MATHEMATICS

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THE APPROXIMATION BY ALGEBRAIC POLYNOMIALS OF SOLUTIONS OF ONE BOUNDARY VALUE PROBLEM WITH A PARAMETER

Abstract

In the work the boundary-valued problem is investigated. The existence and uniqueness of the solution of this equation is proved and the evasion of evasion approximate solution is installed.

In the given paper the approximate method of Dzyadyk V.K. [1] is applied to approximation by polynomials of solutions of the boundary value problem

$$\left.\begin{aligned} &y' = f_0(x,y) + \lambda f_1(x,y), \\ &y(x^0) = y_0 ,\ y(\tilde{x}) = y(x^*),\ x^0 \le \tilde{x} < x^* \le x^0 + h \end{aligned}\right\}, \tag{1}$$

where λ is a real parameter, and functions $f_0(x,y)$ and $f_1(x,y)$ are defined and continuous in the rectangle $D = \{(x,y): x^0 \le x \le x^0 + h = x^1 ,\ |y - y_0| \le b\}$ and satisfies Lipschitz condition with respect to y with constants N_0 and N_1 respectively. Besides, $f_1(x,y) \ne 0$ in $D_1 = \{(x,y): \tilde{x} \le x \le x^*, |y - y_0| \le b\}$. We notice that then function $f_1(x,y)$ preserves the sign in D_1 and therefore, changing λ by $-\lambda$ in can of necessary be assumed that $f_1(x,y) > 0$ in D_1.

We note paper [2], where using the method of successive approximations the existence and uniqueness of solution of problem (1) were proved.

Denote

$$\max_D |f_0(x,y)| = \Omega_0 ,\ \max_{D_1} |f_0(x,y)| = \tilde{\Omega}_0 ,\ \max_D |f_1(x,y)| = \Omega_1 ,\ \max_{D_1} f_1(x,y) = \tilde{\Omega}_1 ,$$

$$\min_{D_1} f_1(x,y) = \omega_1 > 0 .$$

We reduce problem (1) to the integral equation

$$\left.\begin{aligned} &y(x) = y_0 + \int_{x^0}^{x^1} e(x,t) f_0(t,y(t))dt + \lambda \int_{x^0}^{x^1} e(x,t) f_1(t,y(t))dt , \\ &\lambda = \lambda(y) = -\int_{\tilde{x}}^{x^*} f_0(t,y(t))dt \Big/ \int_{\tilde{x}}^{x^*} f_1(t,y(t))dt \end{aligned}\right\}, \tag{2}$$

where

$$e(x,t) = \begin{cases} 1, & \text{if } t \le x , \\ 0, & \text{if } t > x . \end{cases}$$

1. In mathematics the investigated in more detail and convenient for approximation of functions are sequences of different linear polynomial operators $U_n(\psi; x)$. Each of such operators will associate any function $\psi(x) \in L^\infty[x^0, x^1]$ with (generalized) polynomial [1] of type

$$U_n[\psi(\xi); x] = \sum_{k=0}^{n} a_k \varphi_k(x), \tag{3}$$

where $a_k = a_k(\psi)$ are linear functionals in $L^\infty[x^0, x^1]$, and system $\{\varphi_k(x)\}_0^n$ from $n+1$ linear independent functions $\varphi_k \in C^r[x^0, x^1]$. Number r must be not less than the order of the given equation.

We'll change equation (2) by the approximate one:

$$\left.\begin{aligned} &y_n(x) = U_n\left(y_0 + \int_{x^0}^{x^1} e(\xi,t) f_0(t, y_n(t))dt + \lambda_n \int_{x^0}^{x^1} e(\xi,t) f_1(t, y_n(t))dt; x\right), \\ &\lambda_n = \lambda(y_n) = -\int_{\tilde{x}}^{x^*} f_0(t, y_n(t))dt \Big/ \int_{\tilde{x}}^{x^*} f_1(t, y_n(t))dt \end{aligned}\right\}, \tag{4}$$

where U_n is a linear polynomial operator of type (3).

