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ON LOCAL APPROXIMATION OF FUNCTIONS BY HARMONIC POLYNOMIALS

Abstract

In the given paper on continuums of type H^*, which were introduced by Andriyevsky V.V. the localizated direct theorem of approximation theory was obtained.

One of the important tendency of the approximation theory is the approximation of harmonic functions by harmonic polynomials. The papers of Walsh J.L., Sewell W.E. and Elliott [21], V.K. Dzyadyk [11], V.A. Borodin [9], M.Z. Dveyrin [13], V.V. Andriyevsky [4], [6] and others were devoted to study of these questions.

In the paper [4] of Andriyevsky V.V. on continuums E of more general form (of type H^*) on approximation of continuous on E and harmonic in interval points of E functions has been proved.

J.I. Mamedkhanov drew our attention on the fact that, in accordance with these results, which were mentioned above, it is of scientific interest to obtain localizated direct theorems of approximation theory on continuum E.

In the present paper on continuums of the type H^* which were introduced by V.V. Andriyevsky the localizated direct theorem of approximation theory was obtained. The obtained result is a complete analogue of the corresponding localizated direct theorem on approximation of analytical functions (see [17], [18]).

1. Main definitions, notations and result.

Let E be an arbitrary finite continuum of complex plane $\mathbb{C}$, $diam E > 0$ with simply connected complement $\Omega = CE = \overline{\mathbb{C}} | E$, where $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$, and $\Gamma = \partial\Omega = \partial E$ is their common boundary.

Function $w = \phi(z)$ conformally and univalently maps Ω on $\Omega' = \{w : |w| > 1\}$ and is normalized by conditions $\phi(\infty) = \infty,\ \phi'(\infty) > 0$. By the same symbol ϕ we'll denote homeomorphism between compactification $\widetilde{\Omega}$ of domain Ω by prime ends by Carateodory (see [14]) and $\overline{\Omega}'$ which coincides with $\phi(z)$ in Ω. Let $\psi = \phi^{-1}$, $\widetilde{\Gamma} = \widetilde{\Omega} | \Omega$ be a set of all boundary prime ends, $\Gamma_R = \{\zeta : |\phi(\zeta)| = R\},\ R > 1$.

Definition 1. *We'll say that* $E \in H$ *if any points* z *and* $\zeta \in E$ *can be connected by the arc* $\gamma(z,\zeta) \subset E$ *whose length satisfies the following conditions*

$$mes\gamma(z,\zeta) \le c|z-\zeta|, \quad c = c(E) = const \ge 1.$$

Note that if $E \in H$ then all the prime ends $Z \in \widetilde{\Gamma}$ are of the first kind, i.e. have single-point bodies $|Z| = z \in \Gamma$.

Let $E \in H$, $z = |Z| \in \Gamma$ be a body of the prime end $Z \in \widetilde{\Gamma}$. Consider circles $\{\xi : |\xi - z| = r\},\ r > 0$. On this circle one or finite number of arcs, dividing Ω into two subdomains can be found. Denote by $\gamma_Z(r)$ such of them for which unbounded subdomain is a largest one. The arc $\gamma_Z(r)$ separates the prime end Z from ∞. Note, that if $0<r<R<\frac{1}{2}diamE$, then $\gamma_Z(r)$ and $\gamma_Z(R)$ are sides of some quadrilateral $Q_Z(r,R)\subset\Omega$, whose other two sides are parts of the boundary Γ. Denote by $m_Z(r,R)$ the module of the quadrilateral $Q_Z(r,R)$ (see [1], [15]) and namely the module of family of arcs, separating in $Q_Z(r,R)$ sides $\gamma_Z(r)$ and $\gamma_Z(R)$.

Definition 2. *We say that $E\in H^*$ if $E\in H$ and such positive numbers $\varepsilon=\varepsilon(E)<\frac{1}{2}diamE$ and $c=c(E)$ can be found that for any prime ends Z and $\mathrm{Z}\in\widetilde{\Gamma}$ with the property $|z-\zeta|<\varepsilon$ $(z=|Z|,\ \zeta=|\mathrm{Z}|)$ the following inequality is fulfilled*

$$\left|m_Z(|z-\zeta|,\varepsilon)-m_{\mathrm{Z}}(|z-\zeta|,\varepsilon)\right|\le c.$$

Note, that classes of continuums H and H^* were introduced and well studied by V.V. Andriyevsky (see [2], [3]). Domains with the quasi-conformal boundary (see [1], [15], [7], [8]) and classes of domains B_k^*, introduced by V.K. Dzyadyk (see [12, p.392, 440]) belong to H^*.

