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PERIODIC SOLUTIONS OF A PROBLEM ORIGINATING IN THE CONVECTION THEORY

Abstract

In the present paper we consider a periodic by time homogeneous Dirichlet problem for the system of equations which is obtained by corresponding simplifications from a hydrothermodynamic equation and describes convective processes of different types ([1], [2]), in particular the thermics (the ordered motions of type of ascending jets and floating bubbles of hot air).

Here using the methods and results of paper [5, 7, 8] and applying the known Faedo-Galerkin approximate method, we prove the solvability of the formulated problem.

In the domain $Q \equiv D \times (o,T)$, where $D \equiv \{(x,z) \in R^2 \,|\, 0 < x < 1; 0 < z < 1\}$, we consider a system of equations

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nu(t)\Delta u, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \lambda\vartheta + \nu(t)\Delta w, \\ \frac{\partial \vartheta}{\partial t} + u\frac{\partial \vartheta}{\partial x} + w\frac{\partial \vartheta}{\partial z} = \nu(t)\Delta\vartheta + f(x,z), \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \end{cases} \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ is two-dimensional laplacian, u, w are velocity components of wind, ϑ is potential temperature, t is time, $\nu > 0$ is a turbulence coefficient, λ is a convection parameter, p is pressure analogy, and $f(x,z)$ is some function (exterior force).

System of equations (1) is obtained by means of corresponding simplifications from the hydrothermodynamic equation and describes convective processes of different types ([1], [2]), in particular the thermics (the ordered motions of type of ascending jets and Hoating bubbles of warm air).

In different statements the thermodynamics problems of convect processes are considered by a lot of authors (for example, [1], [3], [7] etc.). In the present paper a periodic by time homogeneous Dirichlet problem is considered. Here using the methods and results of papers [5, 7, 8] and applying the known Faedo-Galerkin approximate method, the solvability of the formulated problem is proved.

Thus in the domain Q we'll investigate the next problem for the system (1):

Find a solution of the system of the equation (1) satisfying the following conditions:

$$u\big|_{\substack{x=0,1\\ z=0,1}} = w\big|_{\substack{x=0,1\\ z=0,1}} = \vartheta\big|_{\substack{x=0,1\\ z=0,1}} = 0\,, \tag{2}$$

$$\begin{aligned} u(0,x,z) &= u(T,x,z), \\ w(0,x,z) &= w(T,x,z), \\ \vartheta(0,x,z) &= \vartheta(T,x,z) \end{aligned} \tag{3}$$

For simplifications of notation we reduce the problem (1)-(3) to the vector form. Assume

$$A\equiv\begin{pmatrix}0&0&0\\0&0&\lambda\\0&0&0\end{pmatrix},\quad U\equiv\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\equiv\begin{pmatrix}u\\w\\\vartheta\end{pmatrix};\quad F=\begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix}\equiv\begin{pmatrix}0\\0\\f\end{pmatrix};\ x_1=x,\ x_2=z,$$

$$\frac{\partial U}{\partial t}\equiv U'\equiv\begin{pmatrix}u_1'\\u_2'\\u_3'\end{pmatrix}\equiv\begin{pmatrix}\partial u/\partial t\\\partial w/\partial t\\\partial\vartheta/\partial t\end{pmatrix};$$

$$\sum_{k=1}^{2}u_kU_{x_k}\equiv u_1U_{x_1}+u_2U_{x_2}\equiv u_1\begin{pmatrix}u_{1x_1}\\u_{2x_1}\\u_{3x_1}\end{pmatrix}+u_2\begin{pmatrix}u_{1x_2}\\u_{2x_2}\\u_{3x_2}\end{pmatrix}\equiv u\begin{pmatrix}u_x\\w_x\\\vartheta_x\end{pmatrix}+w\begin{pmatrix}u_z\\w_z\\\vartheta_z\end{pmatrix};$$

$$\Delta U\equiv\begin{pmatrix}\Delta u_1\\\Delta u_2\\\Delta u_3\end{pmatrix}\equiv\begin{pmatrix}\Delta u\\\Delta w\\\Delta\vartheta\end{pmatrix};\quad \nabla\equiv\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},0\right);\quad \Delta\equiv\sum_{k=1}^{2}\frac{\partial^2}{\partial x_k^2}\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}.$$

