatrix}^{-1} =$$

$$=\begin{pmatrix} \gamma_{11}(\lambda)+\gamma_{21}(\lambda)+...+\gamma_{n-1,1}(\lambda) & \gamma_{12}(\lambda)+\gamma_{22}(\lambda)+...+\gamma_{n-1,2}(\lambda) & ... & ... & \gamma_{1,n-1}(\lambda)+\gamma_{2,n-1}(\lambda)+...+\gamma_{n-1,n-1}(\lambda) \\ \gamma_{21}(\lambda) & \gamma_{22}(\lambda) & ... & ... & \gamma_{2,n-1}(\lambda) \\ \vdots & \vdots & ... & ... & \vdots \\ \gamma_{n-1,1}(\lambda) & \gamma_{n-1,2}(\lambda) & ... & ... & \gamma_{n-1,n-1}(\lambda) \end{pmatrix}. \qquad (34)$$

From (33) and (34) at $A_i^1+A_i^2+...+A_i^{n-1}=0 \ (i=2,...n-1)$ it follows that

$$(\gamma_{11}(\lambda)+\gamma_{21}(\lambda)+...+\gamma_{n-1,1}(\lambda))(A_1^1+A_1^2+...+A_1^{n-1})=B_n^1+B_n^2+...+B_n^{n-1}. \qquad (35)$$

Comparing (32) and (35) we find (24).

4. We'll prove formulas (25). When $k=1$, $\Delta_1^{-1}=S_{11}^{-1}(\lambda)$. From lemma 2 and lemma 3 for arbitrary B_n^1 we obtain

$$(1+A_{1-}(\lambda))A_1^1=(1+A_{1+}(\lambda))B_n^1.$$

From here we find

$$\Delta_1^{-1}(\lambda)=(1+A_{1+}(\lambda))^{-1}(1+A_{1-}(\lambda)).$$

When $k=2,3,...,n-1$ from lemma 2 and lemma 5 at $A_1^1+A_1^2+...+A_1^{n-1}=0$, …, $A_{k-1}^1+A_{k-1}^2+...+A_{k-1}^{n-1}=0$ at $B_n^{k+1}=...=B_n^{n-1}=0$ it follows that

$$(1+A_{k-}(\lambda))(A_k^1+...+A_k^{n-1})=(1+A_{k+}(\lambda))\ B_n^k, \qquad (36)$$

On the other hand we have

$$\begin{pmatrix} S_{11}(\lambda) & \dots & S_{1k}(\lambda) & \dots & S_{1,n-1}(\lambda) \\ S_{21}(\lambda) & \dots & S_{2k}(\lambda) & \dots & S_{2,n-1}(\lambda) \\ \dots & \dots & \dots & \dots & \dots \\ S_{k1}(\lambda) & \dots & S_{kk}(\lambda) & \dots & S_{k,n-1}(\lambda) \\ S_{k+1,1}(\lambda) & \dots & S_{k+1,k}(\lambda) & \dots & S_{k+1,n-1}(\lambda) \\ \dots & \dots & \dots & \dots & \dots \\ S_{n-1,1}(\lambda) & \dots & S_{n-1,k}(\lambda) & \dots & S_{n-1,n-1}(\lambda) \end{pmatrix} \begin{pmatrix} B_n^1 \\ B_n^2 \\ \vdots \\ B_n^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ A_k^1 + A_k^2 + \dots + A_k^{n-1} \\ A_{k+1}^1 + A_{k+1}^2 + \dots + A_{k+1}^{n-1} \\ \vdots \\ A_{n-1}^1 + A_{n-1}^2 + \dots + A_{n-1}^{n-1} \end{pmatrix}.$$

In particular we obtain

$$\Delta_k \begin{pmatrix} B_n^1 \\ \vdots \\ B_n^{k-1} \\ B_n^k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_k^1 + \dots A_k^{n-1} \end{pmatrix}.$$

From here it follows that

$$\left(\Delta_k^{-1}\right)_{kk} \left(A_k^1 + \dots + A_k^{n-1}\right) = B^k . \tag{37}$$

Using (36) and (37) we obtain (25) at $k = 2,\dots,n-1$.

