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ON SOME BOUNDARY PROPERTIES OF GENERALIZED ANALYTIC FUNCTIONS

Abstract

At the paper some classes of generalized analytic functions in multi-connected domains, bounded with mean modules are introduced and their boundary properties are studied.

Let's consider a class of generalized analytic functions $U_{p,2}(A,B,G)$ in the sense of H.H. Vekua, i.e. a class of regular solutions of equation

$$\partial_{\bar z}W(z)+A(z)W(z)+B(z)\overline{W}(z)=0, \qquad (1)$$

where $A(z),B(z)\in L_{p,2}(G)$, $p>2$, $\partial_{\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$ (see [1], p.143).

Let's consider also conjugate class $U_{p,2}(-A,-\overline{B},G)$ to the class $U_{p,2}(A,B,G)$ (see [1], p.179).

Let G be a bounded n-connected domain. By $C(A,B,\overline{G})$ we denote a class of continuous generalized analytic functions in $\overline{G}$, by $M(A,B,G)$ a class of bounded generalized analytic functions in G.

Assume that finite n-connected domain G is bounded by n closed rectifiable Jordan curves $\gamma_1,\gamma_2,...,\gamma_n$ from which not a curve is degenerated to a point. The contour γ_1 will be external, and $\gamma_2,...,\gamma_n$ will be interval. The complete boundary of domain G we denote by $\Gamma:\Gamma=\bigcup_{i=1}^{n}\gamma_i$.

Definition. *We say that the generalized analytic function $W(z)$ from the class $U_{p,2}(A,B,G)$ $\left(U_{p,2}(-A,-\overline{B},G)\right)$ belongs to the class $E_\delta(A,B,G)\left(E_\delta(-A,-\overline{B},G)\right)$, $\delta>0$ if there exists a sequence of rectifiable curves $\Gamma^\nu=\bigcup_{i=1}^{n}\gamma_i^\nu$ such that*

1) *γ_1^ν lies inside of γ_1, but γ_i^ν $(i=2,3,...,n)$ for any n contains γ_i inside itself*
2) *$\gamma_i^\nu\to\gamma_i$ when $\nu\to\infty$ $(i=1,2,...,n)$*
3) *There exists such m, that* $\sup_\nu[long.\ \Gamma^\nu]\le m$
4) $\sup_\nu\int_{\Gamma^\nu}|W(z)|^\delta|dz|<\infty$.

We note that the class $E_\delta(A,B,G)$ is an analogue of a class of analytic functions E_δ introduced and studied by M.V.Keldysh, M.A.Lavrentev, V.I.Smirnov in the case of singleconnected domains (see [2], [3]) and by S.Ja.Havinson in the case of n-connected domains (see [4]).

We also note that the class $E_\delta(A,B,G)$ for single-connected domains is introduced and studied in works [6], [7].

As known, all generalized analytic functions $W(z)\in U_{p,2}(A,B,G)$ are represented in the form (see [1], p.156)

$$W(z)=\Phi(z)e^{\omega(z)}, \tag{2}$$

where $\Phi(z)$ is an analytic function in G

$$\omega(z)=\frac{1}{\pi}\iint_G \frac{A(\tau)+B(\tau)\frac{\overline{W(\tau)}}{W(\tau)}}{\tau-z}d\xi d\eta,\ \ \tau=\xi+i\eta. \tag{3}$$

Theorem 1. *Generalized analytic functions $W(z)\in U_{p,2}(A,B,G)$ belongs to the class $E_\delta(A,B,G)$, $\delta>0$ if the function Φ in the representation (2) belongs to the class E_δ.*

Proof. First when $A,B\in L_p(G)$, $p>2$ we study the behavior of function $\omega(z)$. From $A,B\in L_p(G)$ follows that $A(z)+B(z)\frac{\overline{W}(\tau)}{W(z)}\in L_p(G)$, $p>2$

$$\text{since } \left|A(z)+B(z)\frac{\overline{W}(z)}{W(z)}\right|\le|A(z)|+|B(z)|$$

$$\text{that is } \int_\Gamma\left|A(z)+B(z)\frac{\overline{W}(z)}{W(z)}\right|^p|dz|\le C_1<\infty. \tag{4}$$

Let's consider the function $\omega(z)=\frac{1}{\pi}\iint_G\left[\frac{A(\tau)+B(\tau)\frac{\overline{W}(\tau)}{W(\tau)}}{\tau-z}\right]d\xi d\eta$. We have

$$|\omega(z)|=\left|\frac{1}{\pi}\iint_G\frac{\left(A+B\frac{\overline{W}}{W}\right)}{\tau-z}d\xi d\eta\right|\le$$

$$\le\frac{1}{\pi}\left[\iint_G\left|A+B\frac{\overline{W}}{W}\right|^p|d\xi||d\eta|\right]^{\frac{1}{p}}\left[\iint_G\frac{1}{|\tau-z|^q}|d\xi||d\eta|\right]^{\frac{1}{q}},\ \ \frac{1}{p}+\frac{1}{q}=1. \tag{5}$$

The first factor in (5) is bounded by virtue of (4), but the second factor is bounded by virtue of that $q<2$ (since $p>2$).