We'll use symbol $A\preceq B$, which means that $A\preceq CB$, where $c=const>0$ doesn't depend on A and B, and $A \overset{\cup}{\cap} B$. If $A\preceq B$ and $B\preceq A$ simultaneously.

Denote by $B(E)$ the set of real functions continuous on E and harmonic in internal points of E.

In paper [16] of J.I. Mamedkhanov and V.V. Salayev localizated module of continuity of the form

$$\omega_f^{z_0}(\delta,\eta)_E=\sup_{\substack{|z-\tau|\le\delta\\ z,\tau\in O_\eta(z_0)\cap E}}|f(z)-f(\tau)|,$$

where $\delta,\eta>0$ and $O_\eta(z_0)\overset{df}{=}\{z\in\mathbb{C}:|z-z_0|\le\eta\}$ is considered.

Denote by Q a class of positive functions $\varphi(\delta,\eta)$ defined at $0<\delta,\eta<+\infty$ and such, that

1. $\varphi(\delta,\eta)$ doesn't decrease on each argument;
2. $\varphi(\delta,\eta)\cdot\delta^{-1}$ doesn't increase on δ;
3. $\forall\eta\in R_+:\lim\limits_{\delta\to 0}\varphi(\delta,\eta)=0$;
4. $\varphi(\delta,2\eta)\le\varphi(\delta,\eta)$, where in "$\le$" is constant, which doesn't depend on δ and η.

Denote by $C(E)$ a class of continuos on E functions.

Definition 3. *Let* $\varphi\in Q$. *Suppose*

$$H_{\varphi,E}^{z_0}\overset{df}{=}\left\{f\in C(E):\omega_f(z_0;\delta,\eta)\le c\varphi(\delta,\eta),\ \forall\delta,\ \eta:0<\frac{\delta}{2}\le\eta\right\},$$

where $\omega_f(z_0,\delta,\eta)=\delta\sup\limits_{\xi\ge\delta}\xi^{-1}\omega_f^{z_0}(\xi,\eta)$.

Expression of the form

$$Q_n(z)=\mathrm{Re}\sum_{j=0}^{n}a_jz^j,\quad n=0,1,\dots,\tag{1}$$

where a_j, $j=\overline{0,n}$ are generally speaking complex numbers, is accepted to call harmonic polynomial of order $\le n$.

By $\operatorname{int}\Gamma$ we understand a finite domain, whose boundary coincides with Γ, where Γ is a finite Jordan curve and $ext\,\Gamma = \mathbb{C}\setminus\overline{\operatorname{int}\Gamma}$.

Assume

$$\rho_R(z)=\inf_{\zeta\in\Gamma_R}|\zeta - z|, \; z\in\mathbb{C},$$

$$\Omega_R=(\operatorname{int}\Gamma_R)\setminus E, \quad R>1.$$

We formulate the main result of the given paper.

Theorem. *Let* $E\in H^*$, $\varphi\in Q$ *and* $u(z)\in H^{z_0}_{\varphi,E}\cap B(E)$. *Then for each natural* $n=1,2,...$ *there exists such a harmonic polynomial of order* $\le n$, *that at* $z\in\Gamma=\partial E$ *the following inequality is fulfilled*

$$|u(z)-Q_n(z)|\le c\varphi\left[\rho_{1+1/n}(z),\ \rho_{1+1/n}(z)+|z-z_0|\right), \tag{2}$$

where constant c *doesn't depend on* z_0, z, n.

This theorem is an analogue of direct theorem on approximation by analytical polynomials, which were obtained in papers [17], [18].

In papers [17], [18] considered domains have boundaries of set of type B_k^* and quasi-conformal curve. In these papers apparatus of approximation are analytical polynomials.

In the given paper a set of type H^* is considered and apparatus of approximation are harmonic polynomials.

