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ON A QUESTION OF EXISTENCE IN APPROXIMATION BY MIXED BILINEAR FORMS

Abstract

The existence of the best approximate functions in approximation functions of much variables with mixed bilinear forms are established.

The present paper is devoted to the questions of the best approximation by variety of mixed bilinear forms in space L^2 with the help of exact annihilator of approximate apparatus, which was constructed earlier in [1].

The existence of the best approximating bilinear form is established.

By $L^2(I^3)$, $I=[0;1]$ we denote a space of real square integrable functions $h(x,y,z): I^3 \to R$

$$\|h\|^2 \overset{def}{=} \int_{I^3} |h(x,y,z)|^2 \, dx\, dy\, dz < \infty .$$

Denote

$$\Phi_n = \left\{ S_n = \sum_{k=1}^{n} f_k(x,y) g_k(y,z);\ f_k g_k \in L^2(I^3) \right\}, \tag{1}$$

where $f_k \in L^2(I^2)$, $g_k \in L^2(I^2)$.

We define "scalar product" by the following rule:

for all $y \in I$

$$< u, \mathrm{v}; y >= \int_{I^2} u(x,y)\mathrm{v}(y,z)dx\; dz\ , \tag{2}$$

where $u \in L^2(I^2)$, $\mathrm{v} \in L^2(I^2)$.

Consider the best approximation of the function $h \in L^2(I^3)$ by the set Φ_n

$$\rho_n(h) = \inf_{n \in \Phi_n} \|h - S_n\| . \tag{3}$$

Since $\Phi_n \subset \Phi_{n+1}$, $n = 1,2,...$, then

$$\|h\| \ge \rho_1(h) \ge \rho_2(h) \ge \dots . \tag{4}$$

The main result is the following

Theorem. *For every* $h(x,y,z) \in L^2(I^3)$ *and* $n \in N$ *there exists at least one function* $S_n \in \Phi_n$ *satisfying the following correlation*

$$\|h - S_n\| = \rho_n(h).$$

In order to prove the theorem we need the quantity

$$\det C_n h = \begin{vmatrix} h(x_1,y,z_1) & \dots & h(x_1,y,z_n) \\ \vdots & & \vdots \\ h(x_n,y,z_n) & & h(x_n,y,z_n) \end{vmatrix}, \tag{5}$$

where $x_i \in I$, $z_j \in I$, $i,j = \overline{1,n}$.

We'll establish series of auxiliary results:

Lemma 1. *1) For every function* $h(x,y,z)\in\Phi_n$ *the following correlation is fulfilled*

$$\det C_{n+1}h\left(x^*,y,z^*\right)=0 \text{ for any}$$
$$x^*=\left(x_1,...,x_{n+1}\right)\in I^{n+1},\ z^*=\left(z_1,...,z_{n+1}\right)\in I^{n+1},\ y\in I; \tag{6}$$

2) if (6) is fulfilled for some $n\ge 1$ *an additionally*

$$\det C_n h\left(x^0,y,z^0\right)\neq 0 \tag{7}$$

holds for some $x^0\in I^n$ *and* $z^0\in I^n$, *then* h *has the following form*

$$\sum_{k=1}^{n} f_k(x,y)g_k(y,z) \text{ on } I^3,$$

where $f_k\in L^2\left(I^2\right)$, $g_k\in L^2\left(I^2\right)$.

Proof of part I. Let h have the form

$$h\equiv\sum_{k=1}^{n} f_k(x,y)g_k(y,z) \text{ on } I^3.$$

We have to prove that for any $x^*\in I^{n+1}$, $z^*\in I^{n+1}$ determinant

$$\begin{vmatrix} h(x_1,y,z_1) & h(x_1,y,z_2) & ... & h(x_1,y,z_n) & h(x_1,y,z_{n+1}) \\ \vdots & \vdots & & \vdots & \vdots \\ h(x_n,y,z_1) & h(x_n,y,z_2) & ... & h(x_n,y,z_n) & h(x_n,y,z_{n+1}) \\ h(x_{n+1},y,z_1) & h(x_{n+1},y,z_2) & ... & h(x_{n+1},y,z_n) & h(x_{n+1},y,z_{n+1}) \end{vmatrix}=0.$$