That is

$$|\omega(z)|<C_2<\infty, \tag{6}$$

where C_R depends only on domain G.

Also estimating the difference $|\omega(z_1)-\omega(z_2)|$ for $z_1,z_2\in\overline{G}$ we are convinced that $\omega(z)\in C_\alpha(\overline{G})$, $\alpha=\frac{p-2}{p}$ (C_α is a Hölder class with the index $\alpha=\frac{p-2}{p}<1$).

We obtain that when $A,B\subset L_p(G)$, $p>2$, $|\omega(z)|<C_2<\infty$ and $\omega(z)\in C_\alpha(\overline{G})$. It means that

$$\begin{cases} \left|e^{\omega(z)}\right| \le C_4 < \infty, \ e^{\omega(z)} \in C(\overline{G}); \\ \left|e^{-\omega(z)}\right| \le C_5 < \infty, \ e^{-\omega(z)} \in C(\overline{G}). \end{cases} \tag{7}$$

From (2) we have

$$\Phi(z) = e^{-\omega(z)} W(z). \tag{8}$$

Let now $\Phi(z) = E_\delta(G)$. Then according to the work [4]

$$\sup_\nu \int_{\Gamma^\nu} |\Phi(z)|^\delta |dz| \le C_6 < \infty. \tag{9}$$

We have:

$$\sup_\nu \int_{\Gamma^\nu} |W(z)|^\delta |dz| = \sup_\nu \int_{\Gamma^\nu} \left|\Phi(z) e^{\omega(z)}\right|^\delta |dz| =$$

$$= \sup_\nu \int_{\Gamma^\nu} |\Phi(z)|^\delta \left|e^{\omega(z)}\right|^\delta |dz| = C_4^\delta \sup_\nu \int_{\Gamma^\nu} |\Phi(z)|^\delta |dz| < \infty.$$

(we used [7] and [9]).

According to the definition $E_\delta(A,B,G)$ means that $W(z) \in E_\delta(A,B,G)$. Conversely, if $W(z) \in E_\delta(A,B,G)$ then

$$\sup_\nu \int_{\Gamma^\nu} |W(z)|^\delta |dz| \le C_7 < \infty. \tag{10}$$

Using the correlation (8) and inequality (10) we get that $\Phi(z) \in E_\delta(G)$.

Theorem is proved.

Property 1. *If $W(z) \in E_\delta(A,B,G)$, then $W(z)$ has angular boundary values $W(t)$ and $W(t) \in L_\delta(\Gamma)$ almost everywhere on Γ.*

Indeed, since $W(z) \in E_\delta(A,B,G)$, then in the representation (2) analytic in G function $\Phi(z)$ belongs to the class $E_\delta(G)$ (according to the theorem 1) and has angular boundary values almost everywhere on Γ (see [4]) and $\Phi(z) \in L_\delta(\Gamma)$.

Taking into account (7) that the function $l^{\omega(z)}$ is continuous on $\overline{G}$ we get that $W(z)$ has angular boundary values almost everywhere on Γ. Therefore,

$$\int_\Gamma |W(t)|^\delta |dt| = \int_\Gamma |\Phi(t)|^\delta \left|e^{\omega(t)}\right|^\delta |dt| \le C_8 \int_\Gamma |\Phi(t)|^\delta |dt| < \infty,$$

i.e.

$$W(t) \in L_\delta(\Gamma).$$

It is clear that the interior of γ_1 is single-connected domain (if disregard the curves $\gamma_2,...,\gamma_n$), but the exterior of every γ_i $(i \ge 2)$ disregarding the other γ_j $(i \ne j)$ is single-connected domain containing infinity. We denote them by $G_1, G_2, ..., G_n$ correspondingly.

Theorem 2. *If $W(z) \in E_\delta(A,B,G)$, $\delta > 0$ then $W(z)$ is presented in the form*

$$W(z) = W_1(z) + W_2(z) + ... + W_n(z) \tag{11}$$

moreover, $W_i(z) \in E_\delta(A,B,G_i)$.

Proof. Since

$$W(z) = \Phi(z) e^{\omega(z)} \text{ and } W(z) = E_\delta(A,B,G)$$

then according to the theorem 1 $\Phi(z) \in E_\delta(G)$, then according to the paper [4].

$$\Phi(z) = \Phi_1(z) + \Phi_2(z) + \ldots + \Phi_n(z), \tag{12}$$

where

$$\Phi_i(z) \in E_\delta(G_i). \tag{13}$$

We have

$$W(z) = (\Phi_1(z) + \Phi_2(z) + \ldots + \Phi_n(z))e^{\omega(z)}. \tag{14}$$

Taking into account that $e^{\omega(z)}$ is bounded in $\overline{G}$ and

$$\int_{\Gamma''} |\Phi_i(z)|^\delta |dz| < \infty \tag{15}$$

also

$$|W(z)|^\delta = |\Phi_1(z) + \Phi_2(z) + \ldots + \Phi_n(z)| \left| e^{\delta\omega(z)} \right| \le$$
$$\le C_8 \left[|\Phi_1(z)|^\delta + |\Phi_2(z)|^\delta + \ldots + |\Phi_n(z)|^\delta \right]$$

we have

$$\int_{\Gamma''} |W(z)|^\delta |dz| \le C_9 \int_{\Gamma''} \sum_{i=1}^{n} |\Phi_i(z)|^\delta |dz| = C_9 \sum_{1}^{n} \int_{\Gamma''} |\Phi_i(z)|^\delta |dz| < \infty .$$

