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A method for solving dynamic problems for cylindrical domains

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Abstract. *This article explores the transient dynamic problems of the theory of elasticity for bodies with cylindrical symmetry. Problems of harmonic vibrations draw more attention than non-stationary problems in dynamic theory of elasticity. If we consider that in most cases, the sources of excitation of wave propagation in elastic media are shock or explosive nature, it becomes clear that in practical applications of dynamic elasticity theory one has to deal, as a rule, with non stationary problems. The existing analytical solutions cylindrical problems of dynamic elasticity theory are rather few, among these, the most popular is the work [6], where an asymptotic solution of the problem of a cylinders stroke at a fixed rigid barrier is obtained. The purpose of the paper is to give methods for constructing the solutions of an elastic problem of non-stationary dynamics in the general form for cylindrical wave notions. It was stated, for the sake of simplicity, in the example of a semi-infinite, circular cylinder, and on the basis of this method, an exact solution for the initial stages of the process Was constructed. It is appropriate to notice that the proposed method makes possible to obtain analytical solutions for any finite period of time and for any body with an axially symmetric cylindrical configuration (in the special case for layered cylindrical bodies as well).*

Keywords. wave motion, circular cylinder, Lame equations, longitudinal impact and integral transforms.

Mathematics Subject Classification (2010): 74B05

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1 Formulation of the problem

Impact to the end of a circular cylinder.

The problem of wave propagation in semi-infinite circular cylinder of radius$a$, under the action of axial shock forces applied to the end area of the cylinder. In the cylindrical coordinate system z, φ, z related with the end of the cylinder, the solution of the axially symmetric problem is related to the solution of the following system of Lame equations:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{rz}}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} \tag{1.1}$$

$$\sigma_{rr} = 2\mu\varepsilon_{rr} + \lambda e;\ \ \sigma_{\theta\theta} = 2\mu\varepsilon_{\theta\theta} + \lambda e;\ \ \sigma_{zz} = 2\mu\varepsilon_{zz} + \lambda e.$$

$$\sigma_{rz} = 2\mu\varepsilon_{rz};\ \ e = \varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\theta\theta};$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r};$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r};$$

$$\varepsilon_{zz} = \frac{\partial u_r}{\partial z}; \tag{1.2}$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right);$$

$$\varepsilon_{z\theta} = \varepsilon_{r\theta} = 0.$$

for $z > 0;\ \ 0 \le r \le a;\ \ t \ge 0$

Here u_r and u_z are radial and axial displacement of particles, ρ is density.

This system joins the initial and boundary conditions; they are:

$$\left.\begin{array}{c} u_r = u_z = 0 \\ \dfrac{\partial u_r}{\partial t} = \dfrac{\partial u_z}{\partial t} = 0 \end{array}\right\} \text{ in } \ t = 0. \tag{1.3}$$

$$\left.\begin{array}{c} \sigma_{zz} = \sigma_0(r) f(t) \\ u_r = 0 \end{array}\right\} \text{ in } \ \ z = 0 \tag{1.4}$$

$$\sigma_{rr} = \sigma_{rz} = 0 \ \text{ in } \ r = a;\ \ \ 0 < z < \infty \tag{1.5}$$

(1.1) and (1.2) leads to an axially symmetric system of Lame equations:

$$\begin{cases} (2\mu + \lambda)\dfrac{\partial^2 u_r}{\partial r^2} + (\lambda + \mu)\dfrac{\partial^2 u_z}{\partial r \partial z} + \mu\dfrac{\partial^2 u_r}{\partial z^2} + \dfrac{(\lambda + 2\mu)}{r}\dfrac{\partial u_r}{\partial r} - \dfrac{(\lambda + 2\mu)}{r^2} u_r = \rho\dfrac{\partial^2 u_r}{\partial t^2} \\ (\mu + \lambda)\dfrac{\partial^2 u_r}{\partial r \partial z} + (\lambda + 2\mu)\dfrac{\partial^2 u_r}{\partial z^2} + \mu\dfrac{\partial^2 u_z}{\partial r^2} + \dfrac{(\lambda + \mu)}{r}\dfrac{\partial u_r}{\partial z} + \dfrac{\mu}{r}\dfrac{\partial u_z}{\partial r} = \rho\dfrac{\partial^2 u_z}{\partial t^2}. \end{cases}$$

By applying $\sin$ and $\cos$ Fourier in the coordinate z, to the functions u_r and u_z respectively and then the Laplace transform in t, system (1.1) - (1.5) can be reduced to the form:

$$\begin{array}{l} (\lambda + 2\mu)\dfrac{\partial^2 u_r^{(s)}}{\partial r} + (\lambda + 2\mu)\dfrac{1}{r}\dfrac{\partial \bar{u}_r^{(s)}}{\partial r} - \left(\mu q^2 + \rho p^2\right)\ \bar{u}_r^{(s)} - (\lambda + 2\mu)\, q\dfrac{\partial \bar{u}_z^{(c)}}{\partial r} - (\lambda + 2\mu)\dfrac{\bar{u}_r^s}{r^2} = 0 \\ (\lambda + \mu)\, q\dfrac{\partial u_r^{(s)}}{\partial r} + (\lambda + \mu)\, q\dfrac{u_r^{(s)}}{r} + \mu\dfrac{1}{r}\dfrac{\partial u_z^{(c)}}{\partial r} - \left[\rho p^2 + (\lambda + 2\mu)\, q^2\right]\ \bar{u}_z^{(c)} = \sigma_0 f(p) \end{array} \tag{1.6}$$

Here p and q - are the parameters of Laplace and Fourier transforms respectively.