This follows from [1], where in more general case, using exact annihilator, when bilinear form represents the sum of dual products of functions, whose sets of variables have non-empty intersection. A family of continuous operators ∇_θ from $C(T)$ in $C(H)$ such, that the following correlation is valid $f\in B^{M-1}_{C(T)}\Leftrightarrow\nabla_\theta f=0\ \forall\theta\in(H)$

$$\overset{M}{\nabla} f=\left(\overset{M}{\nabla} f\right)_\theta=def\left\|f\left(x_i;y,z_j\right)\right\|_{i,j=1}^{M}=\begin{vmatrix} f(x_1,y,z_1) & ... & f(x_1,y,z_M) \\ ... & ... & ... \\ f(x_M,y,z_1) & ... & f(x_M,y,z_M) \end{vmatrix}$$

is called on exact annihilator of the set

$$Б^{M-1}_C(\Gamma,T)=\left\{\beta\left|\beta=\sum_{k=1}^{M-1}\varphi_k(x,y)\psi_k(y,z),\ \varphi_k\in C(X\times Y),\ \psi_k\in C(Y\times Z)\right.\right\}.$$

In [1] it was proved that $\overset{M}{\nabla} f$ is an exact annihilator of bilinear form.

Proof of part II. Suppose that h satisfied (6) and (7) for some integer $n\ge 1$. We fix $x^0=\left(x_1^0,x_2^0,...,x_n^0\right)\in I_n$ and $z^0=\left(z_1^0,z_2^0,...,z_n^0\right)\in I^n$, satisfying (7) and use (6) with arbitrary $\left(x_1,...,x_n,x_{n+1}\right)=x^*$ and $\left(z_1,z_2,...,z_n,z_{n+1}\right)=z^*$, $z=z_{n+1}$, $x=x_{n+1}$.

Then equality (6) will be written in the following form:

$$\begin{vmatrix} h(x_1,y,z_1) & h(x_1,y,z_2) & ... & h(x_1,y,z_n) & h(x_1,y,z) \\ \vdots & \vdots & & \vdots & \vdots \\ h(x_n,y,z_1) & h(x_n,y,z_2) & ... & h(x_n,y,z_n) & h(x_n,y,z) \\ h(x,y,z_1) & h(x,y,z_2) & ... & h(x,y,z_n) & h(x,y,z) \end{vmatrix}=0. \tag{8}$$

Decomposing this determinant relative to the last row, we obtain

$$\sum_{i=1}^{n}(-1)^{n+i+1}h(x,y,z_i)\varphi_i(y,z)+h(x,y,z)\det C_n h(x^0,y,z^0)=0, \tag{9}$$

where functions $\varphi_i : I^2 \to R$ have the following form

$$\varphi_1(y,z)=\begin{vmatrix} h(x_1,y,z_2) & \dots & h(x_1,y,z) \\ \vdots & & \vdots \\ h(x_n,y,z_2) & \dots & h(x_n,y,z) \end{vmatrix},$$

$$\varphi_i(y,z)=\begin{vmatrix} h(x_1,y,z_1) & \dots & h(x_1,y,z_{i-1}) & h(x_1,y,z_{i+1}) & \dots & h(x_1,y,z) \\ \vdots & & \vdots & \vdots & & \vdots \\ h(x_n,y,z_1) & \dots & h(x_n,y,z_{i-1}) & h(x_n,y,z_{i+1}) & \dots & h(x_n,y,z) \end{vmatrix}, \tag{10}$$

where $i=\overline{2,n}$.

