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BUCKLING OF MULTILAYER NON-LINEARLY- ELASTIC RODS

Abstract

Results of theoretical investigation of the problem on loss of load-carrying capacity of multilayer non-linearly-elastic roads for different forms of attaching are given in the paper. Stated problem is solved by modified variational method of mixed type in combination with Rayleigh -Ritz method. Theoretically definition of critical force is reduced to solving Cauchy problem. As example we can consider two-layer rod. Influence of physico-mechanical, geometrical parameters of system and boundary conditions on the value of warping critical force is obtained. Particularly, for zero eccentricity Shenly critical forces are obtained for received attachings.

By describing the properties of composite materials the elasticity linear theory is often applied. However not all the structural heterogeneous materials can be described with position of linear elasticity theory. In various constructions the thin-shelled elements made of composite materials with piecewise non- homogeneity property on thickness are used as carried elements, and their behaviour is characterized by physical non-linearity. And what is more in the buckling problems it's also necessary to consider the geometrical non-linearity. By virtue of mathematical complexity here the obtaining of effective analytical solutions is very difficult and it's occasionally impossible. It's associated with necessity of determination of solutions of non-linear boundary value problems with disconnected coefficients. Therefore the necessity in application of the approximated methods of solutions to such problems in particular variational, in combination with the Rayleigh-Ritz method) arises. The stability of non-homogeneous by thickness linearly elastic rods for different form of fastening is explicitly studied in [1]. In the suggested paper the generalization of these problems in the case of non-linear elasticity is given. The influence of geometrical and physical parameters to the quantity of critical force of buckling is numerically shown.

1. Introduce in consideration a rectangular with the length l and the thickness $2h$ in plane of rod. Assume that it consists of S different layers by thickness with the different elasticity modulus E_{k+1} and the proportionality limits σ_{k+1}^{0} $[k = 0,1,2,...,(S-1)]$. In addition it's assumed that the division of layers is parallel realized to its lateral bounds. We'll assume that in every layer the elasticity modulus and the proportionality limit depend on the cross coordinate z, i.e. $E_{k+1} = E_{k+1}(z)$, and $\sigma_{k+1} = \sigma_{k+1}(z)$. We denote the thickness of every layer by δ_k. Thus $\sum_{k=1}^{S} \delta_k = 2h$. The considered here approach is based on the following assumptions: a) the contact condition between the layers of pile is in their rigid coupling from this the equality on their displacements, stresses and the absence of mutual pressure of layers follows; b) by virtue of thin-shellness the normal stress σ by thickness of rod varies by a linear low. The accuracy estimation of this approximation is given in [2]. If the hypothesis of plane cross section is guided, then the assumptions a) are automatically satisfied. Taking the hypothesis b) we'll have

$$\sigma = \frac{N(x)}{2h} + \frac{3}{2h^3} M z, \tag{1.1}$$

where N is force and M is moment. By made assumptions the state equation for pile on the whole we write in the form of one equality

$$e^{\vee}=\frac{\sigma}{E_{k+1}(z)}\left\{1+\left[\frac{\sigma}{\sigma_{k+1}^{0}(z)}\right]^{2}\right\} \quad a_k\le z\le a_{k+1}. \tag{1.2}$$

Here for brevity of notation we introduce the designation $a_k=-h+\sum_{j=0}^{k}\delta_j$ $(\delta_0=0)$. The state equation (1.2) is sufficiently good approximation of the non-linear elasticity law for a pile composed of reinforced plastic or some aluminum alloys and duralumins. Now consider the stability of chosen rod with the centrally compressed force T. For further solution of the stated problem we use the variational method of mixed type [3]. We introduce the Cartesian system of coordinates with origin at the point $z=0$ and we direct the axis along the length of rod. In this case taking into account that one measure is unit and allowing only for non-linearity of deflection we can write the used function in the form of [1]:

$$R=R_y+R_H\,, \tag{1.3}$$

where R_y is its expression for linear elasticity

$$R_y=-\int_0^l w_{,x}\dot{w}_{,x}dx+\int_0^l \dot{M}\dot{w}_{,xx}dx-\frac{T}{2}\int_0^l \dot{w}_{,x}^2dx-\frac{1}{2}\int_0^l\sum_{k=0}^{S-1}\int_{a_k}^{a_{k+1}}\frac{\dot{\sigma}^2}{E_{k+1}(z)}dxdz\,, \tag{1.4}$$

and R_H is its non-linear component which gets the form

$$R_H=-\frac{3}{2}\int_0^l\sum_{k=0}^{S-1}\int_{a_k}^{a_{k+1}}\frac{\dot{\sigma}^2}{E_{k+1}(z)}\left\{\frac{\sigma}{\sigma_{k+1}(z)}\right\}^2dxdz\,. \tag{1.5}$$

In the expressions of R_y and R_y under the dot we understand the differentiation with respect to T, i.e. $\dot{T}=1$.