It is known that [1, lemma 1.1] solution of equation (4) (if it exists) represents itself a generalized polynomial by the system of functions $\{\varphi_k(x)\}_0^n$, i.e.

$$y_n(x) = \sum_{k=0}^{n} a_k \varphi_k(x). \tag{5}$$

Let $U_n[y_0, x] = \sum_{k=0}^{n} \alpha_k \varphi_k(x)$, $U_n[e(\xi,t); x] = \sum_{k=0}^{n} \mu_k(t)\varphi_k(x)$ substituting these expressions and expression for $y_n(x)$, defined by equality (5) in (4) for defining the coefficients a_k we obtain the following system of transcendental equations:

$$\left.\begin{aligned} &a_k = \alpha_k + \int_{x^0}^{x^1} \mu_k(t)\left[f_0\left(t, \sum_{i=0}^{n} a_i \varphi_i(t)\right) + \lambda_n f_1\left(t, \sum_{i=0}^{n} a_i \varphi_i(t)\right)\right]dt, \ \left(k = \overline{0,n}\right), \\ &\lambda_n = -\int_{\tilde{x}}^{x^*} f_0\left(t, \sum_{i=0}^{n} a_i \varphi_i(t)\right)dt \Big/ \int_{\tilde{x}}^{x^*} f_1\left(t, \sum_{i=0}^{n} a_i \varphi_i(t)\right)dt, \end{aligned}\right\} \tag{6}$$

From introduced reasoning it is clear that question on solvability of equation (4) is equivalent to the question on solvability of system of equations (6).

The following theorem on the existence and uniqueness of solution of approximate equation (4) takes place.

Theorem 1. *Let the following conditions be satisfied*

1) $\sigma_0 + h\sigma\left(\Omega_0 + \dfrac{\tilde{\Omega}_0 \Omega_1}{\omega_1}\right) \le b$;

2) $q_0 = \sigma h\left[N_0 + \dfrac{\tilde{\Omega}_0 N_1}{\omega_1}\left(1 + \dfrac{\Omega_1}{\omega_1}\right) + \dfrac{\tilde{\Omega}_1 N_0 \Omega_1}{\omega_1^2}\right] < 1$,

where

$$\sigma_0 = \max_{[x_0, x^1]} \left|U_n[y_0; x] - y_0\right|, \ \sigma = \max_{x^0 \le x, t \le x^1} \left|U_n[e(\xi,t); x]\right|.$$

Then equation (4) has a unique solution $y_n(x)$ *and this solution can be found by the successive approximations method.*

Proof. Let $y = y_n^0(x)$ be any admissible function. Consider the following successive approximations

$$\left.\begin{aligned} y_n^k(x)&=U_n\left(y_0+\int_{x^0}^{x^1}e(\xi,t)f_0\left(t,y_n^{k-1}(t)\right)dt+\lambda_n^k\int_{x^0}^{x^1}e(\xi,t)f_1\left(t,y_n^{k-1}(t)\right)dt;x\right),\\ \lambda_n^k&=-\int_{\tilde{x}}^{x^*}f_0\left(t,y_n^{k-1}(t)\right)dt\bigg/\int_{\tilde{x}}^{x^*}f_1\left(t,y_n^{k-1}(t)\right)dt,\ k=1,2,...\end{aligned}\right\}. \tag{7}$$

By virtue of condition 1)

$$\left|y_n^k(x)-y_0\right|\le\sigma_0+\sigma\Omega_0h+\frac{\tilde{\Omega}_0\sigma\Omega_1h}{\omega_1}=\sigma_0+h\sigma\left(\Omega_0+\frac{\tilde{\Omega}_0\Omega_1}{\omega_1}\right)\le b,$$

i.e. approximation process leads to admissible values of $y_n^k(x)$.

