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THE CONSTRUCTION OF HOMOGENEOUS SOLUTIONS FOR TRANSVERSALLY-ISOTROPIC HOLLOW SPHERE

Abstract

The asymmetric problem of elasticity for theory of transversally-isotropic spherical shell is considered. It is supposed that the right part of the boundary spherical shell is free from stresses. The homogeneous solutions are constructed, allowing to satisfy the arbitrary boundary conditions on profile surface of the shell.

Let's consider a problem on elastic equilibrium for transversally-isotropic spherical shell. The state of sphere points in the space is determined by the spherical coordinates r,θ,φ $\left(R_1 \le r \le R_2,\, \theta_1 \le \theta \le \theta_2,\, 0 \le \varphi \le 2\pi\right)$.

Let's show the full system of equations describing the spatial stress-strain state of spherical shell.

The equation of equilibrium in stresses when mass forces are absent in spherical system of coordinate have the following form [1]

$$\begin{gathered}\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{2\sigma_r-\sigma_\theta-\sigma_\varphi+\tau_{r\theta}ctg\theta}{r}=0\ ,\\ \frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\left(\sigma_\theta-\sigma_\varphi\right)ctg\theta+3\tau_{r\theta}}{r}=0\,.\end{gathered} \tag{1}$$

The relation of generalized Hook principle have the following form:

$$\begin{aligned}&\sigma_r=G_1\left[b_{11}e_r+b_{12}\left(e_\theta+e_\varphi\right)\right],\\ &\sigma_\theta=G_1\left[b_{12}e_r+b_{22}e_\theta+b_{23}e_\varphi\right],\\ &\sigma_\varphi=G_1\left[b_{12}e_r+b_{23}e_\theta+b_{22}e_\varphi\right],\\ &\tau_{r\theta}=G_1e_{r\theta}\,.\end{aligned} \tag{2}$$

The components of tensor deformation:

$$\begin{gathered}e_r=\frac{\partial u_r}{\partial r}\ ,\ e_\theta=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\ ,\\ e_\varphi=\frac{u_r}{r}+ctg\theta\frac{u_\theta}{r}\ ,\ e_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\,,\end{gathered} \tag{3}$$

$u_r=u_r(r,\theta)$, $u_\theta=u_\theta(r,\theta)$ are components of vector permutations; b_{ij}, G_1 are material constants:

$$\begin{aligned}&mb_{11}=2G_0E_0\left(1-\nu^2\right),\ mb_{22}=2G_0\left(1-\nu_1\nu_2\right),\\ &mb_{12}=2G_0\nu_1\left(1+\nu\right),\ mb_{33}=2G_0\left(\nu+\nu_1\nu_2\right),\\ &m=1-\nu-2\nu_1\nu_2\,,\ G_0=G\cdot G_1^{-1}\,,\ E_0=E_1\cdot E^{-1}.\end{aligned}$$

Here $\nu,\nu_1,\nu_2,G,G_1,E,E_1$ are technical constants of materials.

Substituting (3), (2) in (1) after the simple calculations we'll get:

$$b_{11}\frac{\partial^2u_r}{\partial r^2}+\frac{2}{r}b_{11}\frac{\partial u_r}{\partial r}-\frac{2}{r^2}\left(b_{22}+b_{23}-b_{12}\right)u_r+\frac{1}{r^2}\frac{\partial^2u_r}{\partial\theta^2}+\frac{ctg\theta}{r^2}\cdot\frac{\partial u_r}{\partial\theta}+$$

$$+\frac{b_{12}+1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u_\theta}{\partial\theta}+ctg\theta\, u_\theta\right)+\frac{1}{r^2}(b_{12}-b_{22}-b_{23}-1)\left(\frac{\partial u_\theta}{\partial\theta}+ctg\theta\, u_\theta\right)=0\,,$$

$$\frac{b_{12}+1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{b_{22}+b_{23}+2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial u_\theta}{\partial r}+\frac{b_{22}-b_{23}-2}{r^2}u_\theta+$$

$$+\frac{b_{22}}{r^2}\frac{\partial}{\partial\theta}\left[ctg\theta u_\theta+\frac{\partial u_\theta}{\partial\theta}\right]=0\,, \tag{4}$$

Let's note that the surfaces $r=R_s$ $(s=1,2)$ we'll be called face surfaces, and the remaining parts of the boundary $(\theta=\theta_n\ ,\ n=1,2)$ as profile surfaces of sphere.