Decomposing determinants $\varphi_i(y,z)$, $i=\overline{1,n}$ relative to the last column, we obtain

$$\varphi_i(y,z)=\sum_{i=1}^{n}(-1)^{i+j}h(x_j,y,z_i)\left(\sum_{j=1}^{n}\alpha_{ij}\det C_n h(x^0,y,z^0)\right), \tag{11}$$

where

$$\alpha_{ij}=\frac{\begin{vmatrix} h(x_1,y,z_1) & \dots & h(x_1,y,z_{i-1}) & h(x_1,y,z_{i+1}) & \dots & h(x_1,y,z_n) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h(x_{j-1},y,z_1) & \dots & h(x_{j-1},y,z_{i-1}) & h(x_{j-1},y,z_{i+1}) & \dots & h(x_{j-1},y,z_n) \\ h(x_{j+1},y,z_1) & \dots & h(x_{j+1},y,z_{i-1}) & h(x_{j+1},y,z_{i+1}) & \dots & h(x_{j+1},y,z_n) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h(x_n,y,z_1) & \dots & h(x_n,y,z_{i-1}) & h(x_n,y,z_{i+1}) & \dots & h(x_n,y,z_n) \end{vmatrix}}{\det C_n h(x^0,y,z^0)},$$

$$i=\overline{2,n}\,;\quad j=\overline{2,n}.$$

From (9) we have

$$h(x,y,z)=\frac{\sum_{i=1}^{n}(-1)^{n+i}h(x,y,z_i)\varphi_i(y,z)}{\det C_n h}.$$

Using (10) and mentioned above α_{ij} from the last correlation, we obtain

$$h(x,y,z)=\sum_{i=1}^{n}(-1)^{i+j}h(x,y,z_i)\left(\sum_{j=1}^{n}\alpha_{ij}h(x_j,y,z)\right). \tag{12}$$

Note that (12) is decomposition at the form

$\sum_{i=1}^{n} f_i(x,y)g_i(y,z)$ with $f_i=h(x,y,z_i)$ and $g_j=\sum_{j=1}^{n}\alpha_{ij}h(x_j,y,z)$, $i,j=\overline{1,n}$.

Lemma 1 is proved.

Lemma 2. *Let $h\in L^2(I^3)$. We define mapping $B: L^2(I^3)\to K_y$ by the following rule*

$$B(u,\mathrm{v})=B(u,\mathrm{v})_y=\int_{I^2}h(x,y,z)u(x,y)\mathrm{v}(y,z)\,dx\,dz. \tag{13}$$

Then h is contained in Φ_n for given n iff

$$\begin{vmatrix} B(u_1, \mathrm{v}_1) & \dots & B(u_1, \mathrm{v}_{n+1}) \\ \dots & \dots & \dots \\ B(u_{n+1}, \mathrm{v}_1) & \dots & B(u_{n+1}, \mathrm{v}_{n+1}) \end{vmatrix} = 0 \qquad (14)$$

holds for any $u_1, u_2, \dots, u_{n+1} \in L^2(I^2)$, $\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_{n+1} \in L^2(I^2)$.

Proof. Necessity. Let $h \in \Phi_n$, i.e.

$$h = \sum_{k=1}^{n} f_k(x,y) g_k(y,z),$$

where $f_k \in L^2(I^2)$, $g_k \in L^2(I^2)$, $1 \le k \le n$. Using this in (13), we obtain

$$B(u,\mathrm{v})_y = \int_{I^2} hu\mathrm{v}dx\,dz = \int_{I^2} \sum_{k=1}^{n} f_k(x,y) g_k(y,z) u(x,y) \mathrm{v}(y,z) dx\,dz =$$

$$= \sum_{k=1}^{n} \int_I \left(\int_I [f_k(x,y) u(x,y)] dx \right) [g_k(y,z) \mathrm{v}(y,z)] dz = \sum_{k=1}^{n} F_k(u)_y G_k(\mathrm{v})_y ,$$

where

$$F_k(u)_y = \int_I f_k(x,y) u(x,y) dx , \quad G_k(\mathrm{v})_y = \int_I g_k(y,z) \mathrm{v}(y,z) dz .$$

So, $B(u,\mathrm{v})_y \in \Phi_n$ consequently by lemma 1 equality (14) is fulfilled each time as soon as

$$u_1, u_2, \dots, u_{n+1} \in L^2(I^2) \text{ and } \mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_{n+1} \in L^2(I^2)$$

holds.