Therefore

$$W(z) \in E_\delta(A, B, G).$$

By virtue of (13)

$$\int_{\gamma''} |\Phi_i(z)|^\delta \left| e^{\omega(z)} \right|^\delta |dz| < +\infty, \ i.e. \quad \Phi_i(z) e^{\omega(z)} \in E_\delta(A, B, G_i).$$

In other words

$$W_i(z) \in E_\delta(A, B, G_i)$$

Theorem 3. *Let the generalized analytic function* $F_1(z) \in U_{p,2}(A, B, G)$ *belongs to the class* $E_1(A, B, G)$ *and* $F_2(z) \in U_{p,2}(-A, -\overline{B}, G)$ *be bounded in* G.

Then

$$\mathrm{Re}\left(\frac{1}{2i} \int_\Gamma F_1(z) F_2(z) dz \right) = 0 .$$

Proof. Under the conditions of the theorem in the case when G is a single-connected domain in work [7] it is proved that

$$\mathrm{Re}\left(\frac{1}{2i} \int_\Gamma F_1(z) F_2(z) dz \right) = 0 . \tag{16}$$

By virtue of theorem 2

$$F_1(z) = W_1(z) + W_2(z) + \ldots + W_n(z), \text{ where}$$
$$W_i(z) \in E_1(A, B, G_i) \quad \left(i = \overline{1,n}\right).$$

Then taking into account that the function $F_2(z)$ is bounded in G (and in G_i), applying (16) we have:

$$\mathrm{Re}\left(\frac{1}{2i} \int_\Gamma F_1(z) F_2(z) dz \right) = \mathrm{Re}\left(\frac{1}{2i} \int_{\bigcup_{i=1}^{n} \gamma_i} F_1(z) F_2(z) dz \right) =$$

$$= \operatorname{Re}\left(\frac{1}{2i}\sum_{i=1}^{n}\int_{\gamma_i} F_1(z)F_2(z)dz\right) = \sum_{i=1}^{n}\left(\operatorname{Re}\frac{1}{2i}\int_{\gamma_i}[W_1(z)+\ldots+W_n(z)]F_2(z)dz\right) = 0$$

since with the opening of brackets we obtain the integrals of the form

$$\operatorname{Re}\left(\frac{1}{2i}\int_{\gamma_i} W_k(z)F_2(z)dz\right), \quad k = 1,2,\ldots,n,$$

where $W_k(z)$ are functions from the class $E_1(A,B,G_k)$, and $F_2(z)$ is bounded in G and $F_2(z) \in U_{p,2}(\_A,\_\overline{B},G_k)$.

Theorem is proved.

Let's note that this theorem is a spreading to the more general case (with respect to the functions $F_1(z)$ and $F_2(z)$) so called «Green's identity» proved by U.H.Vekua (1, [1], p. 179) about that if $W_1(z) \in C(A,B,\overline{G})$, $W_2(z) \in M(\_A,\_\overline{B},G)$ then

$$\operatorname{Re}\left(\frac{1}{2i}\int_{\Gamma} W_1(z)W_2(z)dz\right) = 0$$

since $E_1(A,B,G) \supset C(A,B,G)$

In the conclusion let's note that the considered case of n-connected domains prepares the ground for the investigation of external problems in a class of generalized analytic functions (as considered in the class of analytic functions [4]).

References

[1]. Векуа И.Н. *Обобщенные аналитические функции.* М., 1959, 628 с.

[2]. Привалов И.И. *Граничные свойства аналитических функций.* М.Н., 1950, 336 с.

[3]. Келдыш М.В., Лаврентьев *Sur la representation conforme des domaines limites par des courbes rectficables.* Ann. l'Ecole Norm. Sup., 59, №1, 1957, p.1-38.

[4]. Хавинсон С.Я. *Экстремальные задачи для некоторых классов аналитических функций в конечносвязных областях.* Матем. сборник, т.36(78), №3, 1955, с.444-478.

[5]. Хавинсон С.Я. *Теория экстремальных задач для ограниченных аналитических функций.* Успехи мат. наук, 1963, т.VIII, вып. 2(110), 25-98.

[6]. Мусаев К.М. *О некоторых экстремальных свойствах обьощенных аналитических функций.* ДАН СССР, т.203, №2, 1972, с.289-292.

[7]. Мусаев К.М. *Некоторые замечания к тождеству Грина.* Известия АН Азерб., 1971, №5,6, с.74-79.

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