At the given paper we'll suppose that the face parts of boundary are free from stresses

$$\sigma_r=0\ ,\ \ \tau_{r\theta}=0\,. \tag{5}$$

Now we shalln't revise the boundary conditions of profile surface but we'll suppose them such that the shells are in equilibrium.

The solution of the equations (4) we'll search in the following form:

$$u_r=a(\xi)m(\theta)\ ,\quad u_\theta=b(\xi)\frac{dm(\theta)}{d\theta}\,, \tag{6}$$

where $m(\theta)$ is the solution of Lagrangian equation

$$m''(\theta)+ctg\theta\cdot m'(\theta)+\mu(\mu+1)m(\theta)=0\,. \tag{7}$$

The parameter μ is determined from the condition (5)

$$\xi=\frac{1}{\varepsilon}\ln\frac{r}{r_0},\ \ r_0=\sqrt{R_1\cdot R_2}\ ,\ \xi\in[-1,1]\ ,\ \varepsilon=\frac{1}{2}\ln\frac{R_2}{R_1}\,.$$

Substituting (6) in (4) subject to (7) after the dividing of variable with respect to the pairs of the functions $a(\xi)$, $b(\xi)$ we'll get:

$$b_{11}a''+\varepsilon b_{11}a'+\left[2(b_{12}-b_{22}-b_{33})-\left(z^2-\frac{1}{4}\right)\right]a\varepsilon^2-\left(z^2-\frac{1}{4}\right)(b_{12}+1)\varepsilon b'-$$

$$-(b_{12}-b_{22}-b_{23}-1)\left(z^2-\frac{1}{4}\right)\varepsilon^2 b=0\,,$$

$$(b_{12}+1)\varepsilon a'+\varepsilon^2(b_{22}+b_{23}+2)a+b''+\varepsilon b'-\left[b_{22}\left(z^2-\frac{1}{4}\right)-(b_{22}-b_{23}-2)\right]b\varepsilon^2=0\,. \tag{8}$$

The solution of the system (8) we'll search by the following form:

$$a(\lambda,\xi)=Ae^{\varepsilon\lambda\xi}\ ,\ b(\lambda,\xi)=Be^{\varepsilon\lambda\xi}\ ,\ z=\mu+\frac{1}{2}\,. \tag{9}$$

The solution (8) has the form:

$$a(\lambda,\xi)=A_1e^{S_1\varepsilon\xi}+A_2e^{-S_1\varepsilon\xi}+A_3e^{S_2\varepsilon\xi}+A_4e^{-S_2\varepsilon\xi}\ ,$$

$$b(\lambda,\xi)=B_1e^{S_1\varepsilon\xi}+B_2e^{-S_1\varepsilon\xi}+B_3e^{S_2\varepsilon\xi}+B_4e^{-S_2\varepsilon\xi}\ ,$$

where λ_i are roots of the biquadratic equation

$$b_{11}\lambda^2(\lambda+1)^2-\left[\left(b_{11}b_{22}-b_{12}^2-2b_{12}\right)\left(z^2-\frac{1}{4}\right)-2(b_{12}-b_{22}-b_{23})-2b_{11}(G_0-1)\right]\times$$

$$\times\lambda(\lambda+1)+\left(z^2-\frac{9}{4}\right)\left[b_{22}\left(z^2-\frac{1}{4}\right)-2(b_{12}-b_{22}-b_{23})(G_0-1)\right]=0. \tag{10}$$

