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THE INVERSE BOUNDARY VALUE PROBLEM FOR A THIRD ORDER PSEUDOPARABOLIC EQUATION WITH NON-LOCAL BOUNDARY CONDITION

Abstract

The inverse non-selfadjoint boundary value problem is investigated for third order linear pseudo-parabolic equations with non-local boundary conditions.

On the domain $Q=\{0<x<1,\,0<t<T\}$ we consider the following inverse problem:

$$u_t - a(t)u_{xx} - bu_{xxt} + c(t)u = F(x,t),\ (x,t)\in Q, \tag{1}$$

$$u(x,0)=\varphi(x),\ x\in[0,1], \tag{2}$$

$$u(0,t)=0,\ \ u_x(0,t)=u_x(1,t),\ t\in[0,T], \tag{3}$$

$$\beta u(1,t)+\int_0^1 u(x,t)dx = h(t),\ \ t\in[0,T], \tag{4}$$

here $c(t),h(t),\varphi(x)$ and $F(x,t)$ are given functions, $b>0$ and β given numbers, and functions $u(x,t)$ and $a(t)>0$ are to be defined. The inverse scattering problems were considered in papers [1], [3], [4] and others.

Under the solution of the inverse scattering problem (1)-(4) we understand a pair of functions $\{u(x,t),a(t)\}$ satisfying the following conditions:

1) The functions $u(x,t),\,u_t,u_{xx},u_{xxt}$ are continuous on $\overline{Q}$;
2) The function $a(t)$ is continuous and positive on $[0,T]$;
3) The equation (1) and all conditions of (2)-(4) are fulfilled in the ordinary sense.

To investigate the problems (2)-(4) we need some auxiliary facts from the spectral theory of differentiable equations (see [5]).

Here we cite these facts.

Consider the following Sturm-Liouville problem:

$$X''(x)+\lambda X(x)=0,\ 0<x<1, \tag{5}$$

$$X(0)=0,\ \ X'(0)=X'(1). \tag{6}$$

It is easy to find eigen values and eigen functions of the problems (5). They will have the following form:

$$\lambda_k=(2\pi k)^2,\ k=0,1,2,..., \tag{7}$$

$$\overline{X}_0(x)=x,\ \overline{X}_k(x)=\sin 2\pi kx,\ k=1,2,... \tag{8}$$

The system $\{\overline{X}_k(x)\}$ doesn't form a basis in $L_2(0,1)$.

Now define the adjoint functions of the system $\{\overline{X}_k(x)\}$. They are defined as the solutions of the problem

$$X_k''(x)+(2\pi k)^2X_k(x)=-4\pi kX_{k-1}(x), \tag{9}$$

$$X_k(0)=0,\ X_k'(0)=X_k(1) \tag{10}$$

and are of the form:

$$\overline{\overline{X}}_k(x)=x\cos 2\pi kx,\ k=1,2,.... \tag{11}$$

This system also doesn't form a basis in $L_2(0,1)$.

Make a new system

$$X_0(x)=x,\ \ X_{2k-1}(x)=x\cos 2\pi kx,\ \ X_{2k}(x)=\sin 2\pi kx \tag{12}$$

The system (12) forms basis in $L_2(0,1)$ (see [2]).

The eigen values and eigen functions of the corresponding adjoint problem (5), (6)

$$Y''(x)+\lambda Y(x)=0, \tag{13}$$

$$Y(0)=Y(1),\ \ Y'(1)=0 \tag{14}$$

have the form:

$$\lambda_k=(2\pi k)^2,\ \ k=0,1,2,..., \tag{15}$$

$$\overline{Y}_0(x)=2,\ \ \overline{Y}_k(x)=4\cos 2\pi kx,\ \ k=1,2,.... \tag{16}$$