Consider the difference $y_n^{k+1}(x)-y_n^x(x)$. It's clear that

$$y_n^{k+1}(x)-y_n^k(x)=\int_{x^0}^{x^1}U_n[e(\xi,t);x]\left[f_0\left(t,y_n^k(t)\right)-f_0\left(t,y_n^{k-1}(t)\right)\right]dt+\left(\lambda_n^{k+1}-\lambda_n^k\right)\times$$

$$\times\int_{x^0}^{x^1}U_n[e(\xi,t);x]f_1\left(t,y_n^k(t)\right)dt+\lambda_n^k\int_{x^0}^{x^1}U_n[e(\xi,t);x]\cdot\left[f_1\left(t,y_n^k(t)\right)-f_1\left(t,y_n^{k-1}(t)\right)\right]dt. \tag{8}$$

We estimate the difference $\lambda_n^{k+1}-\lambda_n^k$. It's easy to check that

$$\left|\lambda_n^{k+1}-\lambda_n^k\right|\le\frac{\left(\tilde{\Omega}_0N_1+\tilde{\Omega}_1N_0\right)\int_{\tilde{x}}^{x^*}\left|y_n^k(t)-y_n^{k-1}(t)\right|dt}{\omega_1^2\left(x^*-\tilde{x}\right)}. \tag{9}$$

Then from (8) we have

$$\left|y_n^{k+1}(x)-y_n^k(x)\right|\le\sigma N_0\int_{x^0}^{x^1}\left|y_n^k(t)-y_n^{k-1}(t)\right|dt+\frac{\left(\tilde{\Omega}_0N_1+\tilde{\Omega}_1N_0\right)\sigma\Omega_1h}{\omega_1^2\left(x^*-\tilde{x}\right)}\int_{\tilde{x}}^{x^*}\left|y_n^k(t)-y_n^{k-1}(t)\right|dt+$$

$$+\frac{\tilde{\Omega}_0}{\omega_1}\sigma N_1\int_{x^0}^{x^1}\left|y_n^k(t)-y_n^{k-1}(t)\right|dt.$$

Consequently,

$$\max_{[x^0,x^1]}\left|y_n^{k+1}(x)-y_n^k(x)\right|\le\left[\sigma N_0+\frac{\left(\tilde{\Omega}_0N_1+\tilde{\Omega}_1N_0\right)\sigma\Omega_1}{\omega_1^2}+\frac{\tilde{\Omega}_0\sigma N_1}{\omega_1}\right]h\times$$

$$\times\max_{[x^0,x^1]}\left|y_n^k(x)-y_n^{k-1}(x)\right|=q_0\max_{[x^0,x^1]}\left|y_n^k(x)-y_n^{k-1}(x)\right|. \tag{10}$$

The following estimation follows from (9)

$$\left|\lambda_n^{k+1}-\lambda_n^k\right|\le\frac{\tilde{\Omega}_0N_1+\tilde{\Omega}_1N_0}{\omega_1^2}\max_{[x^0,x^1]}\left|y_n^k(x)-y_n^{k-1}(x)\right|. \tag{11}$$

Further because of (10) we obtain

$$\max_{[x^0,x^1]}\left|y_n^{k+1}(x)-y_n^k(x)\right|\le q_0^k\max_{[x^0,x^1]}\left|y_n^1(x)-y_n^0(x)\right|. \tag{12}$$

By virtue of (11) and (12) the series

$$y_n^0(x)+\sum_{k=0}^{\infty}\left[y_n^{k+1}(x)-y_n^k(x)\right],\quad \lambda_n^0+\sum_{k=0}^{\infty}\left(\lambda_n^{k+1}-\lambda_n^k\right)$$

converge absolutely and moreover convergence of the first of them is uniform relative to x. This proves the existence of the solution of equation (4).

We prove the uniqueness of the existing solution.

Let $(z_n(x),\lambda_n^*)$ be some different from $(y_n(x),\lambda_n)$ solution of equation (4). Then by analogy with (10) we have

$$\max_{[x^0,x^1]}\left|y_n(x)-z_n(x)\right|\le q_0\max_{[x^0,x^1]}\left|y_n(x)-z_n(x)\right|,$$

from which $y_n(x)\equiv z_n(x)$ follows.