Then the system of equations (1) gets the next form

$$\begin{cases}U_t+\sum_{k=1}^{2}u_kU_{x_k}-\nu(t)\Delta U+\nabla p=AU+F,\\ \nabla U=0\end{cases}. \tag{4}$$

Everywhere below by $L_2\left(0,T;\left(W_2^1(D)\right)^3\right)$ we'll denote a space of the measurable vector functions $t\to U(t):(0,T)\to\left(W_2^1(D)\right)^3$ getting the values from R^3 and such that

$$\left(\int_0^T\|U(t)\|^2_{\left(W_2^1\right)^3}dt\right)^{\frac{1}{2}}\equiv\|U\|_{L^2\left(0,T;\left(W_2^1\right)^3\right)}<\infty,\ where\ \|U(t)\|^2_{\left(W_2^1\right)^3}\equiv\sum_{i=1}^{3}\|u_i\|^2_{W_2^1}.$$

Following book [5] denote by $\overset{0}{H}{}^1(D)$ a space determined in the form of

$$\overset{\circ}{H}{}^1(D)=\left\{U\,\middle|\,U\in\left(W_2^1(D)\right)^3,\quad U_{|x_1=x_2=0}=0,\ U_{|x_1=x_2=1}=0\right\},$$

and by $V(D)$, by analogy with determination of that type *spaces from [5] denote a space* determined in the form of

$$V(D)=\left\{U\,\middle|\,U\in\overset{0}{H}{}^1(D),\ \nabla U=0\right\}$$

and we supply this space with the norm

$$\|U\|_{V(D)}\equiv\left(\sum_{i=1}^{2}\int_D\left(\frac{\partial U}{\partial x_i},\left(\frac{\partial U}{\partial x_i}\right)^*\right)dD\right)^{\frac{1}{2}},\ \text{here}\ \left(\frac{\partial U}{\partial x_i}\right)^*\equiv\left(\frac{\partial u_1}{\partial x_i},\frac{\partial u_2}{\partial x_i},\frac{\partial u_3}{\partial x_i}\right).$$

Definition. *The pair* $(U(t,x),p(t,x))\in L_2(0,T;V(D))\cap L^\infty(0,T;L_2(D))\times L_2(Q)$ *is called a generalized solution of the periodic problem (4), (or the same as the problem (1)-(3)), if the vector function* $U(t,x)$ *satisfies the equality*

$$\left(U',V^*\right)+\left(\sum_{k=1}^{2}u_kU_{x_k},V^*\right)-\nu(t)\left(\Delta U,V^*\right)=\left(AU,V^*\right)+\left(F,V^*\right),\ \ \forall V\in L_2(0,T;V(D))$$

and the equality $U(0,x)=U(T,x)$, $x\in D\subset R^2$, *and the scalar function* $p(t,x)\in L_2(Q)$ *is an arbitrary fixed function; in addition the first equality is understood in the sense of* $L_2(0,T)$, *and the second equality is understood in the sense of* $(L_2(D))^3$.

Theorem. *Let* $\nu(t)$ *be a non-negative continuous function and there exists a number* $\delta>0$ *such that* $\nu(t)\geq\delta>0$ *and* $\lambda<\delta\cdot\lambda_1$, *where* λ_1 *is a first eigen value of laplacian operator in* $\overset{0}{H}{}^1(D)$. *Then for any* $f\in W_2^{-1}(D)$ *there exists such a vector function* $U(t,x)$ *that*

$$U\in L_2(0,T;V(D))\cap L^\infty\left(0,T;(L_2(D))^3\right), \tag{5}$$

$$\left(U',V^*\right)+\left(\sum_{k=1}^{2}u_kU_{x_k},V^*\right)-\nu(t)\left(\Delta U,V^*\right)=\left(AU,V^*\right)+\left(F,V^*\right), \tag{6}$$

$\forall V\in L_2(0,T;V(D))$, *and*

$$U(0,x)=U(T,x),\quad x\in D\subset R^2, \tag{7}$$

where $U\equiv\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ *and* $V\equiv\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, *and* $(\cdot,\cdot)$ *is a scalar product in* $(L_2(D))^3$ *in addition the equality (6) is understood in the sense of* $L_2(0,T)$, *and equality (7) is understood in sense of* $(L_2(D))^3$.