We obtain the corresponding adjoint functions of the system (16) in the form

$$\overline{Y}_k(x)=4(1-x)\sin 2\pi kx,\ \ k=1,2,.... \tag{17}$$

They are found as the solutions of the problem

$$Y_k''+(2\pi k)^2 Y_k(x)=-4\pi k\overline{Y}_{k-1}(x), \tag{18}$$

$$Y_k(0)=Y_k(1),\ \ \ Y_k'(1)=0. \tag{19}$$

By means of the systems (16) and (17) we form a new system

$$Y_0(x)=2,\ Y_{2k-1}(x)=4\cos 2\pi kx,\ Y_{2k}(x)=4(1-x)\sin 2\pi kx. \tag{20}$$

The systems (12) and (20) form a biorthogonal system such that

$$(X_i,Y_j)=\int_0^1 X_i(x)Y_j(x)dx=\delta_{ij},\ \ i,j=0,1,2,..., \tag{21}$$

where δ_{ij} is Kronecker's symbol.

Theorem. *Let*

1) $\varphi(x)\in C^{(2)}([0,T]),\ \ \varphi'''(s)\in L_2(0,1),\ \ \varphi(0)=\varphi''(0)=0,\ \ \varphi'(0)=\varphi'(1)$

$$\varphi_k=\int_0^1\varphi(x)Y_k(x)dx\ge 0\quad (k=0,1,2,...)$$

and there exists at least one entire number k_0, *such that* $\varphi_{2k_0-1}>0$;

2) $h(t)\in C^{(1)}([0,T])$ and $h(t)\le 0,\ h'(t)\le 0,\ c(t)\in C([0,T]),\ \ c(t)\ge 0\ \ \forall t\in[0,T]$;

3) $F(x,t)\in C(\overline{Q}),\ F(x,\cdot)\in C^{(2)}([0,T]),\ \ F'''(x,\cdot)\in L_2(0,1)$

$F(0,t)=F_{xx}(0,t),\ \ F_x(0,t)=F_x(1,t)$,

$$F_k(t)=\int_0^1 F(x,t)\varphi_k(x)dx\ge 0\quad (k=0,1,2,...);$$

4) $h(0)-\int_0^1\varphi(x)dx-\beta\varphi_0-\beta\sum_{k=1}^{\infty}\varphi_{2k-1}=0,\quad b>0,\ \ \beta>0$

$$\Phi(t)=\int_0^1 F(x,t)dx+\beta\sum_{k=1}^{\infty}\frac{\lambda_k}{(1+b\lambda_k)^2}F_{2k-1}(t)-h'(t)-h(t)c(t)\ge\Phi_*>0,\ \forall t\in[0,T];$$

5) $\left\{bc^*+\Phi^*e^{c^*T+1}\left[b\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}\right]^{-1}\right\}T\le 1,$

where by f^* *and* f_* *we denote* $\max_{Q}|f(x,t)|$ *and* $\min_{Q}|f(x,t)|$ *respectively.*

Then the inverse problem has unique solution $\{u(x,t),a(t)\}$.

Proof. Construct the first component of the vector function $\{u(x,t),a(t)\}$.

Obviously, it has the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x) \qquad (22)$$

for each fixed t from $[0,T]$, where

$$u_k(t)=\int_0^1 u(x,t)\varphi_k(x)dx,\ \ (k=0,1,2,\ldots). \qquad (23)$$

To find the unknowns $u_k(t)\,(k=0,1,2,\ldots)$ we multiply the both sides of the equation (1) by $Y_k(x)$ and integral according to x from zero to unit. Then we obtain the following system of equations:

$$u_0'(t)+c(t)u_0(t)=F_0(t), \qquad (24)$$

$$(1+b\lambda_k)u_{2k-1}'(t)+[\lambda_k a(t)+c(t)]u_{2k-1}(t)=F_{2k-1}(t), \qquad (25)$$

$$(1+b\lambda_k)u_{2k}'(t)+[\lambda_k a(t)+c(t)]u_{2k}(t)=F_{2k}(t)-4\pi k[u_{2k-1}(t)+u_{2k-1}'(t)]. \qquad (26)$$