2. Auxiliary facts.

We mention some auxiliary facts which are necessary for proving the main result.

Lemma 1 (see [3], [4], [5]). *Let* $E\in H^*$, $Z\in\tilde{\Gamma}$, $\mathrm{Z}\in\tilde{\Omega}$, $|Z|=z$, $|\mathrm{Z}|=\zeta$, $R>1$. *Then for points* $\tilde{z}_R \overset{df}{=} \psi[R\Phi(Z)]$ *and* $\tilde{\zeta}_R \overset{df}{=} \psi[R\Phi(\mathrm{Z})]$. *The following correlations are valid*

$$\rho_R(z)\asymp|z-\tilde{z}_R|,$$

$$\rho_R(\zeta)\asymp\rho_R(z), \quad if \quad |\zeta - z|\preceq\rho_R(z),\ \zeta\in\Gamma,$$

$$\left|\frac{\zeta_R-\zeta}{\tilde{\zeta}_R - z}\right|^{\beta}\preceq\frac{\rho_R(z)}{|\tilde{\zeta}_R-z|}\preceq\left|\frac{\tilde{\zeta}_R-\zeta}{\tilde{\zeta}_R-z}\right|^{\alpha}, \quad \beta>\alpha>0.$$

Lemma 2. *Let* E *be an arbitrary finite continuum with simply connected complement* $u(z)\in H^{z_0}_{\varphi,E}\cap B(E)$. *For any* $n=1,2,...$ *function* $u_n(z)$, $z\in\mathbb{C}$ *can be found with the following properties:*

1) $|u(z)-u_n(z)|\preceq\varphi\left[\rho_{1+1/n}(z),\ \rho_{1+1/n}(z)+|z-z_0|\right]$, $z\in\Gamma$,

$u_n(z)=u(z)$, $z\in E$, $d(z,\Gamma)>\dfrac{1}{2}\rho_{1+1/n}(z)$;

2) $u_n(z)\big|_{z\in\mathbb{C}|\Gamma_{1+1/n}}\in C^{\infty}(\mathbb{C}|\Gamma_{1+1/n})$, $u_n(z)=0$, $z\in ext\,\Gamma_{R_0}$, $R_0=const>1$;

3) *functions*

$$\vartheta_x(z)\overset{df}{=}\lim_{\substack{\xi+i\eta=\zeta\to z\\ \zeta\in\Gamma_{1+1/n}}}\frac{\partial u_n(\zeta)}{\partial\xi},\ \ z\in\Gamma_{1+1/n}\ ,$$

$$\vartheta_y(z)\overset{df}{=}\lim_{\substack{\xi+i\eta=\zeta\to z\\ \zeta\in\Gamma_{1+1/n}}}\frac{\partial u_n(\zeta)}{\partial\eta},\ \ z\in\Gamma_{1+1/n}$$

are continuous on $\Gamma_{1+1/n}$*;*

4) $|\Delta u_n(z)|\overset{df}{=}\left|\frac{\partial^2 u_n(z)}{\partial x^2}+\frac{\partial^2 u_n(z)}{\partial y^2}\right|\preceq\varphi[t_n(z),t_n(z)+|z-z_0|]t_n^{-2}(z), z\in x+iy\in \mathbb{C}\,|\,\Gamma_{1+1/n}$,

where $t_n(z)\overset{df}{=}\max\{d(z,\Gamma),\ \rho_{1+1/n}(z)\}$.

Proof. We extend $u(z)$ on a whole plane up to real function $u^*(z), z\in\mathbb{C}$, satisfying the conditions (see, e.g. [19, p.204-209])

$$u^*(z)=u(z),\ z\in E\ ;$$

$$\left|u^*(z_1)-u^*(z_2)\right|\preceq\varphi(|z_1-z_2|,\max\{|z_1-z_0|,|z_2-z_0|\}),\ z_1,z_2\in\mathbb{C}\ ;$$

$$u^*(z)\Big|_{z\in\Omega}\in C^\infty(\Omega), \tag{3}$$

$$\left|\frac{\partial^{i+j}u^*(z)}{\partial x^i\partial x^j}\right|\le c_1\frac{\varphi[d(z,\Gamma),d(z,\Gamma)+|z-z_0|]}{d^{i+j}(z,\Gamma)},\ \ z=x+iy\in\Omega,$$

where $i,j=0,1,...,i+j>0$, constant $c_1>0$ doesn't depend on z.