5. We'll prove formulas (26). From lemma 2 and lemma 4 for $B_n^1 = B_n^2 = \dots = = B_n^{n-1} \equiv B_n$ we obtain

$$\left(1 - A_{nk+}^{2n}(\lambda)\right)^{-1}\left(1 + A_{nn+}^{2n}(\lambda)\right)B_n = \left(1 - A_{n,k+}^{2n-1}(\lambda) + A_{n-1,k+}^{2n-1}(\lambda)\right)^{-1} \times$$
$$\times \left(1 + A_{n-1,n-1+}^{2n-1}(\lambda) - A_{n,n-1+}^{2n-1}(\lambda)\right)\left(A_{n-1}^{n-1} - A_{n-1}^{k}\right).$$

Taking into account that $A_{n-1}^{n-1} - A_{n-1}^{k} = \left(S_{n-1,n-1}(\lambda) - S_{n-1,k}(\lambda)\right)B_n$ and using the arbitrariness of the choice of B_n we find, that

$$S_{n-1,n-1}(\lambda) - S_{n-1,k}(l) = 1 + Q_{k-}(\lambda), \quad k = 1,\dots,n-2 .$$

6. We'll prove formulas (27). Indeed, from lemma 2 and lemma 3

$$A_1^k = G_{2-}^k(\lambda)\left(1 + A_{k+}(\lambda)\right)B_n^k .$$

From here we find

$$S_{1k}(\lambda) = G_{2-}^k(l)\left(1 + A_{k+}(\lambda)\right), \quad k = 2,\dots,n-1 .$$

7. Finally, we'll prove formulas (28). Starting from (8) at $A_i^1 + \dots + A_i^{n-1} = 0$, $i = 1,\dots,n-2$ by the direct checking we are convinced in validity of the equality

$$\gamma_{k,n-1}(\lambda)\left(A_{n-1}^1 + \dots + A_{n-1}^{n-1}\right) = B_n^k . \tag{38}$$

From (16) and (19) at $p = 1,2,\dots,n-2$ it follows that

$$\left(1 - A_{pn-}^n(\lambda)\right)^{-1} A_{p,n-1-}(\lambda)\left(A_{n-1}^1 + \dots + A_{n-1}^{n-1}\right) = \left(1 + A_{k+}(\lambda)\right)B_n^k . \tag{39}$$

Comparing (38) and (39) we find (28).

The theorem is proved.

Theorem 3. *Let* $S(\lambda) = \left\|S_{ij}(\lambda)\right\|_{i,j=1}^{n-1}$ *be a scattering matrix for the system (1) with coefficients* $c_{ij}(x)\,(i,j = 1,\dots,n)$ *satisfying conditions (2) and let singular numbers be* *absent. Then the coefficients are uniquely defined by the known scattering matrix* $S(\lambda)$ *on a semi-axis.*

Proof. We reduce the inverse scattering problem on a semi-axis (i.e. the problem on restoration of potential of the coefficients by the known scattering matrix $S(\lambda)$) to the inverse problem for the system (1) on a whole axis with additional condition that the coefficient equals zero at $x<0$.

We introduce into consideration the transition matrix

$$\Pi(\lambda):(A_1,...,A_{n-1},B_n)^t \to (y_1(0,\lambda),...,y_k(0,\lambda))^t. \tag{40}$$

The inverse scattering problem for the linear systems of the first order on a whole axis is studied in papers [5-8]. Therefore, in order to prove the theorem it is sufficient to prove that the matrix $\Pi(\lambda)$ is unique restored by $S(\lambda)$. The matrix $\Pi(\lambda)$ for the system (1) coincides with a scattering matrix for the whole axis, when the coefficients in the system (1) are equal to zero identically at $x<0$.