Solving these equations under initial conditions

$$u_k(0)=\varphi_k,\ \ k=0,1,2,\ldots, \qquad (27)$$

we get

$$u_0(t)=\varphi_0 e^{-\int_0^t c(s)ds}+\int_0^t F_0(\tau)e^{-\int_\tau^t c(s)ds}d\tau, \qquad (28)$$

$$u_{2k-1}(t)=\varphi_{2k-1}e^{-\frac{1}{1+b\lambda_k}\int_0^t(\lambda_k a(s)+c(s))ds}+\frac{1}{1+b\lambda_k}\int_0^t F_{2k}(\tau)e^{-\frac{1}{1+b\lambda_k}\int_\tau^t(\lambda_k a(s)+c(s))ds}d\tau, \qquad (29)$$

$$u_{2k}(t)=\varphi_{2k}e^{-\frac{1}{1+b\lambda_k}\int_0^t(\lambda_k a(s)+c(s))ds}+$$

$$+\frac{1}{1+b\lambda_k}\int_0^t[F_{2k}(\tau)-4\pi k(u_{2k-1}(\tau)+u_{2k-1}'(\tau))]e^{-\frac{1}{1+b\lambda_k}\int_\tau^t(\lambda_k a_k(s)+c(s))ds}d\tau. \qquad (30)$$

It is easy to show that for any $a(t)\in C[0,T]$ the function

$$u(x,t)=u_0(t)X_0(x)+\sum_{k=1}^{\infty}u_{2k-1}(t)X_{2k-1}(x)+\sum_{k=1}^{\infty}u_{2k}(t)X_{2k}(x) \qquad (31)$$

formally satisfies the equations (1), initial condition (2) and boundary conditions (3). Now choose the function $a(t)$ so that complimentary condition (4) also be satisfied.

Integrating the both sides of (1) according to x from zero to unit, we obtain

$$\frac{d}{dt}\int_0^1 u(x,t)dx+c(t)\int_0^1 u(x,t)dx=\int_0^1 F(x,t)dx. \qquad (32)$$

To fulfill the condition (4), the equality

$$h(t)c(t)=\int_0^1 F(x,t)dx-h'(t)+\beta[c(t)u(1,t)+\beta u_t(1,t)]. \qquad (33)$$

should hold.

As

$$u(1,t)=u_0(t)+\sum_{k=1}^{\infty}u_{2k-1}(t), \qquad (34)$$

then

$$h(t)c(t)=\int_0^1 F(x,t)dx-h'(t)+\beta[c(t)u_0(t)+u_0'(t)]+\beta\left[c(t)\sum_{k=1}^{\infty}u_{2k-1}(t)+\sum_{k=1}^{\infty}u_{2k-1}'(t)\right]. \quad (35)$$

It is easy to see $c(t)u_0(t)+u_0'(t)=0$. Therefore

$$h(t)c(t)=\int_0^1 F(x,t)dx-h'(t)+\beta\left[c(t)\sum_{k=1}^{\infty}u_{2k-1}(t)+\sum_{k=1}^{\infty}u_{2k-1}'(t)\right]. \quad (36)$$

So, to determine a pair $\{u(x,t),a(t)\}$ we obtain a system of equations (31) and (36). Then substituting from (25) and (29) the values $u_{2k-1}'(t)$ and $u_{2k-1}(t)$ into (36) we obtain

$$\begin{aligned} h(t)c(t)=&\int_0^t F(x,t)dx-h'(t)+\beta c(t)\sum_{k=1}^{\infty}\varphi_{2k-1}e^{-\frac{1}{1+b\lambda_k}\int_0^t[\lambda_k a(s)+c(s)]ds}+\\ &+\beta c(t)\sum_{k=1}^{\infty}\frac{1}{1+b\lambda_k}\int_0^t F_{2k-1}(\tau)e^{-\frac{1}{1+b\lambda_k}\int_\tau^t[\lambda_k a(s)+c(s)]ds}d\tau-\\ &-\beta\sum_{k=1}^{\infty}\frac{\varphi_{2k-1}}{1+b\lambda_k}[\lambda_k a(t)+c(t)]e^{-\frac{1}{1+b\lambda_k}\int_\tau^t[\lambda_k a(s)+c(s)]ds}+\\ &+\beta\sum_{k=1}^{\infty}\frac{1}{1+b\lambda_k}F_{2k-1}(t)-\beta\sum_{k=1}^{\infty}\frac{\lambda_k a(t)+c(t)}{(1+b\lambda_k)^2}\int_0^t F_{2k-1}(\tau)e^{-\frac{1}{1+b\lambda_k}\int_\tau^t[\lambda_k a(s)+c(s)]ds}d\tau\ . \end{aligned} \quad (37)$$