Assume that $u^*(z)=0$, $z\in ext\Gamma_R$, where $R=const>1$. Let $K(z)$ be an arbitrary averaging kernel. Then $K(z)\in C^\infty(\mathbb{C})$, $K(z)=0, |z|\ge1, K(z)=K(|z|)$, $\iint\limits_{\mathbb{C}}K(z)d\sigma_z=1$.

Denote by $\delta_n(z), z\,\overline{\in}\,\Gamma_{1+1/n}$ regularized distance to the level line (see [19, p. 203]).

For function $\delta_n(z)$ the following conditions are fulfilled:

$$\delta_n(z)\in C^\infty(\mathbb{C}|\Gamma_{1+1/n}),$$

$$\delta_n(z)\overset{\cup}{\frown}\rho_{1+1/n}(z),\ \ z\,\overline{\in}\,\Gamma_{1+1/n}\ ,$$

$$\left|\frac{\partial^{i+j}\delta_n(z)}{\partial x^i\partial x^j}\right|\le c_2\rho_{1+1/n}^{1-i-j}(z),\ \ z=x+iy\,\overline{\in}\,\Gamma_{1+1/n}\,,$$

where $i,j=0,1,...$ constant $c_2>0$ doesn't depend on z and n.

Denote $h=h(z,n)=\varepsilon\delta_n(z)$, where $\varepsilon>0$ was chosen such that for all points $z\in E$ with property $d(z,\Gamma)>\frac{1}{2}\rho_{1+1/n}(z)$ conditions

$$u(z,h)=\{\zeta:|\zeta-z|<h\}\subset E$$

are fulfilled.

We construct function $u_n(z)$ in the following form (see [4])

$$u_n(z)=h^{-2}\iint\limits_{\mathbb{C}}u^*(\zeta)K\left(\frac{\zeta-z}{h}\right)d\sigma_\zeta=\iint\limits_{\mathbb{C}}u^*(z+h\zeta)K(\zeta)d\sigma_\zeta\ ,$$

$$z \overline{\in} \Gamma_{1+1/n}\ ,\ u_n(z)=u^*(z),\ z\in\Gamma_{1+1/n}\,. \tag{4}$$

Statements 1) and 2) of the lemma are obvious. At first we'll prove the estimations

$$\left|\frac{\partial^2 u_n(z)}{\partial x^2}\right| \le \frac{\varphi\left[t_n(z),t_n(z)+|z-z_0|\right]}{t_n^2(z)},\quad z\in \mathbb{C}\,|\,\Gamma_{1+1/n}\,. \tag{5}$$

We fix point $z_1 \in u(z,h)$. Suppose $r \overset{df}{=} \dfrac{|\zeta - z|}{h}$.

We have

$$\frac{\partial u_n(z)}{\partial x}=\frac{\partial}{\partial x}\left[u_n(z)-u^*(z_1)\right]=-\frac{2h'_x}{h^3}\iint\limits_{\mathbb{C}}\left[u^*(\zeta)-u^*(z_1)\right]K(r)d\sigma_\zeta +$$

$$+h^{-2}\iint\limits_{\mathbb{C}}\left[u^*(\zeta)-u(z_0)\right]K'_r(r)\left(\frac{\left|\zeta-z\right|'_x}{h}-\frac{rh'_x}{h}\right)d\sigma_\zeta\ , \tag{6}$$

$$\frac{\partial^2 u_n(z)}{\partial x^2}=2\left(\frac{3h'_x}{h^4}-\frac{h''_{xx}}{h^3}\right)\iint\limits_{\mathbb{C}}\left[u^*(\zeta)-u^*(z_1)\right]K(r)d\sigma_\zeta-\frac{4h'_x}{h^3}\iint\limits_{\mathbb{C}}\left[u^*(\zeta)-u^*(z_1)\right]K'_r(r)\times$$

$$\times\left(\frac{\left|\zeta-z\right|'_x}{h}-\frac{r\,h'_x}{h}\right)d\sigma_\zeta+h^{-2}\iint\limits_{\mathbb{C}}\left[u^*(\zeta)-u^*(z_1)\right]K''_{rr}(r)\left(\frac{\left|\zeta-z\right|'_x}{h}-\frac{r\,h'_x}{h}\right)^2 d\sigma_\zeta +$$