Let $y^1(x,\lambda)$ be a solution of the first problem. Then

$$A_1^1 = S_{11}(\lambda)B_n^1,\ A_2^1 = S_{21}(\lambda)B_n^1,...,\ A_{n-1}^1 = S_{n-1,1}(\lambda)B_n^1$$

by virtue of lemma 2 and by the formula (3)

$$y_1^1(0,\lambda)=(1+A_{1+}(\lambda))B_n^1,\ y_2^1(0,\lambda)=...=y_{n-1}^1(0,\lambda)=0,\ y_n^1(0,\lambda)=(1+A_{1+}(\lambda))B_n^1.$$

Substituting these values in (40), we have

$$\Pi(\lambda)\begin{pmatrix} S_{11}(\lambda) \\ S_{21}(\lambda) \\ \vdots \\ S_{n-1,1}(\lambda) \\ 1 \end{pmatrix} = \begin{pmatrix} 1+A_{1+}(\lambda) \\ 0 \\ \vdots \\ 0 \\ 1+A_{1+}(\lambda) \end{pmatrix}. \tag{41}$$

Analogously, for the i-th problem $(i=2,...,n-1)$ we obtain

$$A_1^i = S_{1i}(\lambda)B_n^i,\ A_2^i = S_{2i}(\lambda)B_n^i,\ ...,\ A_{n-1}^i = S_{n-1,i}(\lambda)B_n^i$$

and from lemma 2 $y_i^i(0,\lambda)=y_n^i(0,\lambda)=(1+A_{i+}(\lambda))B_n^i$.

Using (40) we have

$$y_1^i(0,\lambda)=...=y_{i-1}^i(0,\lambda)=y_{i+1}^i(0,\lambda)=...=y_{n-1}^i(0,\lambda)=0.$$

From (40) repeating the previous arguments we obtain

$$\Pi(\lambda)\begin{pmatrix} S_{1i}(\lambda) \\ \vdots \\ S_{ii}(\lambda) \\ \vdots \\ S_{n-1,i}(\lambda) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1+A_{i+}(\lambda) \\ \vdots \\ 0 \\ 1+A_{i+}(\lambda) \end{pmatrix},\quad i=2,...,n-1. \tag{42}$$

We combine equalities (41), (42) $(i=2,...,n-1)$ into one matrix equality and obtain

$$\Pi(\lambda)\begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda) & \ldots & S_{1,n-1}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) & \ldots & S_{2,n-1}(\lambda) \\ \vdots & \vdots & \ldots & \vdots \\ S_{n-1,1}(\lambda) & S_{n-1,2}(\lambda) & \ldots & S_{n-,n-1}(\lambda) \\ 1 & 1 & \ldots & 1 \end{pmatrix}=$$

$$=\begin{pmatrix} 1+A_{1+}(\lambda) & 0 & \ldots & 0 \\ 0 & 1+A_{2}(\lambda) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1+A_{n-1+}(\lambda) \\ 1+A_{1+}(\lambda) & 1+A_{2+}(\lambda) & \ldots & 1+A_{n-1+}(\lambda) \end{pmatrix}. \tag{43}$$

On the other hand, from the representations (12) at $x=0$ we have

$$\Pi(\lambda)=\begin{pmatrix} 1+A_{11+}^{n+1}(\lambda) & A_{12}^{n+1}(\lambda) & \ldots & A_{1,n-1}^{n+1}(\lambda) & A_{1n-}^{n+1}(\lambda) \\ A_{21+}^{n+1}(\lambda) & 1+A_{22}^{n+1}(\lambda) & \ldots & A_{2,n-1}^{n+1}(\lambda) & 1+A_{2n-}(\lambda) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1+}^{n+1}(\lambda) & A_{n2}^{n+1}(\lambda) & \ldots & A_{n,n-1}^{n+1}(\lambda) & 1+A_{nn-}(\lambda) \end{pmatrix}, \tag{44}$$

$$\Pi(\lambda)\begin{pmatrix} S_{11}(\lambda) & \ldots & S_{1,n-1}(\lambda) & 0 \\ S_{21}(\lambda) & \ldots & S_{2,n-1}(\lambda) & 0 \\ \ldots & \ldots & \ldots & \ldots \\ S_{n-1,1}(\lambda) & \ldots & S_{n-1,n-1}(\lambda) & 0 \\ 1 & \ldots & 1 & 1 \end{pmatrix}=$$