2. At first we consider the stability of multi-layer rod when the both ends are rigidly fastened. Then according to the Rayleigh-Ritz method for monomial approximation [1] we write the deflection function w and the moment M in the form of

$$w(x,T)=a(T)\sin\frac{\pi x}{l}\cdot\sin\pi\left(1-\frac{x}{l}\right), \tag{2.1}$$

$$M(x,T)=m(T)\cos\frac{\pi x}{l}\cdot\cos\pi\left(1-\frac{x}{l}\right). \tag{2.2}$$

The further operation of calculations is in that we substitute the relations (1.1), (2.1) and (2.2) to the expression of the functional (1.3) and after the integration we find it as the function a, m and the derivatives of these quantities with respect to T. Further if we equate

$$\partial R/\partial\dot{a}=0 \text{ to } \partial R/\partial\dot{m}=0$$

we obtain a system of two ordinary differential equations:

$$-(aT)^{\bullet}+\dot{m}=0\,, \tag{2.3}$$

$$\frac{\pi^2\dot{a}}{2l}+\frac{3l\eta_{1,2}}{64h^4}-\frac{27l\dot{m}}{32h^6}\eta_{2,2}-\frac{9T^2l\gamma_{1.2}}{32h^6}-\frac{81Tml}{64h^8}\gamma_{2,2}-\frac{405m^2l}{256h^{10}}\gamma_{3,2}-$$
$$-\frac{81T^2\dot{m}l}{128h^8}\gamma_{2,2}-\frac{405Tm\dot{m}l}{128h^{10}}\gamma_{3,2}-\frac{8505m^2\dot{m}l}{2048}\gamma_{4,2}=0\,. \tag{2.4}$$

Here for brevity of notation the following designations are introduced

$$\eta_{i,s}=\sum_{k=0}^{S-1}\int_{a_k}^{a_{k+1}}\frac{z^i dz}{E_{k+1}(z)}\quad,\qquad \gamma_{i,s}=\sum_{k=0}^{S-1}\int_{a_k}^{a_{k+1}}\frac{z^i dz}{E_{k+1}(z)\sigma_{k+1}^{0\,2}(z)}\,. \tag{2.5}$$

It's necessary to complete a system of the equations (2.3)-(2.4) by initial conditions which starting from physics of the phenomenons are concluded in absence of moment for $T=0$ and in the presence of initial imperfection that we can write in the following form

$$m(0)=0\,;\quad w(0)=w_0\,, \tag{2.6}$$

where w_0 is prescribed amplitude of initial imperfection. Using the first condition of (2.6) and introducing the pure quantities

$$c=\frac{a}{h},\ \tau=\frac{T}{T_0}\,,\ \xi=\frac{h}{l}\,,\ \chi_1=\frac{\eta_{1,2}}{h^3}T_0\,,\ \chi_2=\frac{E_1\eta_{2,2}}{h^3},\chi_3=\frac{E_1\gamma_{1,2}}{h^4}T_0^2\,,$$

$$\chi_4=\frac{\gamma_{2,2}T_0^3}{h^6}\,,\ \chi_5=\frac{\gamma_{3,2}}{h^7}T_0^3\,,\ \chi_6=\frac{\gamma_{4,2}}{h^8}T_0^3 \tag{2.7}$$

by means of substitution of the system (2.3)-(2.4) after the series of transformations we lead to the equation

$$\frac{d\tau}{dc}=0{,}5\pi^2\xi-0{,}56\pi^2\xi\tau\chi_2-0{,}63\xi^{-1}\tau^3\chi_4-3{,}16\xi^{-1}c\tau^3\chi_5-4{,}15\xi^{-1}c^2\tau^3\chi_6\Big/$$

$$\Big/0{,}56\pi^2\xi c\chi_2+0{,}05\xi^{-1}\chi_1+0{,}19\pi^2\xi\tau^2\chi_3+1{,}89\xi^{-1}c\tau^2\chi_4+$$

$$+4{,}74\xi^{-1}c^2\tau^2\chi_5+4{,}15\xi^{-1}c^3\tau^2\chi_6\,, \tag{2.8}$$

where we can write the Euler force T_0 as

$$T_0=\frac{2}{3}\pi^2\frac{h^3}{l^2}E_1\,.$$

At zero eccentricity the critical force of stability is determined from the solution of the cubic equation (the vanishing condition of the numerator of equation (2.8))

$$0{,}5\pi^2\xi-0{,}56\pi^2\xi\chi_2\tau-0{,}63\xi^{-1}\chi_4\tau^3=0\,. \tag{2.9}$$

It's the Shenli critical force.