$$+h^{-2}\iint\limits_{\mathbb{C}}\left[u^*(\zeta)-u^*(z_1)\right]K'_r(r)\left(\frac{\left|\zeta-z\right|'_{xx}}{h}-\frac{2\left|\zeta-z\right|'_x h'_x+\left|\zeta-z\right|h''_{xx}}{h^2}+\frac{2rh'^2_x}{h^2}\right)d\sigma_\zeta\,. \tag{7}$$

The following estimation holds

$$\left|u^*(\zeta)-u^*(z_1)\right|\preceq\omega_f\left(z_0,\left|\zeta-z_1\right|,\max\left\{\left|\zeta-z_0\right|,\left|z_1-z_0\right|\right\}\right)\preceq$$

$$\preceq\omega_f\left(z_0;\left|\zeta-z_1\right|,\left|\zeta-z_1\right|+\left|z_1-z_0\right|\right)\preceq\varphi\left(h,h+\left|z_1-z_0\right|\right). \tag{8}$$

Taking into account that $|z_1-z_0|\le|z_1-z|+|z-z_0|$, $|z_1-z|<h$ we have

$$\left|u^*(\zeta)-u^*(z_1)\right|\preceq\varphi\left(h,h+\left|z_1-z\right|+\left|z-z_0\right|\right)\preceq\varphi\left(h,h+\left|z-z_0\right|\right).$$

Substituting inequality (8) into (7) and taking into account properties of the function $\delta_n(z)$ we obtain

$$\left|\frac{\partial^2 u_n(z)}{\partial x^2}\right|\le\varphi\left[h,h+|z-z_0|\right]\cdot h^{-4}\iint\limits_{\mathbb{C}}K(r)d\sigma_\zeta+\varphi\left[h,h+|z-z_0|\right]\times$$

$$\times h^{-4}\iint\limits_{u(z,h)}d\sigma_\zeta+\varphi\left[h,h+|z-z_0|\right]h^{-2}\iint\limits_{u(z,h)}\left(\frac{1}{h^2}+\frac{1}{|\zeta-z|}\right)d\sigma_\zeta\preceq\frac{\varphi\left[h,h+|z-z_0|\right]}{h^{-2}}\,. \tag{9}$$

It's obvious that if $\rho_{1+1/n}\ge d(z,\Gamma)$ then $t_n(z)=\rho_{1+1/n}(z)$. In this case inequality (5) follows from inequality (9).

Consider the case when $\rho_{1+1/n}<d(z,\Gamma)$.

Let $z=x+iy$ and $\zeta=\xi+i\eta\in\Omega$ and $|\zeta-z|<\dfrac{1}{2}d(z,\Gamma)$.

Note that (see [10])

$$u^*(\zeta)=u^*(z)+|\zeta-z|\frac{\partial u^*(z)}{\partial l}+\int\limits_{[z,\zeta]}|\zeta-\tau|\frac{\partial^2 u^*(\tau)}{\partial l^2}dl\,, \tag{10}$$

where vector $\zeta-z$ defines the direction l, $\frac{\partial}{\partial l}$ is operator of differentiation on this direction.

Using (10) and (3) we have

$$u^*(\zeta)-\left[u^*(z)-u_x^{*'}(z)(\xi-x)+u_y^{*'}(\eta-y)\right]\overset{df}{=}R(\zeta,z)=$$
$$=O\left(\frac{\varphi[d(z,\Gamma),d(z,\Gamma)+|z-z_0|]}{d^2(z,\Gamma)}|z-\zeta|^2\right).$$

Let $z_1=x_1+i\,y_1\in u(z,h)$ be some fixed point. Consider function (see [4])

$$r(z,z_1)\overset{df}{=}h^{-2}\iint\limits_{\mathrm{C}}R(\zeta,z_1)K\left(\frac{\zeta-z}{h}\right)d\sigma_\zeta=$$
$$=u_n(z)-\left[u^*(z_1)+u_x^{*'}(z_1)(x-x_1)+u_y^{*'}(z_1)(y-y_1)\right]. \tag{11}$$