Let's suppose $\lambda_n(\lambda_n+1)=\tau_n$

$$\tau^2-2q_1\tau+q_2=0, \tag{11}$$

$$2q_1=b_{11}^{-1}\left[\left(b_{11}b_{22}-b_{12}^2-2b_{12}\right)\left(z^2-\frac{1}{4}\right)-2\left(b_{12}-b_{22}-b_{23}\right)-2b_{11}\left(G_0-1\right)\right],$$

$$q_2=b_{11}^{-1}\left(z^2-\frac{1}{4}\right)\left[b_{22}\left(z^2-\frac{1}{4}\right)-2\left(b_{12}-b_{22}-b_{23}\right)\left(G_0-1\right)\right],$$

$$S_n=\sqrt{q_1-(-1)^n\sqrt{q_1^2-q_2}}\,,\quad \tau_n=\pm S_n\quad (n=1,2),$$

$$a(\lambda,\xi)=d_1e^{\varepsilon S_1\xi}C_1+\left(d_1-2S_1\right)e^{-\varepsilon\xi S_1}C_2+d_2e^{\varepsilon S_2\xi}C_3+\left(d_2-2S_2\right)e^{-\varepsilon S_2\xi}C_4,$$

$$b(\lambda,\xi)=B_1e^{\varepsilon S_1\xi}C_1+L_1e^{-\varepsilon S_1\xi}C_2+B_2e^{\varepsilon S_2\xi}C_3+L_2\,e^{-\varepsilon S_2\xi}C_4,$$

$C_i\ \left(i=\overline{1,4}\right)$ are any constants;

$$d_i=S_i(S_i+1)+b_{22}-b_{23}-2-b_{22}\left(z^2-\frac{1}{4}\right)\ (i=1,2),$$

$$B_k=-\left[\left(b_{12}+1\right)S_k+b_{22}+b_{23}+2\right]\ (k=1,2), \tag{12}$$

$$L_n=\left[\left(b_{12}+1\right)S_n-b_{22}-b_{23}-2\right]\ (n=1,2).$$

Satisfying the homogeneous boundary conditions (5) we get the following characteristics equation with respect to the parameter z:

$$\Delta(S_1,S_2,\varepsilon)=4\left[\left(A_{11}B_{12}-A_{12}B_{11}\right)\left(B_{21}A_{22}-A_{21}B_{22}\right)sh^2\left(S_2+S_1\right)\varepsilon+\right.$$
$$\left.+\left(A_{11}B_{22}-A_{22}B_{11}\right)\left(A_{21}B_{12}-A_{12}B_{21}\right)sh^2\left(S_2-S_1\right)\varepsilon\right]=0, \tag{13}$$

where

$$A_{1k}=b_{11}S_k^2(S_k+1)+2b_{12}S_k(S_k+1)+\left[\left(b_{12}^2+b_{12}-b_{11}b_{22}\right)\left(z^2-\frac{1}{4}\right)+2b_{11}\left(G_0-1\right)\right]S_k+$$
$$+2b_{12}\left(\frac{9}{4}-z^2\right)\left(G_0-1\right),$$

$$A_{2k}=-b_{11}S_k^2(S_k-1)+2b_{12}S_k(S_k-1)-\left[\left(b_{12}^2+b_{12}-b_{11}b_{22}\right)\left(z^2-\frac{1}{4}\right)+2b_{11}\left(G_0-1\right)\right]S_k+$$
$$+2b_{12}\left(\frac{9}{4}-z^2\right)\left(G_0-1\right),$$

$$B_{1k}=-b_{12}S_k^2+\left(b_{12}-b_{22}-b_{23}\right)S_k+b_{22}\left(\frac{9}{4}-z^2\right),$$

$$B_{2k}=-b_{12}S_k^2-\left(b_{12}-b_{22}-b_{23}\right)S_k+b_{22}\left(\frac{9}{4}-z^2\right)\ (k=1,2).$$

The equation (13) has the countable set of roots with the point of condensation on infinity. The roots of the equation (13) can be found numerically or as shown in [2] for thin shells in more effective asymptotic method. But at the given paper our aim is to construct only homogeneous solutions in common case therefore we shan't stop on this.