This theorem actually is a solvability theorem in generalized sense for the formulated problem and, precisely, from the theorem the correctness of statement immediately follows.

Corollary. *In the conditions of the theorem the problem (1)-(3) is solvable in generalized sense, i.e. there exists the pair* $\{U(t,x),p(t,x)\}$ *such that the first of them is a vector function satisfing the statement of the theorem, and the second is an arbitrary fixed function from the space* $L_2(Q)$.

The proof of the theorem. We lead the proof in two steps:

1) at first using the Faedo-Balirkin method the solvability of Cauchy problem for the considered system with given initial conditions from some class will be proved;
2) then it'll shown that the obtained in this way solution completely continuously depends on the initial data, and then using this and the theorem on the existence of a fixed point we prove the solvability of the formulated problem.

I step. Note that the proof of solvability of an initial boundary value problem is led as well as in paper [7] (as [8]). Therefore we reduce only that part of proof which we need in later on reasonings.

Thus we consider a mixed problem for the system (1) with homogeneous boundary conditions (2) with the Cauchy condition in the form of $U(0,x)=U_0(x)$, $x\in D$. For the proof of the solvability of initial boundary value problem the Faedo-Galerkin method is used and therefore the complete system of vector function is chosen from $V(D)$ which is denoted by $\{\bar{y}_j\}_{j=1}^{\infty}$.

We'll look for the approximate solution in the form of

$$U_m(x,t)\equiv\sum_{j=1}^{m}\overline{g}_j(t)\cdot\overline{y}_j(x)\equiv\begin{pmatrix}\sum_{j=1}^{m}g^1_{jm}(t)y^1_j\\ \sum_{j=1}^{m}g^2_{jm}(t)y^2_j\\ \sum_{j=1}^{m}g^3_{jm}(t)y^3_j\end{pmatrix},$$

i.e. as element of linear manifold over $\{\overline{y}_j\}_{j=1}^{m}$, we denote this manifold later on by $\{\overline{y}_j\}_{j=1}^{m}$.

In addition the unknown functions $g^i_{jm}(t),\ i=1,2,3$ are determined from the next system of arbitrary differential equations

$$\left(u'_{im}(t),y^i_j\right)_0+\nu(t)\left(\nabla u_{im}(t),\nabla y^i_j\right)_0+\left(\sum_{k=1}^{2}u_{mk}u_{imx_k},y^i_j\right)_0=\left(\sum_{\ell=1}^{3}a_{i\ell}u_{\ell m}(t),y^i_j\right)_0+ \tag{8}$$
$$+\left(F,y^i_j\right),\quad j=\overline{1,m},\ i=1,2,3,$$

to which we add the following Cauchy condition

$$U_m(0)=U_{0m}\ ,\ \text{ where }\ U_{0m}\in\{\overline{y}_j\}_{j=1}^{m}\subset\left(\overset{\circ}{H}{}^1(D)\right)^3. \tag{9}$$

Assume, that $U_{0m}\to U_0=U(0)$ in $(L_2(0))^3$, where $\|U_{0m}\|\le\|U_0\|$.

Thus for determination of the function $g^i_{jm}(t),\ \left(i=1,2,3;\ j=\overline{1,m}\right)$ we have to solve the Cauchy problem (8)-(9), whose local solvability follows from the general theory of ordinary differential equations by virtue of that, all the conditions of general theorem on local solvability for the considered Cauchy problem are satisfied. Consequently, we obtain that problem (8)-(9) has the solution $\overline{g}_{jm}(t)$ in some interval $(0,t_m)$. Hence again by virtue of known theorems of the general theory of ordinary differential equations and obtained later on estimations it follows that $\overline{g}_{jm}(t)$ is a solution of the problem on the whole interval in $[0,T]$.