It may be assumed that $h\le\frac{1}{5}\rho_{1+1/n}(z)$. Then $u(z,h)\subset u\left(z_1,\frac{1}{2}d(z_1,\Gamma)\right)$. As for the formulas (6) and (7) formulas for $r'_x(z,z_1)$ and $r'_{xx}(z,z_1)$ can be obtained. By repeating the proof of (9) we obtain the following estimations

$$|r'_x(z,z_1)|\preceq\frac{\varphi[d(z,\Gamma),\,d(z,\Gamma)+|z-z_0|]}{d^2(z,\Gamma)}h\,, \tag{12}$$

$$|r''_{xx}(z,z_1)|\preceq\frac{\varphi[d(z,\Gamma),\,d(z,\Gamma)+|z-z_0|]}{d^2(z,\Gamma)}\,. \tag{13}$$

The following equalities are also valid

$$r'_x(z,z_1)=\frac{\partial u_n(z)}{\partial x}-u_x^{*'}(z_1)\,,$$

$$r''_{xx}(z,z_1)=\frac{\partial^2 u_n(z)}{\partial x^2}\,.$$

Thus, in case $\rho_{1+1/n}<d(z,\Gamma)$ we obtain estimation (5).

Validity of the inequality (5) for $\frac{\partial u_n(z)}{\partial y^2}$ is established analogously.

The statement 3) of the lemma 2 was proved in paper [4].

Lemma 3 (see [4]). *Let* $E\in H^*,\zeta_0\in\Gamma_{R_0},\,R_0>1,m>0$ *be fixed. For any* $n=1,2,...$ *there exists a harmonic kernel* $\pi_n(\zeta,z)=\operatorname{Re}\sum\limits_{j=0}^{n}a_j(\zeta)z^j$ *satisfying at* $\zeta\in D_n\bigcup\Omega_{R_0}$ *and* $z\in\Gamma$ *the inequality*

$$\left|\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right|\preceq\begin{cases}\dfrac{\rho^m_{1+1/n}(z)}{|\zeta-z|^m}, & |\zeta-z|\geq\rho_{1+1/n}(z),\\ 1+\ln\dfrac{\rho_{1+1/n}(z)}{|\zeta-z|}, & |\zeta-z|<\rho_{1+1/n}(z).\end{cases} \tag{14}$$

3. Proof of the main result.

By virtue of lemma 2 function $u(z)$ can be approximated by function $u_n(z)$. With the help of Green's formula (see, e.g. [4], [10, p.363-367]) function $u_n(z)$ can be represented at $z\in E$ in the form

$$u_n(z)=\frac{1}{2\pi i}\iint_{\operatorname{int}\Gamma_{1+1/n}}\Delta u_n(\zeta)\ln|\zeta-z|\,d\sigma_\zeta+$$

$$+\frac{1}{2\pi}\int_{\Gamma_{1+1/n}}\left(u_n(\zeta)\frac{\partial\ln|\zeta-z|}{\partial n_\zeta}-\frac{\partial u_n(\zeta)}{\partial n_\zeta}\ln|\zeta-z|\right)|d\xi|=\frac{1}{2\pi}\iint_{G_n}\Delta u_n(\zeta)\ln|\zeta-z|\,d\sigma_\zeta\ , \tag{15}$$

where $\dfrac{\partial}{\partial n_\zeta}$ is an operator of differentiation on external normal line to the curve $\Gamma_{1+1/n}$ at the point ζ,

$$D_n=\left\{\zeta:\zeta\in E,\ d(\zeta,\Gamma)<\frac{1}{2}\rho_{1+1/n}(\zeta)\right\},$$
$$G_n=D_n\cup\left[\left(\operatorname{int}\Gamma_{R_0}\right)\setminus\left(E\cup\Gamma_{1+1/n}\right)\right],$$

$R_0>0$ is a constant from lemma 2.