Summing of all roots of the equation (13) and allowing for (2) and (6) we'll get homogeneous solutions of the following form:

$$u_r = r_0 \sum_{k=1}^{\infty} C_k u_k(\xi) m_k(\theta),$$
$$u_\theta = r_0 \sum_{k=1}^{\infty} C_k \vartheta_k(\xi) \frac{dm_k(\theta)}{d\theta},$$
$$\sigma_r = G_1 \varepsilon^{-1} e^{-\varepsilon\xi} \sum_{k=1}^{\infty} C_k Q_{rk}(\xi) m_k(\theta),$$
$$\sigma_\theta = G_1 \varepsilon^{-1} e^{-\varepsilon\xi} \sum_{k=1}^{\infty} C_k \left[Q_{\theta k}^{(1)}(\xi) m_k(\theta) + Q_{\theta k}^{(2)}(\xi) \frac{dm_k(\theta)}{d\theta} \right],$$
$$\sigma_\varphi = G_1 \varepsilon^{-1} e^{-\varepsilon\xi} \sum_{k=1}^{\infty} C_k \left[Q_{\varphi k}^{(1)}(\xi) m_k(\theta) + Q_{\varphi k}^{(2)}(\xi) \frac{dm_k(\theta)}{d\theta} \right],$$
$$\tau_{r\theta} = G_1 \varepsilon^{-1} e^{-\varepsilon\xi} \sum_{k=1}^{\infty} C_k T_k(\xi) \frac{dm_k}{d\theta}(\theta), \qquad (14)$$

where

$$u_k(\xi) = d_1 e^{\varepsilon S_1 \xi} \Delta_{11} + (d_1 - 2S_1) e^{-\varepsilon S_1 \xi} \Delta_{12} + d_2 e^{\varepsilon S_2 \xi} \Delta_{13} + (d_2 - 2S_2) e^{-\varepsilon S_2 \xi} \Delta_{14},$$
$$\vartheta_k(\xi) = B_1 e^{\varepsilon S_1 \xi} \Delta_{11} + L_1 e^{-\varepsilon S_1 \xi} \Delta_{12} + B_2 e^{\varepsilon S_2 \xi} \Delta_{13} + L_2 e^{-\varepsilon S_2 \xi} \Delta_{14},$$
$$\Delta_{11} = e^{\varepsilon S_1} (A_{12} B_{22} - B_{12} A_{22}) B_{21} - (A_{21} B_{22} - B_{21} A_{22}) B_{12} e^{-\varepsilon(S_1 + 2S_2)} +$$
$$+ (A_{21} B_{12} - B_{21} A_{12}) B_{22} e^{\varepsilon(2S_2 - S_1)},$$
$$\Delta_{12} = -(A_{12} B_{22} - B_{12} A_{22}) B_{11} e^{-\varepsilon S_1} + (A_{11} B_{22} - B_{11} A_{22}) B_{12} e^{\varepsilon(S_1 - 2S_2)} -$$
$$- (A_{11} B_{22} - A_{12} B_{11}) B_{22} e^{\varepsilon(2S_1 + 2S_2)},$$
$$\Delta_{13} = (A_{11} B_{21} - B_{11} A_{21}) B_{22} e^{\varepsilon S_2} + (A_{21} B_{22} - B_{21} A_{22}) B_{11} e^{-(2S_1 + S_2)\varepsilon} -$$
$$- (A_{11} B_{12} - B_{11} A_{22}) B_{21} e^{\varepsilon(2S_1 -- S_2)},$$
$$\Delta_{14} = -(A_{11} B_{21} - A_{21} B_{11}) B_{12} e^{-\varepsilon S_2} - (A_{21} B_{12} - B_{21} A_{12}) B_{11} e^{-\varepsilon(2S_1 - S_2)} -$$
$$- (A_{11} B_{12} - B_{11} A_{12}) B_{21} e^{\varepsilon(2S_1 + S_2)},$$
$$Q_{rk}(\xi) = b_{11} u_k'(\xi) + \varepsilon \left[2u_k(\xi) - \left(z_k^2 - \frac{1}{4} \right) \vartheta_k(\xi) \right] b_{12},$$
$$Q_{\theta k}^{(1)}(\xi) = b_{12} u_k'(\xi) + \varepsilon (b_{22} + b_{23}) u_k(\xi) - b_{22} \left(z_k^2 - \frac{1}{4} \right) \varepsilon \vartheta_k(\xi),$$
$$Q_{\theta k}^{(2)}(\xi) = (b_{23} - b_{22}) \varepsilon \cdot ctg\theta\, \vartheta_k(\xi),$$
$$Q_{\varphi k}^{(1)}(\xi) = b_{12} u_k'(\xi) + \varepsilon (b_{22} + b_{23}) u_k(\xi) - \varepsilon \cdot b_{23} \left(z_k^2 - \frac{1}{4} \right) \vartheta_k(\xi),$$
$$Q_{\varphi k}^{(2)}(\xi) = (b_{22} - b_{23}) \varepsilon \cdot ctg\theta\, \vartheta_k(\xi),$$
$$T_k(\xi) = \vartheta_k'(\xi) + \varepsilon [u_k(\xi) - \vartheta_k(\xi)].$$