Hence, to define the second component of the solution $\{u(x,t),a(t)\}$ we obtain a functional integral equation

$$\begin{aligned} &a(t)=\Phi(t)[P(a)]^{-1}+bc(t),\\ &P(a)=\beta\left[\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}e^{-\frac{1}{1+b\lambda_k}\int_0^t[\lambda_k a(s)+c(s)]ds}+\right.\\ &\left.+\sum_{k=1}^{\infty}\frac{\lambda_k}{(1+b\lambda_k)^2}\int_0^t F_{2k-1}(\tau)e^{-\frac{1}{1+b\lambda_k}\int_0^t[\lambda_k a(s)+c(s)]ds}d\tau\right]. \end{aligned} \quad (38)$$

Let

$$\mathcal{F}=\{a(t)\in C([0,T]),\ a_*\le a(t)\le a^*\quad \forall t\in[0,T]\},$$

where $a^*=bc^*+\Phi^* e^{c^*T+1}\left[\beta\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\right]^{-1}$

$$a_*=bc_*+\frac{\Phi_*}{\beta\left[\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}+\sum_{k=1}^{\infty}\frac{\lambda_k}{(1+b\lambda_k)^2}\int_0^T F_{2k-1}(\tau)d\tau\right]}.$$

Obviously $a_*<a^*$.

It is easy to see for each $a(t)\in\mathcal{F}$

$$P(a)\le \beta\left[\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}+\sum_{k=1}^{\infty}\frac{\lambda_k}{(1+b\lambda_k)^2}\int_0^t F_{2k-1}(\tau)d\tau\right]$$

and $P(a)\ge \beta\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}e^{-\frac{1}{1+b\lambda_k}\left[\lambda_k a^* T+Tc^*\right]}\ge \beta\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}e^{-Ta^*b^{-1}}e^{-Tc^*}=\delta$ ..

By denoting the right hand side of (38) by $B(a)$ we obtain an operator equation of the form

$$a(t)=B(a). \tag{39}$$

It is not difficult to see that $B(a)\in C([0,T])$ and

$$B(a)\le bc_*+\frac{\Phi_* e^{a^*Tb^{-1}+c^*T}}{\beta\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}}\le bc^*+\Phi^* C^{1+c^*T}\left[\beta\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}\right]^{-1}=a^* ,$$

$$B(a)\le bc_*+\Phi_*\left[\beta\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}+\sum_{k=1}^{\infty}\frac{1}{(1+b\lambda_k)^2}\int_0^T F_{2k-1}(\tau)d\tau\right]^{-1}=a_*$$

consequently the operator $B(a)$ is defined in $\mathcal{F}$ and transfers $\mathcal{F}$ to $\mathcal{F}$.

For any $a_1(t)$ and $a_2(t)$ from $\mathcal{F}$

$$|B(a_1)-B(a_2)|\le \delta^{-2}\Phi^*|P(a_1)-P(a_2)| \tag{40}$$

since

$$|P(a_1)-P(a_2)|\le \beta\left[\sum_{k=1}^{\infty}\frac{\lambda_k}{1+b\lambda_k}\varphi_{2k-1}\left|e^{-\frac{1}{1+b\lambda_k}\int_0^t[\lambda_k a_1(s)+c(s)]ds}-e^{-\frac{1}{1+b\lambda_k}\int_0^t[\lambda_k a_2(s)+c(s)]ds}\right|+\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{\lambda_k}{(1+b\lambda_k)^2}\int_0^T F_{2k-1}(\tau)\left|e^{-\frac{1}{1+b\lambda_k}\int_0^t[\lambda_k a_1(s)+c(s)]ds}-e^{-\frac{1}{1+b\lambda_k}\int_\tau^t[\lambda_k a_2(s)+c(s)]ds}\right|d\tau\right]\le \tag{41}$$