Note, that (see [4])

$$D_n\subset\bigcup_{z\in\Gamma}u\left(z,\rho_{1+1/n}(z)\right).$$

Consider the function (see [4])

$$g_n(z)=\frac{1}{2\pi i}\iint_{G_{n0}}\Delta u_n(\zeta)\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|d\sigma_\zeta\ ,\quad z\in E, \tag{16}$$

where $\zeta_0\in\Gamma_{R_0}$ is a fixed point ($R_0>1$ is a constant from lemma 2), we define the following harmonic polynomial

$$T_n(z)\overset{df}{=}\frac{1}{2\pi i}\iint_{G_n}\Delta u_n(\zeta)\pi_n(\zeta,z)\,d\sigma_\zeta\ ,\quad z\in E, \tag{17}$$

where $\pi_n(\zeta,z)$ is a kernel from lemma 3.

It's obvious, that

$$u_n(z)-g_n(z)=\frac{1}{2\pi i}\ln|\zeta_0-z|\iint_{G_n}\Delta u_n(\zeta)\,d\sigma_\zeta\ . \tag{18}$$

Using lemma 2 and properties of distance between continuums and their level lines (see [20, p.181])

$$\left|\iint_{G_n}\Delta u_n(\zeta)\,d\sigma_\zeta\right|\preceq\frac{\varphi\left(n^{-2},n^{-2}+|z-z_0|\right)}{h^{-4}}. \tag{19}$$

We obtain, that function $u_n(z)-g_n(z)$ can be approximated on E by sequence of harmonic polynomials with the rate of geometrical progression. In order to prove the theorem its sufficient to prove the following inequality

$$|g_n(z)-T_n(z)|=\frac{1}{2\pi}\left|\iint_{G_n}\Delta u_n(\zeta)\left[\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right]d\sigma_\zeta\right|\preceq$$

$$\preceq\varphi\left(\rho_{1+1/n}(z),|z-z_0|+\rho_{1+1/n}(z)\right), \tag{20}$$

where $z\in\Gamma$.

It's obvious that $G_n=\left[G_n\cap O_\eta(z_0)\right]\cup\left[G_n\,|\,O_\eta(z_0)\right]$.

Denote $\rho=\rho_{1+1/n}(z)$. Suppose

$$u_n=u(z,\rho/2),\ B_n=G_n\cap\left(O_\eta(z_0)\,|\,u_n\right),\ V_n=G_n\,|\,O_\eta(z_0).$$

We obtain

$$|g_n(z)-T_n(z)|=\frac{1}{2\pi}\left|\iint_{u_n}\Delta u_n(\zeta)\left[\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right]d\sigma_\zeta\right|+$$

$$+\frac{1}{2\pi}\left|\iint_{B_n}\Delta u_n(\zeta)\left[\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right]d\sigma_\zeta\right|+$$

$$+\frac{1}{2\pi}\left|\iint_{V_n}\Delta u_n(\zeta)\left[\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right]d\sigma_\zeta\right|=J_1+J_2+J_3. \tag{21}$$

By virtue of lemmas 2 and 3 we have

$$J_1\preceq\left|\iint_{u_n}\Delta u_n(\zeta)\left[\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right]d\sigma_\zeta\right|\preceq$$

$$\preceq\iint_{u_n}\frac{\varphi(\rho,\rho+|z-z_0|)}{\rho^2}\left(1+\ln\frac{\rho}{|\zeta-z|}\right)d\sigma_\zeta\preceq\varphi(\rho,\rho+|z-z_0|). \tag{22}$$

For estimation of J_i $(i=2,3)$ previously we notice the following correlations (see [4]).

Let $\zeta\in(G_n\,|\,u_n)\cap E$. We define point $\zeta\in\Gamma\setminus u(z,\rho)$ from the condition $|\zeta_1-z|=d(\zeta,\Gamma\setminus u(z,\rho))$. We obtain

$$|\zeta_1-z|\geq\rho_{1+1/n}(\zeta_1)\overset{df}{=}\rho_1,\qquad \rho_2\overset{df}{=}\rho_{1+1/n}(\zeta)\overset{\cup}{\cap}\rho_1.$$

By virtue of lemma 1 and 2 we obtain

$$|\Delta u_n(\zeta)|\preceq\frac{\varphi(\rho_2,\rho_2+|\zeta-z_0|)}{\rho_2^2}\preceq\frac{\varphi(|\zeta_1-z|,|\zeta_1-z|+|\zeta-z_0|)}{|\zeta_1-z|}\frac{1}{|\zeta_1-z|}\left|\frac{\zeta_1-z}{\rho_1}\right|^2\preceq$$

$$\preceq\frac{\varphi(\rho,\rho+|\zeta-z_0|)}{\rho}\frac{1}{|\zeta_1-z|}\left|\frac{\zeta_1-z}{\rho}\right|^{2\gamma}\preceq\frac{\varphi(\rho,\rho+|\zeta-z_0|)}{\rho^{1+2\gamma}}|\zeta-z|^{2\gamma-1}, \tag{23}$$

where $\gamma=\alpha^{-1}=const>1$.

