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ON THE DEPENDENCE OF EIGEN-VALUES ON POTENTIAL OF THE STURM-LIOUVILLE SINGULAR PROBLEM

Abstract

At the paper the question on the dependence of eigen-values on the potential of the Neyman-Liouville singular problem is considered. The condition of convergence of difference of eigen-values of two singular Sturm-Liouville operators, considered on the whole axis and in a semi-axis, differing by finite continuous disturbance, and also the formula for this difference, being the analogy of Gelfand-Levitan formula in the regular case are found.

Consider a linear differential equation of the second order in the normal form of Luioville

$$y'' + (\lambda - q(x))y = 0 \tag{1}$$

on the segment $[0,\pi]$ with the real potential $q(x)$, satisfying the condition

$$x^{1-\varepsilon}|q(x)| \in L_1(0;\pi) \ (\varepsilon \geq 0), \tag{2}$$

and with the bounded conditions of Neyman

$$y(0) = 0\,, \ \ y'(\pi) - h\,y(\pi) = 0\,, \tag{3}$$

where h is a fixed real number.

Under formulated conditions, by virtue of [1], the boundary value problem (1), (3) has an infinite sequence $\{y_n(x)\}$ of normalized eigen-functions from $L_2(0,\pi)$, where the n-th eigen-function has exactly the n single zeros inside the interval $(0;\pi)$. The corresponding sequence of the eigen-values $\{\lambda_n\}$ is strictly increasing to the infinity and satisfy the next asymptotics at $n \to \infty$

$$\lambda_n = n - \frac{1}{2} + O(r_n) \tag{4}$$

at the first approximation and

$$\lambda_n = n - \frac{1}{2} - \frac{h}{\pi\left(n - \frac{1}{2}\right)} \int_0^{\pi} \sin^2\left(n - \frac{1}{2}\right)t \cdot q(t)dt + O(r_n^2)$$

at the second approximation where

$$r_n = \frac{1}{n} + \int_0^{1/n-1} t|q(t)|dt + \frac{1}{n}\int_{1/n+1}^{\pi} |q(t)|dt\,.$$

In [1] it is proved, that for a class of potentials of the form (2) there exist a sequence of eigen-functions $\{s(x,\lambda_n)\}$ so, that $s(0,\lambda_n) = 0$, $s'(0,\lambda_n) = 1$, and satisfying the asymptotics at $n \to \infty$

$$s(x,\lambda_n) - \frac{\sin\lambda_n x}{\lambda_n} + O\left(\frac{r_n}{n}\right) \tag{5}$$

(this formula in the paper [1] is precised at the second approximation).

Numbers

$$\alpha_n = \int_0^{\pi} s^2(x,\lambda_n)dx \qquad n = 0,1,2,... \tag{6}$$

are called the normalized numbers of the boundary value problem of Neyman-Liouville (1), (3). Let's find an asymptotic formula for them. By the virtue of the (4) and (5) from the (6) we get:

$$\alpha_n=\int_0^\pi s^2(x,\lambda_n)dx=\int_0^\pi\left[\frac{\sin\left[n-\frac{1}{2}+O(r_n)\right]x}{n-\frac{1}{2}+O(r_n)}+O\left(\frac{r_n}{n}\right)\right]^2dx=$$

$$=\int_0^\pi\frac{\sin^2\left[n-\frac{1}{2}+O(r_n)\right]x}{\left(n-\frac{1}{2}+O(r_n)\right)^2}dx+O\left(\frac{r_n}{n^2}\right)=\int_0^\pi\frac{1-\cos[2n-1+O(r_n)]}{2\left(n-\frac{1}{2}\right)^2}dx+$$

$$+O\left(\frac{r_n}{n^2}\right)=\frac{\pi}{2\left(n-\frac{1}{2}\right)^2}+O\left(\frac{r_n}{n^2}\right).$$

Analogously to [2] it is determined that the functional $\lambda_n(q)$ has the derivative $\lambda_n'(q)$, which is a linear continuous mapping of a class of potential (2) to the real straight line, whereas the effect of the functional $\lambda_n'(q)$ to the function $f(x)\in L_1(0;\pi)$ has the form:

$$\lambda_n'(q)f=\int_0^\pi f(x)y_n^2(q,x)dx.$$

Hence by virtue of the Lagrange formula for the finite increments follows that for any functions $q_1(x)$ and $q_2(x)$ from the class (2), the difference of which is a regular function, and for any number $n=0,1,2,...$ there exists $\theta\in[0;\pi]$ such that the equality

$$\lambda_n(q_1+q_2)-\lambda_n(q_1)=\int_0^\pi q_2(x)y_n^2(q_1+\theta q_2,x)dx$$

is fulfilled.

From this relation it is easy to determine the strict monotone dependence of eigen-values from the potential.