$$\le \beta\left[\sum_{k=1}^{\infty}\frac{\lambda_k}{(1+b\lambda_k)^2}\varphi_{2k-1}\int_0^t|a_1(s)-a_2(s)|ds+\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{\lambda_k^2}{(1+b\lambda_k)^3}\int_0^t F_{2k-1}(\tau)\int_\tau^t|a_1(s)-a_2(s)|ds\right]\le \sigma\int_0^t|a_1(\tau)-a_2(\tau)|d\tau ,$$

where

$$\sigma=\beta\left[\sum_{k=1}^{\infty}\frac{\lambda_k^2}{(1+b\lambda_k)^2}\varphi_{2k-1}+\sum_{k=1}^{\infty}\frac{\lambda_k^2}{(1+b\lambda_k)^3}\int_0^T F_{2k-1}(\tau)d\tau\right],$$

then

$$|B(a_1)-B(a_2)|\le \delta^{-2}\Phi^*\sigma\int_0^t|a_1(\tau)-a_2(\tau)|d\tau .$$

Using mathematical induction method we can prove that for any integer m it holds the inequality

$$\|B^m(a_1)-B^m(a_2)\|\le \frac{(\bar{\sigma}T)^m}{m!}\|a_1-a_2\|_{[C[0,T]]}. \tag{42}$$

where $\overline{\sigma}=\delta^{-2}\Phi^{*}\sigma$. B^{m} is m-th iteration of the operator B.

Choose the number m so sufficiently large that

$$q_m=\frac{(\overline{\sigma}T)^m}{m!}<1.$$

From

$$\left\|P^m(a_1)-P^m(a_2)\right\|_{C[0,T]}\leq q_m\|a_1-a_2\|_{[C[0,T]]} \tag{43}$$

It follows that B^m is a contracting operator in $C[0,T]$. Then according to the generalized principle on a fixed point, the operator $B^m(a)$ has a unique fixed point $\overline{a}(t)$ in $C([0,T])$ and this fixed point is a unique solution of the operator equation (39) and consequently of (38).

Then, substituting $\overline{a}(t)$ into the right hand side of (31) we find $\overline{u}(x,t)$.

We can directly show that by fulfilling the conditions of the theorem, $\overline{u}(x,t)$ satisfies the equation (1), initial condition (2) and boundary condition (3).

Show that a pair $\{\overline{u}(x,t),\overline{a}(t)\}$ satisfies the complementary condition (4).

Obviously this pair satisfies (32) and (33). Then

$$\left(\beta u(1,t)+\int_0^1 u(x,t)dx-h(t)\right)'+c(t)\left(\beta u(1,t)+\int_0^1 u(x,t)dx-h(t)\right)=0.$$

Set

$$z(t)=\beta u(1,t)+\int_0^1 u(x,t)dx-h(t).$$

By the condition of the theorem

$$z(0)=\beta u(1,0)+\int_0^1 u(x,t)dx-h(0)=\beta u_0(0)+\beta\sum_{k=1}^{\infty}u_{2k-1}(0)+\int_0^1 u(x,0)dx-h(0)=$$

$$=\beta\varphi_0+\beta\sum_{k=1}^{\infty}\varphi_{2k-1}+\int_0^1\varphi(x)dx-h(0)=0.$$

Then we obtain the following Cauchy problem for $z(t)$:

$$z'(t)+c(t)z(t)=0,\ z(0)=0.$$

Hence it follows that $z(t)\equiv 0$. Consequently and complementary condition (4) is also satisfied.

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