$$=\begin{pmatrix} 1+A_{1+}(\lambda) & 0 & \ldots & 0 & A_{1n-}^{n+1}(\lambda) \\ 0 & 1+A_{2+}(\lambda) & \ldots & 0 & A_{2n-}^{n+1}(\lambda) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1+A_{n-1+}(\lambda) & A_{n-1n-}^{n+1}(\lambda) \\ 1+A_{1+}(\lambda) & 1+A_{2+}(\lambda) & \ldots & 1+A_{n-1+}(\lambda) & 1+A_{nn-}^{n+1}(\lambda) \end{pmatrix}. \tag{45}$$

From here

$$\Pi(\lambda)=\begin{pmatrix} 1+A_{1+}(\lambda) & 0 & \ldots & 0 & A_{1n-}^{n+1}(\lambda) \\ 0 & 1+A_{2+}(\lambda) & \ldots & 0 & A_{2n-}^{n+1}(\lambda) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1+A_{n-1+}(\lambda) & A_{n-1n-}^{n+1}(\lambda) \\ 1+A_{1+}(\lambda) & 1+A_{2+}(\lambda) & \ldots & 1+A_{n-1+}(\lambda) & 1+A_{nn-}^{n+1}(\lambda) \end{pmatrix}\times$$

$$\times\begin{pmatrix} \gamma_{11}(\lambda) & \gamma_{12}(\lambda) & \ldots \\ \gamma_{21}(\lambda) & \gamma_{22}(\lambda) & \ldots \\ \ldots & \ldots & \ldots \\ \gamma_{n-1,1}(\lambda) & \gamma_{n-1,2}(\lambda) & \ldots \\ -(\gamma_{11}(\lambda)+\ldots+\gamma_{n-1,1}(\lambda)) & -(\gamma_{12}(\lambda)+\ldots+\gamma_{n-1,2}(\lambda)) & \ldots \end{pmatrix}$$

$$\begin{pmatrix} \dots & \gamma_{1,n-1}(\lambda) & 0 \\ \dots & \gamma_{2,n-1}(\lambda) & 0 \\ \dots & \dots & \dots \\ \dots & \gamma_{n-1,n-1}(\lambda) & 0 \\ \dots - (\gamma_{1,n-1}(\lambda)+\dots+\gamma_{n-1,n-1}(\lambda)) & & 1 \end{pmatrix}. \qquad (46)$$

From (25) with the help of Riemann-Hilbert problem we find factorization multipliers $1+A_{k+}(\lambda)\ (k=1,...n-1)$. Therefore, it's sufficient to find functions $A_{in-}^{n+1}\ (i=1,...,n)$.

For this purpose from (30) and (31) using the boundary conditions (3) and lemma 2 at $A_j^1+...+A_j^{n-1}=0\ (j=2,...,n-1)$ we obtain

$$\left(\delta_{k1}+A_{k1+}^{n+1}(\lambda)\right)\left(A_1^1+...+A_1^{n-1}\right)+A_{kn-}^{n+1}(\lambda)\left(B_n^1+...+B_n^{n-1}\right)=\left(1+A_{k+}(\lambda)\right)B_n^k,$$

$$A_{n1+}^{n+1}(\lambda)\left(A_1^1+...+A_1^{n-1}\right)+\left(1+A_{nn-}^{n+1}(\lambda)\right)\left(B_n^1+...+B_n^{n-1}\right)=\sum_{k=1}^{n-1}\left(1+A_{k+}(\lambda)\right)B_n^k. \qquad (47)$$