Now let the case of combined fastening be realized. Without losing generality we assume that when $x=l$ the rod is rigidly fastened, and when $x=0$ the hinge support is realized. In this case we give the bending and moment forms by the follows formulas [1]

$$\begin{aligned} w(x,T)&=a(T)\sin\frac{\pi x}{l}\cdot\sin\frac{\pi}{2}\left(1-\frac{x}{l}\right),\\ M(x,T)&=m(T)\cos\frac{\pi x}{l}\cdot\cos\frac{\pi}{2}\left(1-\frac{x}{l}\right). \end{aligned} \tag{2.10}$$

As a result of procedure fulfilled analogously to previous one we obtain solving equation in the form of

$$\frac{d\tau}{dc}=0{,}25\pi^2\xi-0{,}37\pi^2\chi_2\xi\tau-3{,}02\chi_4\xi^{-1}\tau^3-6{,}04\chi_5\xi^{-1}c\tau^3-1{,}19\chi_6\xi^{-1}c^2\tau^3\Big/$$

$$\Big/0{,}5\pi^{-1}\chi_1\xi^{-1}+0{,}37\pi^2\chi_2\xi c+0{,}25\chi_3\xi\tau^2+3{,}86\chi_4\xi^{-1}\tau^2c+$$

$$+9{,}06\chi_5\xi^{-1}\tau^2c^2+1{,}19\chi_6\xi^{-1}\tau^2c^3\,, \tag{2.11}$$

whence the Shenli force is determined from the cubic equation

$$0,25\pi^2\xi - 0,37\pi^2\chi_2\xi\tau - 3,02\chi_4\xi^{-1}\tau^3 = 0 . \qquad (2.12)$$

The corresponding equation for hinge support has the from [4]

$$\frac{d\tau}{dc} = 0,5\pi^2\xi - 0,75\pi^2\chi_2\xi\tau - 0,84\chi_4\xi^{-1}\tau^3 - 13,5\pi\chi_5\xi^{-1}\tau^3 c - 5,69\chi_6\xi^{-1}\tau^3 c^2 \Big/$$

$$\Big/ 1,5\pi^{-1}\chi_1\xi^{-1} + 0,75\pi^2\chi_2\xi c + 1,25\chi_3\xi\tau^2 + 2,25\chi_4\xi^{-1}\tau^2 c +$$

$$+ 20,25\chi_5\xi^{-1}\tau^2 c^2 + 5,69\chi_6\xi^{-1}\tau^2 c^3 . \qquad (2.13)$$

Here the Shenli force is determined from the solution of the equation

$$0,5\pi^2\xi - 0,75\pi^2\chi_2\xi\tau - 0,84\chi_4\xi^{-1}\tau^3 = 0 . \qquad (2.14)$$

3. Let's assume that every layer is homogeneous $E_{k+1} = const$ and $\sigma_{k+1}^0 = const$. As an example we consider the case of two-layer rod with the thickness δ_1, δ_2 the elasticity modulus E_1, E_2 and the proportionality limits σ_1^0, σ_2^0. Allowing for the said on the basis of formulas (2.5) and (2.7) it's easy to count the values χ_j $\left(j = \overline{1,6}\right)$. They have the forms

$$\chi_1 = \frac{4\xi^2\beta(-1+\alpha)\pi^2}{3(1+\beta)^2}, \quad \chi_2 = \frac{2\left[1+3\beta^2+\alpha\beta\left(3+\beta^2\right)\right]}{3(1+\beta)^3}, \quad \chi_3 = \frac{2^3\nu^2\xi^4\beta\left(-1+\alpha\mu^2\right)\pi^4}{3^2(1+\beta)^2},$$

$$\chi_4 = \frac{2^4\xi^6\nu^2\left[1+3\beta^2+\alpha\beta\mu^2\left(3+\beta^2\right)\right]\pi^6}{3^4(1+\beta)^3}, \quad \chi_5 = \frac{2^4\xi^6\nu^2\beta\left(\alpha\mu^2-1\right)\left(1+\beta^2\right)\pi^6}{3^3(1+\beta)^4},$$