2. Now we'll engage in establishing the estimation of deviation of obtained approximate solution from the exact one. Using such fact, that functions $y(x)$ and $y_n(x)$ satisfy equations (2) and (4) respectively, we'll have:

$$y(x)-y_n(x)=y(x)-U_n\left(y_0+\int_{x^0}^{x^1}e(\xi,t)f_0(t,y_n(t))dt+\lambda_n\int_{x^0}^{x^1}e(\xi,t)f_1(t,y_n(t))dt;x\right)=$$

$$=y(x)-U_n\left(y_0+\int_{x^0}^{x^1}e(\xi,t)f_0(t,y(t))dt+\lambda\int_{x^0}^{x^1}e(\xi,t)f_1(t,y(t))dt;x\right)+\int_{x^0}^{x^1}U_n[e(\xi,t);x]\times$$

$$\times[f_0(t,y(t))-f_0(t,y_n(t))]dt+\lambda\int_{x^0}^{x^1}U_n[e(\xi,t);x][f_1(t,y(t))-f_1(t,y_n(t))]dt+(\lambda-\lambda_n)\times$$

$$\times\int_{x^0}^{x^1}U_n[e(\xi,t);x]f_1(t,y_n(t))dt=y(x)-U_n[y(\xi);x]+U_n[\beta_n(\xi);x]-\beta_n(x)+\beta_n(x)+,$$

$$+\lambda\{U_n[\gamma_n(\xi);x]-\gamma_n(x)\}+\lambda\gamma_n+(\lambda-\lambda_n)\int_{x^0}^{x^1}U_n[e(\xi,t);x]f_1(t,y_n(t))dt, \qquad (13)$$

where

$$\beta_n(x)=\int_{x^0}^{x}[f_0(t,y(t))-f_0(t,y_n(t))]dt,$$

$$\gamma_n(x)=\int_{x^0}^{x}[f_1(t,y(t))-f_1(t,y_n(t))]dt.$$

Since

$$|\beta_n'(x)|=|f_0(x,y(x))-f_0(x,y_n(x))|\le N_0\|y(x)-y_n(x)\|_{C[x^0,x^1]},$$

then function $\beta_n(x)$ satisfies Lipschitz condition with the constant $N_0\|y(x)-y_n(x)\|_{C[x^0,x^1]}$. Then

$$\check{\beta}_n(x)=\beta_n\left[x^0+\frac{h(x+1)}{2}\right]\in B_0H[-1;1],$$

where

$$B_0=\frac{N_0h}{2}\|y(x)-y_n(x)\|_{C[x^0,x^1]},\ \ H=Zip_1 1.$$

Analogously we have

$$\check{\gamma}_n(x)=\gamma_n\left[x^0+\frac{h(x+1)}{2}\right]\in B_1H[-1;1],$$

where

$$B_1 = \frac{N_1 h}{2}\left\| y(x) - y_n(x)\right\|_{C[x^0,x^1]} .$$

Then taking into account the linearity of the operator U_n we obtain the following estimations

$$\left\| U_n[\beta_n(\xi);x] - \beta_n(x)\right\|_{C[x^0,x^1]} = \left\| \check{U}_n\left[\check{\beta}_n;x\right] - \check{\beta}_n(x)\right\|_{C[-1,1]} \le B_0\varepsilon_n\left(\check{U}_n;H\right),$$

$$\left\| U_n[\gamma_n(\xi);x] - \gamma_n(x)\right\|_{C[x^0,x^1]} = \left\| \check{U}_n\left[\check{\gamma}_n;x\right] - \check{\gamma}_n(x)\right\|_{C[-1,1]} \le B_1\varepsilon_n\left(\check{U}_n;H\right),$$

where

$$\varepsilon\left(\check{U}_n;H\right) = \sup_{\check{\psi}\in H[-1;1]}\left\| \check{\psi}(x) - \check{U}_n\left(\check{\psi};x\right)\right\|_{C[-1;1]}$$

and by $\check{U}_n$ we denote an operator, induced by the operator U_n with the help of passage from the segment $\left[x^0,x^1\right]$ to $[-1;1]$.