Since $B_n^k=\gamma_{k1}(\lambda)\left(A_1^1+...+A_1^{n-1}\right), k=1,...,n-1,\ B_n^1+...+B_n^{n-1}=\left(\gamma_{11}(\lambda)+...+\gamma_{n-1,1}(\lambda)\right)\times$ $\times\left(A_1^1+...+A_1^{n-1}\right)$, then (47) takes the form

$$\delta_{k1}+A_{k1+}^{n+1}(\lambda)+A_{kn-}^{n+1}(\lambda)\left(\gamma_{11}(\lambda)+...+\gamma_{n-1,1}(\lambda)\right)=\left(1+A_{k+}(\lambda)\right)\gamma_{k1}(\lambda)\ (k=1,...,n-1),$$

$$A_{n1+}^{n+1}(\lambda)+\left(1+A_{nn-}^{n+1}(\lambda)\right)\left(\gamma_{11}(\lambda)+...+\gamma_{n-1,1}(\lambda)\right)=\sum_{k=1}^{n}\left(1+A_{k+}(\lambda)\right)\gamma_{k1}(\lambda). \qquad (48)$$

Using correlation (24) we obtain

$$A_{k1+}^{n+1}(\lambda)\left(1+R_+(\lambda)\right)^{-1}+A_{kn-}^{n+1}(\lambda)\left(1+R_-(\lambda)\right)^{-1}=\left(1+A_{k+}(\lambda)\gamma_{k1}(\lambda)-\delta_{k1}\right)\left(1+R_+(\lambda)\right)^{-1},$$
$$(k=1,...,n-1),$$

$$A_{n1+}^{n+1}(\lambda)\left(1+R_+(\lambda)\right)^{-1}+\left(1+A_{nn-}^{n+1}(\lambda)\right)\left(1+R_-(\lambda)\right)^{-1}=\sum_{k=1}^{n}\left(1+A_{k+}(\lambda)\gamma_{k1}(\lambda)\right)\left(1+R_+(\lambda)\right)^{-1}, \qquad (49)$$

where

$$\delta_{k1}=\begin{cases}1 & k=1\\ 0 & k\neq 1\end{cases}.$$

Using (24) and (49) we find $A_{kn-}^{n+1}(\lambda)\ (k=1,...,n)$.

The theorem is proved.

References

[1]. Gasymov M.G., Levitan B.M. *Defining of Dirac system by the scattering phase.* DAN SSSR, 1996, v.167, №6, p.1219-1222. (in Russian)

[2]. Nizhnik L.P., Vu F.L. *The inverse scattering problem on a semi-axis with non-self-adjoint.* Ukr.mat.journal, 1974, 26, №4, p.469-486. (in Russian)

[3]. Frolov I.S. *The inverse scattering problem for Dirac system on a whole axis.* DAN SSSR, 1972, v.207, №1, p.44-47. (in Russian)

[4]. Maksudov F.G., Veliyev S,G, *The inverse scattering problem for self-adjoint Dirac operator on a whole axis.* DAN SSSR, 1975, v.225, №6, p.1263-1266. (in Russian)

[5]. Shabat A.B. *Functional analysis and its application.* 1975, v.9, №3, p.75-79; Diff.ur., 1979, v.15, p.1824-1834. (in Russian)

[6]. Kaup D.J. *The three-wave interaction-non-dispersive phenomenon.* Studies in applied mathematics, 55, 1976, p.9-44.

[7]. Zakharov V.E., Manakov S.V., Novikov S.P., Pitayevsky L.P. *Soliton theory. The inverse problem method.* M., Nauka, 1980, 320p. (in Russian)
[8]. Gerdjikov V.S., Kulish P.P. *Expansion by squares of eigen functions of matrix linear system.* Zap.nauch.semin., LOMI AN SSSR, 1981, v.101, p,46-63. (in Russian)
[9]. Beals R., Coifman R.R. *Scattering and inverse scattering for first order systems.* Commun. on Pure and Appl.Math., v.XXXVII, p.39-90.
[10]. Iskenderov N.Sh. *The inverse scattering problem for a system of three first order ordinary differential equations on a semi-axis.* Proceeding of IMM AN Azerb., 1998, v.VIII(XVI), p.91-99. (in Russian)
[11]. Iskenderov N.Sh. *The inverse scattering problem on a semi-axis for first order ordinary differential equations system.* Transactions of AS Azerbaijan, 2000, v.XX, №4, p.97-108.
[12]. Garayev A.P. *The scattering problem for a system of the first order ordinary differential equations on a semi-axis.* Proceedings of Instute of Mathematics and Mechanics, 2001, v.15(XXIII), p.62-67.

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