Sufficiency. Let bilinear form (13) satisfy (14). Throwing out the trivial case $B(u,\mathrm{v}) \equiv 0$ we obtain, that there exists such $1 \le k \le n$, for which

$$\begin{vmatrix} B(u_1, \mathrm{v}_1) & B(u_1, \mathrm{v}_2) & \dots & B(u_1, \mathrm{v}_k) \\ \dots & \dots & \dots & \dots \\ B(u_k, \mathrm{v}_1) & B(u_k, \mathrm{v}_2) & \dots & B(u_k, \mathrm{v}_k) \end{vmatrix} \ne 0 \qquad (15)$$

holds for some $u_1, u_2, \dots, u_k \in L^2(I^2)$, $\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_k \in L^2(I^2)$.

Without losing generality we take $k = n$.

Then by lemma 1 B has the following form

$$B(u,\mathrm{v}) = \sum_{k=1}^{n} F_k(u) G_k(\mathrm{v}), \qquad (16)$$

where

$$F_k(u) = \sum_{i=1}^{n} c_{ki} B(u, \mathrm{v}_i) \text{ and } G_k = \sum_{i=1}^{n} \alpha_{ki} B(u_i, \mathrm{v}),$$

where c_{ki} and d_{ki} are corresponding constants.

Using inequality (13), we obtain

$$F_k(u) = \sum_{i=1}^{n} c_{ki} B(u, \mathrm{v}_i) = \sum_{i=1}^{n} c_{ki} \int_{I^2} h(x,y,z) u(x,y) \mathrm{v}_i(y,z) dx\,dz =$$

$$= \int_{I^2} \sum_{i=1}^{n} c_{ki} h(x,y,z) u(x,y) \mathrm{v}_i(y,z) dxdz = \int_I u(x,y) \left[\sum_{i=1}^{n} c_{ki} \int_I h(x,y,z) \mathrm{v}_i(y,z) dz \right] dx =$$

$$= \int_I f_k(x,y) u(x,y) dx ,$$

where

$$f_k(x,y)=\sum_{i=1}^{n} c_{ki}\int_I h(x,y,z)\mathrm{v}_i(y,z)\,dz\,,$$

$$G_k(\mathrm{v})=\sum_{j=1}^{n} d_{kj}B(u_j,\mathrm{v})=\sum_{j=1}^{n} d_{kj}\int_{I^2} h(x,y,z)u_j(x,y)\mathrm{v}(y,z)dxdz=$$

$$=\int_{I^2}\sum_{j=1}^{n} d_{kj}h(x,y,z)u_j(x,y)\mathrm{v}(y,z)dxdz=\int_I \mathrm{v}(y,z)\left[\sum_{j=1}^{n} d_{kj}\int_I h(x,y,z)u_j(x,y)dx\right]dz=$$

$$=\int_I g_k(y,z)\mathrm{v}(y,z)dz\,,$$

where

$$g_k(y,z)=\sum_{j=1}^{n} d_{kj}\int_I h(x,y,z)u_j(x,y)\,dx\,.$$

Taking into account all that in (16) we obtain

$$B(u,\mathrm{v})=\int_{I^2} h(x,y,z)u(xy)\mathrm{v}(y,z)dx\,dz=\sum_{k=1}^{n} F_k(u)G_k(\mathrm{v})=$$

$$=\sum_{k=1}^{n}\int_I f_k(x,y)u(x,y)dx\left(\int_I g_k(y,z)\mathrm{v}(y,z)dz\right)=\int_{I^2}\left(\sum_{k=1}^{n} f_k(x,y)g_k(y,z)\right)u(x,y)\mathrm{v}(y,z)dxdz$$

for any $u\in L^2(I^2),\ \mathrm{v}\in L^2(I^2)$.