Consider two possible cases:

1) $|\zeta-z_0|\leq|z-z_0|$; 2) $|\zeta-z_0|>|z-z_0|$.

In the first case we have

$$\varphi(\rho,\rho+|\zeta-z_0|)\preceq\varphi(\rho,\rho+|z-z_0|). \tag{24}$$

Using (23) and (24) we obtain

$$|\Delta u_n(\zeta)|\le\frac{\varphi(\rho,\rho+|z-z_0|)}{\rho^{1+2\gamma}}|\zeta-z|^{2\gamma-1}.$$

In the second case, taking into account that

$$|\zeta-z_0|\le|\zeta-z|+|z-z_0|,$$

we have

$$\varphi(\rho,\rho+|\zeta-z_0|)\preceq\varphi(\rho,\rho+|\zeta-z_0|+|z-z_0|)\preceq\varphi(\rho,\rho+|z-z_0|). \tag{25}$$

From correlations (23) and (25) we obtain

$$|\Delta u_n(\zeta)|\preceq\frac{\varphi(\rho,\rho+|z-z_0|)}{\rho^{1+2\gamma}}|\zeta-z|^{2\gamma-1}, \tag{26}$$

where $\gamma>1$.

Let $\zeta\in(G_n|U_n)\cap\Omega$. Using correlations

$$|\Delta u_n(\zeta)|\le\frac{\varphi\left(|\zeta-\tilde{\zeta}_{1+1/n}|,\max\{|\zeta-z_0|,|\tilde{\zeta}_{1+1/n}-z_0|\}\right)}{|\zeta-\tilde{\zeta}_{1+1/n}|},$$

$$|\tilde{\zeta}_{1+1/n}-z|\overset{\cup}{\cap}|\zeta-z| \tag{27}$$

for $\zeta\in(G_n|U_n)\cap\Omega$ similarly to previous we obtain

$$|\Delta u_n(\zeta)|\le\frac{\varphi(\rho,\rho+|z-z_0|)}{\rho^{1+2\gamma}}|\zeta-z|^{2\gamma-1}. \tag{28}$$

Thus, for $\zeta\in(G_n|U_n)\cap E$ and $\zeta\in(G_n|U_n)\cap\Omega$ we have

$$|\Delta u_n(\zeta)|\le\frac{\varphi(\rho,\rho+|z-z_0|)}{\rho^{1+2\gamma}}|\zeta-z|^{2\gamma-1}. \tag{29}$$

So, we obtain that for estimation of J_i $(i=2,3)$ it's sufficient to estimate the integral

$$\iint\limits_{G_n|U_n}\Delta u_n(\zeta)\left[\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right]d\sigma_\zeta . \tag{30}$$

Using correlations (29) and of lemma 3 we have

$$\left|\iint\limits_{G_n|U_n}\Delta u_n(\zeta)\left[\ln\left|\frac{\zeta-z}{\zeta_0-z}\right|-\pi_n(\zeta,z)\right]d\sigma_\zeta\right|\preceq$$

$$\preceq\iint\limits_{G_n|U_n}\frac{\varphi(\rho,\rho+|z-z_0|)}{\rho^{1+2\gamma}}|\zeta-z|^{2\gamma-1}\frac{\rho^m}{|\zeta-z|^m}d\sigma_\zeta\le\varphi(\rho,\rho+|z-z_0|)\times$$

$$\times\rho^{m-1-2\gamma}\int\limits_\rho^\infty\frac{dx}{x^{m-2\gamma}}\le(\rho,\rho+|z-z_0|), \tag{31}$$

where $m>2\gamma+1$.

Using (21), (22) and (31) we obtain inequality (20)

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