Now we show that there exists a number $M\ge 0$ independent of m such that

$$\operatorname*{ess\,sup}_t\|U_m(t)\|_0\le M\ ,\ \|U_m\|_{L_2\left(0,T;\left(\overset{\circ}{H}{}^1(D)\right)^3\right)}\le M. \tag{10}$$

For obtaining these estimations we multiply j-th system from (8) by $\overline{g}_{jm}(t)$ and sum by j from 1 to m. Then we get

$$\frac{1}{2}\frac{d}{dt}\|U_m(t)\|_0^2+\nu(t)\left(\nabla U_m(t),\nabla U^*_m(t)\right)_0=\left(AU_m(t),U^*_m(t)\right)_0+\left(F,U^*_m(t)\right),$$

or

$$\frac{1}{2}\frac{d}{dt}\|U_m(t)\|_0^2+\nu(t)\left(\nabla U_m(t),\nabla U_m(t)\right)_0\le\lambda\|U_m(t)\|_0^2+\|f\|_{W_2^{-1}}\|U_m(t)\|_V. \tag{11}$$

Hence in conditions of the theorem we obtain the next apriori estimations

$$\operatorname*{ess\,sup}_t\|U_m\|_0^2\le C\left(\|U_0\|_0,\|f\|_{W_2^{-1}},\delta,\lambda\right),$$
$$\|\nabla U\|_0^2\le C\left(\|U_0\|_0,\|f\|_{W_2^{-1}},\delta,\lambda\right)$$

here it is considered that $F \equiv \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}$.

As it is known (for example, [8]) hence in standard form the solvability of the considered initial boundary value problem follows for the system of equations (6) at every given pair $U_0(x) \in L_2(D),\ F \in V^*(D)$.

Now we pass to the second step.

II step. From (11) we have

$$\frac{1}{2}\frac{d}{dt}\|U_m(t)\|_0^2 + (v(t) - \varepsilon)\|\nabla U_m(t)\|_0^2 \le \lambda \|U_m(t)\|_0^2 + c(\varepsilon)\|f\|_{W_2^{-1}}^2,$$

here $\varepsilon > 0$ is some number chosen in corresponding form and depending on data of problem. Hence, by virtue of the known inequality $\|U_m\| \le c_1 \|\nabla U_m\|_0$ for any $U_m \in \overset{0}{H}{}^1(D)$ we obtain the validity of the inequality

$$\frac{d}{dt}\|U_m(t)\|_0^2 + 2c_2\|U_m(t)\|_0^2 \le 2\lambda \|U_m(t)\|_0^2 + 2c(\varepsilon)\|f\|_{W_2^{-1}}^2, \tag{12}$$

where $c_2 > 0$ such that $\lambda < c_2 \le (v(t) - \varepsilon)\lambda_1$.

From the differential inequality (12) we obtain that the next inequality is valid

$$e^{2(c_2-\lambda)t}\left\{\|U_m(t)\|_0^2 + k\|f\|_{W_2^{-1}}^2\right\} \le \left\{\|U_0\|_0^2 + k\|f\|_{W_2^{-1}}^2\right\},\quad k = \frac{c(\varepsilon)}{\lambda - c_2},\quad \forall t \in [0,T], \tag{13}$$

which is correct for any $m: m = 1,2,\ldots$.

And correspondingly we have

$$\left\{\|U_m(t)\|_0^2 + k\|f\|_{W_2^{-1}}^2\right\} \le e^{2(\lambda-c_2)t}\left\{\|U_m(0)\|_0^2 + k\|f\|_{W_2^{-1}}^2\right\} \Rightarrow$$

$$\|U_m(t)\|_0^2 \le e^{2(\lambda-c_2)t}\left\{\|U_{0m}\|_0^2 + k\|f\|_{W_2^{-1}}^2\right\} - k\|f\|_{W_2^{-1}}^2. \tag{14}$$

Hence by virtue of conditions of theorem it implies that if the initial function $U_0(x)$ is taken from the closed ball $(B_R(0))^3 \subset (L_2(D))^3$, $R = |k|^{\frac{1}{2}}\|f\|_{W_2^{-1}}$, then for any $t \in [0,T]$ the inclusion $U(x,t) \in (B_R(0))^3$, where $R > 0$ is an above determined number, holds.