The last equality is possible only if representation $h=\sum_{k=1}^{n} f_k\cdot g_k$ takes place on I^2 almost everywhere from which it follows, that $h\in\Phi^n$. Lemma 2 is proved.

Lemma 3. *Family of function* Φ_n *represents a closed subset of a Hilbert space* $L^2(I^3)$ *for each* $n\in N$.

Proof. Let $n\in N$ be fixed and $\{h_k\}_{k=1}^{\infty}\subseteq\Phi_n$ be any sequence weak convergent to the function $h_0\in L^2(I^3)$, i.e. $<h_k,h>\to<h_0,h>$ as soon as $k\to\infty$, for any $h\in L^2(I^3)$.

Particularly, let $B_k(u,\mathrm{v})\to B_0(u,\mathrm{v})$ as soon as $k\to\infty$, where $B_k(u,\mathrm{v})=\int_{I^2} h_k u\mathrm{v}\,dxdz$ for any $u\in L^2(I^2)$ and $\mathrm{v}\in L^2(I^2)$.

Since $h_k\in\Phi_n$ for any $k\ge 1$, then by lemma 2 we obtain that equality (14) is fulfilled at fixed $u_1,u_2,...,u_{n+1}\in L^2(I^2)$ and $\mathrm{v}_1,\mathrm{v}_2,...,\mathrm{v}_{n+1}\in L^2(I^2)$ for each $B=B_k$.

Let $k\to\infty$ in (14) with $B=B_k$.

We obtain

$$\begin{vmatrix} B_0(u_1,\mathrm{v}_1) & \dots & B_0(u_1,\mathrm{v}_{n+1}) \\ B_0(u_2,\mathrm{v}_1) & \dots & B_0(u_2,\mathrm{v}_{n+1}) \\ \vdots & & \vdots \\ B_0(u_{n+1},\mathrm{v}_1) & \dots & B_0(u_{n+1},\mathrm{v}_{n+1}) \end{vmatrix}=0\,.$$

Since $u_1,u_2,...,u_{n+1}\in L^2(I^2)$ and $\mathrm{v}_1,\mathrm{v}_2,...,\mathrm{v}_{n+1}\in L^2(I^2)$ in (14) are arbitrary by lemma 2 we obtain, that $h_0\in\Phi_n$.

The lemma is proved.

Proof of the theorem. We fix $h \in L^2(I^3)$ and $n \in N$. Choose sequence $\{S_n^k\}_{k=1}^{\infty} \subseteq \Phi_n$ such, that

$$\left\| h - S_n^k \right\| \to \rho_n(h) \text{ at } k \to \infty . \tag{17}$$

This sequence is bounded in $L^2(I^3)$ because of

$$\left\| S_n^k \right\| \le \|h\| + \left\| S_n^k - h \right\|, \ k \ge 1 .$$

Since any bounded subset of a Hilbert space is weak compact, there exists sequence $\{S_n^{k_j}\}_{j=1}^{\infty}$ weakly convergent to the function $S_n^0 \in L^2(I^3)$. Then, on the basis of lemma 3, we obtain $S_n^0 \in \Phi_n$.

By definition

$$\rho_n(h) = \inf_{S_n \in \Phi_n} \| h - S_n \|$$

and on the basis of (17)

$$\left\| h - S_n^0 \right\| \ge \rho_n(h) = \inf \left\| h - S_n^{k_j} \right\| \tag{18}$$

holds.

On the other hand, function $h - S_n^0$ is a weak limit of the sequence $\{h - S_n^{k_j}\}_{j=1}^{\infty}$, whence it follows that

$$\lim_{j \to \infty} \left\| h - S_n^{k_j} \right\| = \left\| h - S_n^0 \right\| .$$

Comparing the latter with (18), we obtain

$$\left\| h - S_n^0 \right\| = \rho_n(h)$$

and the theorem is proved.

In conclusion note, that the proved theorem contains corresponding result of Jaromira Шимша [4] who considered case of approximation by bilinear forms with separated variables, i.e. when $f_i(x,y) \equiv f_i(x)$, $g_i(y,z) \equiv g_i(z)$.

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