By virtue of obtained above estimation we get

$$|y(x) - y_n(x)| \le \left\| y(x) - U_n(y(\xi);x)\right\|_{C[x^0,x^1]} + \left(B_0 + \frac{\widetilde{\Omega}_0 B_1}{\omega_1}\right)\varepsilon_n\left(\check{U}_n;H\right) + \left(N_0 + \frac{\widetilde{\Omega}_0 N_1}{\omega_1}\right)\times$$
$$\times\int_{x^0}^{x}|y(t) - y_n(t)|dt + \frac{\sigma\Omega_1 h\left(\widetilde{\Omega}_0 N_1 + \widetilde{\Omega}_1 N_0\right)}{\omega_1^2\left(x^* - \widetilde{x}\right)}\int_{x^0}^{x^1}|y(t) - y_n(t)|dt . \tag{14}$$

Suppose

$$C_n = \left\| y(x) - U_n(y(\xi);x)\right\|_{C[x^0,x^1]} + \left(B_0 + \frac{\widetilde{\Omega}_0 B_1}{\omega_1}\right)\varepsilon_n\left(\check{U}_n;H\right),$$

$$P = \left(N_0 + \frac{\widetilde{\Omega}_0 N_1}{\omega_1}\right),\ N = \frac{\left(\widetilde{\Omega}_0 N_1 + \widetilde{\Omega}_1 N_0\right)\sigma\ \Omega_1 h}{\omega_1^2\left(x^* - \widetilde{x}\right)} . \tag{15}$$

Subject to (15) we find from (14)

$$|y(x) - y_n(x)| \le C_n + N\int_{x^0}^{x^1}|y(t) - y_n(t)|dt + P\int_{x^0}^{x}|y(t) - y_n(t)|dt . \tag{16}$$

Let the following condition be satisfied

$$q = \left(1 + e^{Ph}Ph\right)Nh < 1 . \tag{17}$$

Then applying the well-known theorem from [3] we'll have

$$\left\| y(x) - y_n(x)\right\|_{C[x^0,x^1]} \le C_n R_n ,$$

where

$$R_n = m + \frac{mNh(1+m)/2}{1 + mNh},\ m = 1 + Phe^{Ph} .$$

From here we easily obtain the following estimation

$$\left\| y(x) - y_n(x)\right\|_{C[x^0,x^1]} \le (1+\alpha_n)\left\| y(x) - U_n[y(\xi);x]\right\|_{C[x^0,x^1]}\cdot R_n , \tag{18}$$

where

$$\alpha_n = \begin{cases} \dfrac{\alpha_n^*}{2\omega_1 - \alpha_n^*}, \text{ if } \alpha_n^* < 1, \\ \infty, \quad \text{if } \alpha_n^* \ge 2\omega_1, \end{cases}$$

$$\alpha_n^* = h\left(\omega_1 N_0 + N_1 \widetilde{\Omega}_0\right)\varepsilon_N\left(\check{U};H\right)R_N .$$

Thus, the following theorem is proved.

Theorem 2. *Let all conditions of theorem 1 and condition (17) be satisfied. Then for any linear polynomials operator* U_n *of type (3) the polynomial* $y_n(x)$, *which is a solution of equation (4), approximates on* $\left[x^0, x^1\right]$ *the solution of equation (2) so, that condition (18) is satisfied.*

Remark. In theorem 2 the following properties of operators U_n were used:

a) linearity;
b) permutability in the sense of lemma 1.1 from [1];
c) property of these operators to give at large n good approximations of functions of Hölder classes H^1. The polynomial property of these operators was not used.

Note also that on the basis of mentioned above reasoning a system of transcendental equation (6) has a unique solution by fulfilling the conditions of theorem 1.

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