Here c_k are any constants.

As in [3], we can prove that the system of homogeneous solutions (14) satisfy the generalized conditions of orthogonality, that allows to solve the problem for the sphere only at the mixed boundary conditions on profile surface of sphere.

At all other cases we have to appeal to different approximate methods. Therefore let's consider the question on satisfying the boundary conditions on profile surfaces of sphere by view of the class of homogeneous solutions in common cases. Let's suppose for simplicity that the middle surface represents the sphere with one circular perforation. Let at $\theta=\theta_1$ the system of stresses be given

$$\sigma_\theta = f_1(\xi),\ \tau_{r\theta} = f_2(\xi) \quad \text{for} \quad \theta=\theta_1 \tag{15}$$

moreover the condition of equilibrium

$$\int_{R_1}^{R_2} [f_1(\xi)\sin\theta + f_2(\xi)\cos\theta] e^{2\varepsilon\xi}\, d\xi = 0 \tag{16}$$

be fulfilled.

We'll accept the coefficients C_k as generalized parameters. Since the homogeneous solutions exactly satisfy the equilibrium equations and boundary condition on face, the Lagrangian possible permutation principle has the following form:

$$\int_{R_1}^{R_2} (\sigma_\theta \delta u_\theta + \tau_{r\theta}\delta u_r) r dr = \int_{R_1}^{R_2} [f_1(r)\delta u_\theta + f_2(r)\delta u_r] r dr\,. \tag{17}$$

Equating to zero the coefficients at independent variables we'll get the following infinite system:

$$\sum_{k=1}^{\infty} M_{kp} C_k = N_p \qquad (p = 1,2,\ldots). \tag{18}$$

Here

$$M_{kp} = \int_{-1}^{1} \left[\left[Q_{\theta k}^{(1)} m_k(\theta_1) + Q_{\theta k}^{(2)} \frac{dm_k(\theta_1)}{d\theta} \right] \vartheta_p(\xi) \frac{dm_p(\theta_1)}{d\theta} + \right.$$

$$\left. + T_k(\xi) u_p(\xi) \frac{dm_k(\theta_1)}{d\theta} m_p(\theta_1) \right] e^{\varepsilon\xi} d\xi\,,$$

$$N_p = \int_{-1}^{1} \left[f_1(\xi)\vartheta_p(\xi) \frac{dm_p(\theta_1)}{d\theta} + f_2(\xi) u_p(\xi) m_p(\theta) \right] e^{2\varepsilon\xi} \varepsilon\, d\xi\,.$$

The elements of these matrices don't depend on the form of loading on profile surface and therefore the conversion may be realized once forever.

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Received September 3, 2001; Revised December 5, 2001.
Translated by Mamedova V.I.