Let's calculate the sum of difference of the eigen-values of the operators L and $\widetilde{L}$, generated by the differential equations

$$L:\quad y''+(\lambda-q(x))y=0, \tag{7}$$

$$\widetilde{L}:\quad y''+(\lambda-q(x)-q_0(x))y=0. \tag{8}$$

Theorem 1. *Let the operators L and $\widetilde{L}$ be considered on whole axis, $q(x)$ be the function from the class (2) and $q(x)\xrightarrow[|x|\to\infty]{}\infty$ and $q_0(x)$ be a finitary continuous function. Then if $\int q_0(x)dx=0$, then the series $\left(\sum_{n=1}^{\infty}(\mu_n-\lambda_n)\right)$, from the differences of eigen-values of the operators L and $\widetilde{L}$ converges and equals to zero.*

Proof. Under formulated conditions the spectrums of the operators L and $\widetilde{L}$ are discrete. Let's denote the eigen-values of the operator L and $\widetilde{L}$ by the $\lambda_1<\lambda_2<...$ and $\mu_1<\mu_2<...$, and eigen-functions by $\varphi_1(x),\varphi_2(x),...$ and $\psi_1(x),\psi_2(x),...$ respectively.

We need from [3] the next.

Theorem. *If the tracks of the operators $B = C - A$ (in our case $\widetilde{L} - L$) at the bases $\{\varphi_n\}$ and $\{\psi_n\}$ are equal, then*

$$\lim_{N\to\infty}\sum_{n=1}^{N}(\mu_n - \lambda_n) = \lim_{N\to\infty}\sum_{n=1}^{N}(B\varphi_n, \varphi_n).$$

As the domains of determinations of the operators L and $\widetilde{L}$ are equal, i.e. $D_L = D_{\widetilde{L}}$, then according to the last theorem it is enough to prove that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\int q_0(x)\psi_n^2(x)dx = \lim_{N\to\infty}\sum_{n=1}^{N}\int q_0(x)\varphi_n^2(x)dx.$$

On the other hand

$$\lim_{N\to\infty}\sum_{n=1}^{N}\int q_0(x)\psi_n^2(x)dx = \lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int q_0(x)\psi_n^2(x)dx.$$

From [4] it is known that at any bounded alternation domain x at great λ it holds:

$$\sum_{\mu_n<\lambda}\psi_n^2(x) = \frac{2}{\pi}\sqrt{\lambda} + o(1).$$

As $q_0(x)$ is a finite continuous function, then

$$\lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int q_0(x)\psi_n^2(x)dx = 0.$$

Exactly by the same way it is proved that

$$\lim_{\lambda\to\infty}\sum_{\lambda_n<\lambda}\int q_0(x)\varphi_n^2(x)dx = 0.$$

Theorem 2. *Let the operators L and $\widetilde{L}$ be considered on the semi-axis $(0 \le x < \infty)$ with a bounded condition $y'(0) - hy(0) = 0$, where h is a real number, and $q(x)$ is a function from the class (2) and $q(x)\xrightarrow[x\to\infty]{}\infty$, and $q_0(x)$ is finite continuous function and in small neighbourhood of zero has the derivative and $\int q_0(x)dx = 0$. Then it holds*

$$\lim_{N\to\infty}\sum_{n=1}^{N}(\mu_n - \lambda_n) = \frac{q_0(0)}{4}. \tag{9}$$

Proof. As in the case of theorem 1 the spectrums of the operators L and $\widetilde{L}$ are discrete. Let's save the notations. It is enough to prove that

$$\lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int q_0(x)\psi_n^2(x)dx = \lim_{\lambda\to\infty}\sum_{\lambda_n<\lambda}\int q_0(x)\varphi_n^2(x)dx = \frac{q_0(0)}{4}. \tag{10}$$

From the [4] it is known that for large λ

$$\sum_{\lambda_n<\lambda}\varphi_n^2(x) - \frac{2}{\pi}\int_0^{\sqrt{\lambda}}\cos t\,x\,dt = o(1).$$

Let's denote $\theta(x,\lambda) \equiv \frac{2}{\pi}\int_0^{\sqrt{\lambda}}\cos t\,x\,dt$. Then (10) will be written in the form

$$\lim_{\lambda\to\infty}\sum_{\lambda_n<\lambda}\int q_0(x)\varphi_n^2(x)dx = \lim_{\lambda\to\infty}\int q_0(x)\theta(x,\lambda)dx.$$

It is evident, that

$$\theta(x,\lambda)=\frac{1}{\pi}\sqrt{\lambda}+\frac{1}{2\pi}\frac{\sin 2\sqrt{\lambda x}}{x}.$$

Then from the (10) we'll get

$$\lim_{\lambda\to\infty}\int q_0(x)\theta(x,\lambda)dx=\frac{1}{2\pi}\lim_{\lambda\to\infty}\int q_0(x)\frac{\sin 2\sqrt{\lambda x}}{x}dx=\frac{q_0(0)}{4}.$$

It is proved similarly, that

$$\lim_{\lambda\to\infty}\sum_{\mu_n<\lambda}\int q_0(x)\psi_n^2(x)dx=\frac{q_0(0)}{4}.$$

Theorem is proved.

Remark 1. The formula (9) of the theorem 9 is the analogue of the known formula by Gelfand-Levitan [5].

Remark 2. At the process of proving the theorems 1 and 2, it is determined that for the convergence of the series $\sum_{n=1}^{\infty}(\mu_n-\lambda_n)$ the condition $\int q_0(x)dx=0$, is necessary and sufficient.

Remark 3. It work [6] it is shown that the changing of eigen-values in finite numbers of points, with the condition of preserving the relations between the "effects little" to the changing of potential in the sense that the difference of potentials of such perturbed and non-perturbed problems is absolutely continuous differentiable function.

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