Select two new functions by the formulas:

$$
\begin{aligned}
\bar{u}_r^{(s)} &= \frac{\partial \varphi}{\partial r} - q\frac{\partial \psi}{\partial r} \\
\bar{u}_z^{(s)} &= q\varphi - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)
\end{aligned}
\tag{1.7}
$$

The system (1.6) after applying the substitution (1.7) takes the simplest form:

$$
\left.
\begin{aligned}
&(\lambda+2\mu)\,(B_1\varphi) = q(B_2\psi) \\
&q^2(B_2\psi) - \frac{1}{r}\frac{d}{dr}(B_2\psi) - \frac{d^2(B_2\psi)}{dr^2} = \frac{\sigma_0\tau(p)}{\mu}
\end{aligned}
\right\}
\tag{1.8}
$$

Here B_1 and B_2 are modified Bessel operators (of zero index).

$$
B_k = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{p^2}{c_k^2} + q^2\right) \quad k = 1, 2.
$$

The second equation of system (1.8) is also a modified Bessel equation with respect to the function $(B_2\psi)$:

$$
B_0 B_2 \psi = -\frac{\sigma_0 f(p)}{\mu};
$$

here $B_0 = \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - q^2$.

A bounded solution of the system in the area $0 \le r \le a$ are the functions:

$$
\left.
\begin{aligned}
\varphi &= -\frac{\sigma_0 f(p)}{q\nu_1^2\,(\lambda+2\mu)} + A_0 I_0(\nu_1 r) \\
\psi &= -\frac{\sigma_0 f(p)}{q^2\nu_2^2\mu} + B_0 I_0(\nu_2 r)
\end{aligned}
\right\}
\tag{1.9}
$$

here $\nu_k = \sqrt{\dfrac{p^2}{c_k^2} + q^2}$, $k = 1, 2$ and I_0 are modified Bessel functions.

The unknown coefficients are determined from the lateral conditions on the surface $r = a$;

$$
\left.
\begin{aligned}
\sigma_{rr} &= 0 \\
\sigma_{rz} &= 0
\end{aligned}
\right\} \text{ in } r = a;\ z > 0.
$$

With regard to (1.7) and (1.9), the last relations turn into a system of two linear algebraic equations for determining A_0 and B_0:

$$
\begin{aligned}
&A_0\left[(\lambda+2\mu)\,I_0^{''}(v_1 a) + \frac{\lambda}{a}I_0^{'}(v_1 a) - \lambda q^2 I_0(v_1 a)\right] - \\
&-B_0\left[(\lambda+2\mu)\,qI_0^{''}(v_2 a) + \frac{\lambda}{a}qI_0^{'}(v_2 a)] - \lambda q[\frac{1}{a}I_0^{'}(v_2 a) + I_0^{''}(v_2 a)\right] = \frac{\lambda q\sigma_0 f(p)}{v_1^2\,(\lambda+2\mu)} \\
&2A_0 q I_0^{'}(v, a) - B_0\left[q^2 I_0^{'}(v_2 a) + \frac{1}{a}I_0^{'}(v_2 a) + \frac{1}{a}I_0^{''}(v_2 a) + I_0^{'''}(v_2 a)\right] = 0.
\end{aligned}
\tag{1.10}
$$

The determinant of this system is the function

$$
\begin{aligned}
F = \mu\, v_2 \Bigg\{ &\left[\frac{c_1^2}{c_2^2}v_1^2 - \left(\frac{c_1^2}{c_2^2} - 2\right)q^2\right](q^2 + v_2^2) I_1(av_2) I_0(v_1 a) + 2\frac{p^2}{c_2^2}\frac{v_1}{a}\times \\
&\times I_1(v_1 a) I_1(v_2 a) + 4q^2 v_1 v_2 I_1(v_1 a) I_0(v_2 a)\Bigg\}
\end{aligned}
\tag{1.11}
$$

System (1.10) has the following solution:

$$
\left.
\begin{aligned}
A_0 &= \frac{\lambda\sigma_0 q}{(\lambda+2\mu)v_1^2 F}\left(q^2 + v_2^2\right) v_2 I_1(v_2 a) \\
B_0 &= \frac{-2\lambda\sigma_0 q^2}{v_1(\lambda+2\mu)F} I_1(v_1 a)
\end{aligned}
\right\}
\tag{1.12}
$$

Substituting these expressions in formulas of common solutions (1.9) the solutions in transformations are completely determined.