$$\chi_6 = \frac{2^4\xi^6\nu^2\left[1+10\beta^2+5\alpha\beta\mu^2\left(1+2\beta^2\right)+5\beta^4\right]\pi^6}{5\cdot 3^3(1+\beta)^5},$$

where in addition the following pure quantities are introduced

$$\alpha = \frac{E_1}{E_2}, \quad \mu = \frac{\sigma_1^0}{\sigma_2^0}, \quad \beta = \frac{\delta_2}{\delta_1}, \quad \nu = \frac{E_1}{\sigma_1^0}.$$

Table 1

$\mu = 0,25 ; \beta = 4$

α	0.25	0.75	1.25	1.75	2.25	2.75	3.25	3.75
$\tau_{cr}^{ж}$	0.182	0.171	0.164	0.158	0.152	0.147	0.142	0.137
τ_{cr}^{k}	0.177	0.161	0.150	0.134	0.125	0.119	0.114	0.111
$\tau_{cr}^{ш}$	0.180	0.167	0.157	0.146	0.139	0.133	0.128	0.124

Table 2

$\alpha = 0,25 ; \beta = 4$

μ	0.25	0.75	1.25	1.75	2.25	2.75	3.25	3.75
$\tau_{cr}^{ж}$	0.182	0.162	0.137	0.115	0.101	0.092	0.085	0.079
τ_{cr}^{k}	0.177	0.148	0.126	0.109	0.094	0.081	0.070	0.062
$\tau_{cr}^{ш}$	0.180	0.155	0.131	0.112	0.097	0.086	0.078	0.071

Table 3

$\alpha = \mu = 0{,}25$

β	0.5	1	1.5	2	2.5	3
$\tau_{cr}^{ж}$	0.163	0.167	0.169	0.172	0.174	0.176
τ_{cr}^{k}	0.143	0.157	0.164	0.169	0.174	0.173
$\tau_{cr}^{ш}$	0.153	0.162	0.167	0.170	0.172	0.174

Assuming $\xi = 10^{-1}$, $\nu = 3 \cdot 10^{2}$ the Cauchy problem for the equations (2.8), (2.11) and (2.13) under the initial condition $c(0) = 10^{-1}$ is numerically solved by the Runger-Kuth method. In tables 1,2 and 3 the dependencies of the critical force τ_{cr} on the quantities α, μ and β are mentioned. The values of critical forces are determined from the equality $d\tau/dc = 0$. By homogeneity $(\mu = \alpha = \beta = 1)$ we have

$$\tau_{cr}^{ж} = 0{,}101 \ , \quad \tau_{cr}^{k} = 0{,}069, \quad \tau_{cr}^{ш} = 0{,}085 \ .$$

Taking $\alpha = \beta = 0{,}25$ and $\beta = 4$ we cite the numerical values of the Shenli critical forces

$$\tau_{cr}^{ж} = 0{,}224 \ , \quad \tau_{cr}^{k} = 0{,}193, \quad \tau_{cr}^{ш} = 0{,}202 \ .$$

The corresponding values from the homogeneous case will be the following

$$\tau_{cr}^{ж} = 0{,}132 \ , \quad \tau_{cr}^{k} = 0{,}092, \quad \tau_{cr}^{ш} = 0{,}113 \ .$$

The variant of linear elasticity is automatically obtained when $\chi_j = 0 \ \left(j = \overline{3,6}\right)$.

Thus by chosen values of parameters we can conclude the followings:

- as it follows to expect, accounting the physical non-linearity the value of critical forces decreases in comparison with linear elasticity;
- the buckling critical force in rigid fastening is more than in combined and hinge fastening, and in turn in combined fastening the critical force is less than in hinge support;
- at the fixed β the increase of α leads to decrease of τ_{cr}, that is completely explained by the fixing property of E_1 because the decrease of α is associated with the decrease E_2 the elasticity modulus of the second layer, which leads to decrease of the general rigidity of a rod;
- the variation of the values μ doesn't change the quality picture of critical state at given β;
- from the data of table 3 it follows that the substitution of the last rigid material in the pile to the more rigid, leads to increase of critical force.

In conclusion it's necessary to note that by constructing the non-homogeneity we can increase (decrease) the bucking critical force and by the same token in specific sense to optimize the construction.

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