Thus we obtain that for any m the mapping $A: U_{0m} \to U_m(t)$ ball $(B_R(0))^3$ transforms into itself $\forall t \in [0,T]$.

In other words we have the mapping A operating in $(L_2(D))^3$ and such that for every $m: m = 1,2,\ldots,$ from $U_{0m} \in (B_R(0))^3$ it implies that $U_m(T) = A(U_{0m}) \in (B_R(0))^3$. Then applying the theorem on a fixed point for finite-dimensional spaces we obtain that in the ball $(B_R(0))^3$ there exists an element U_{0m} such that $A(U_{0m}(0)) = U_{0m} = U_m(T)$.

And what is more since for every finite m there exists such element U_{0m}, we get the next sequence $\{U_{0m}\} \in (B_R(0))^3$.

Further how we said above for any $m = 1,2$ the Cauchy problem for the equation (8) with the condition

$$U_m(0) = U_{0m} \tag{15}$$

has the solution $U_m(t)$ that by virtue of above mentioned reasonings satisfies the estimation (10). Consequently we obtain that

$$U_m(t) \text{ are bounded in } L_2(0,T,V)\cap L^\infty\left(0,T;(L_2(D))^3\right) \text{ for } m=1,2 \qquad (16)$$

where the bounded set runs in this space for $m\to\infty$.

Besides from obtained above result that $\bar{g}_{jm}(t)$ is a solution of the problem (8)-(9) it follows that using the system of the equation (8) i.e. the fact $U_m(t)$ satisfies this system of equalities, the a priori estimations (16) and completeness of the system $\{\bar{y}_j\}_{j=1}^{\infty}$ in $V(D)$ in standard form ([5,8] etc) we obtain that

$$U'_m(t)\in L_2\left(0,T;V^*(D)\right) \qquad (17)$$

and the bounded set runs in it for $m\to\infty$.

Further from a priori estimations (16), (17) by virtue of reflexivity of corresponding spaces we obtain that from the sequence of approximate solutions $\{U_m(t)\}$ with the correspondent initial conditions $\{U_{0m}\}$ we can choose the subsequence $\{U_\mu(t)\}$ such that

$$U_\mu\to U \text{ (weakly) in } L_2(0,T;V),$$
$$U'_\mu\to U' \text{ (weakly) in } L_2\left(0,T;V^*\right).$$

Hence it implies that $\{U_\mu(0)\}$ and $\{U_\mu(T)\}\subset(L_2(D))^3$ and what is more $U_\mu\to U$ (i.e. weakly) in $(L_2(D))^3$ and $U_\mu(T)\to U(T)$ (i.e. weakly) in $(L_2(D))^3$.

On the other hand we have $U_\mu(0)=U_{0\mu}$ and as well by virtue of that, $U_{0\mu}$ is a fixed point of the mapping A for every $\mu=1,2,...$ we have $U_\mu(T)=U_{0\mu}$.

Consequently, we obtain that $U(0)=U(T)$ in sense of $(L_2(D))^3$.

Thus finally we obtained that in the first step considering an initial boundary value problem with such initial condition, i.e. a problem containing this function, and finding out those reasonings approximately we get the problem (8)-(9) with the initial values $U_{0\mu}$ where as it was shown above we can pass to the limit when $\mu\to\infty$.

Hence it implies that considering the formulated problem with above determined $U(0)$, finding out a standard procedure ([5,7,8]) as above and passing to the limit in the problem (8)-(9) subject to the equality $U_m(0)=U_m(T)$, we obtain the correctness of the theorem and consequently the solvability of the formulated problem.

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Received January 10, 2001; Revised March 27, 2001.
Translated by Mirzoyeva K.S.