We must now return to the real coordinates, therefore, inverse transformation of the function of the form (1.9), with taken (1.14) into account should be reproduced.

Judging from expressions (1.12) and (1.11), this operation is fraught with difficulties and to this day only the asymptotic solution of the stated problem as $t \to \infty$ is known.

It can be shown that the function$F$, defined by (1.11), and a part of the denominator in (1.12), at the values of real$q$, has no zeros in the complex plane *p,* outside of an imaginary axis. On the imaginary axis $Rep = 0$ by replacement $p = ik$ the equation

$$F = 0 \tag{1.13}$$

is reduced to the well-known Pochhammer equation for the frequencies of vibrations of infinitely long circular cylinder, when the longitudinal waves propagate in it. This equation was first obtained and studied in [4], further analytical and numerical study of this equation was conducted by a number of authors [3, 5].

From these studies it follows that equation (1.13) for every real q has on the axis $Rep = 0$ an infinite number of roots$p_m = if_m(q) \quad (m = \pm 1, \pm 2...)$, all of them are simple and pairwise complex conjugate $(p_{-m} = -p_m)$. But here we will not deal with finding these roots, we use the method developed in [6] for the solution of transient dynamics of a rectangular beam, and in the special case, of a layer.

For simplicity, we show this method on an example of finding the velocity of particles of the central axis, that in transformations is expressed by the formula:

$$\bar{\dot{u}}_z = -\frac{\sigma_0}{v_1^2}\frac{c_2^2}{c_1^2} + \frac{\lambda\sigma_0 c_2^2}{c_1^2}\left[\frac{q^2\left(q^2+v_2^2\right)v_2}{v_1^2 F} I_1\left(v_1 r\right) I_0(v_1 r) - \frac{2q^2 v_2^2}{v_1 F} I_1\left(v_1 a\right) I_0(v_2 r)\right]; \tag{1.14}$$

We modify it to the form:

$$\begin{gathered}\bar{\dot{u}}_z = -\frac{\sigma_0}{v_1^2(\lambda+2\mu)} + \frac{\lambda\sigma_0 q^2}{(\lambda+2\mu)}\left[\frac{q^2+v_2^2}{v_2^3}\frac{I_1\left(av_2\right)}{I_0\left(v_2 a\right)} \times \frac{1}{v_1^2}\frac{I_0\left(v_1 r\right)}{I_0\left(v_1 a\right)} - \right. \\ \left. -2\frac{1}{v_1}\frac{I_1\left(v_1 a\right)}{I_0\left(v_1 a\right)} \times \frac{1}{v_2^2}\frac{I_0\left(v_2 r\right)}{I_0\left(v_2 a\right)}\right] \times \frac{v_2^4}{F^*}\end{gathered} \tag{1.15}$$

here

$$\begin{gathered}F^* = v_2\left\{\left[\frac{c_1^2}{c_2^2}v_1^2 - \left(\frac{c_1^2}{c_2^2} - 2\right)q^2\right]\left(q^2+v_2^2\right)\frac{I_1(av_2)}{I_0(av_2)} + \right. \\ \left. +2\frac{p^2}{c_2^2}\frac{v_1}{a}\frac{I_1(v_1 a)}{I_0(v_1 a)}\frac{I_1(v_2 a)}{I_0(v_2 a)} + 4q^2 v_1 v_2 \frac{I_1(v_1 a)}{I_0(v_1 a)}\right\}\end{gathered}$$

Following [2], the function $\frac{v_2^4}{F^*}$ expands in series:

$$\frac{v_2^4}{F^*} = \sum_{n=1}^{\infty} a_n \frac{1}{v_2^n} \tag{1.16}$$

Each member of this series is the Laplace and Fourier transforms. If we are satisfied with the solutions for small values of t, it is sufficient to retain the first few terms:

$$\frac{v_2^4}{F^*} = \frac{1}{v^2}\left[1 + \frac{c_1}{c_2}\frac{1}{av_2} + \frac{1}{v_2^2}\left[q^2\left(1+\frac{c_2^2}{c_1^2}\right) + \frac{c_1^2}{c_2^2}\frac{1}{a^2}\right]\right].$$

The expressions in square brackets of Formula (1.15) as shown is a meromorphic function in the half plane $Rep > 0$, disappearing at infinity as $|p| = R_n \to \infty$ and also having simple poles at the points of the axis $Rep = 0$. Thus, we can apply to it the second expansion theorem [1], according to which

$$f(t) = \sum_{p_k} res_{p_k} f^*(p) e^{pt}$$

here $f^*(p) \overset{\circ}{\underset{\circ}{=}} L(f(t))$ and it possesses above properties.

First note that for small values t, in (1.15) by the first term dominates:

$$-\frac{\sigma_0}{\lambda+2\mu}\frac{1}{v_1^2} \overset{LF}{\underset{0}{\overset{0}{=}}} \sqrt{\frac{\pi}{2}}\frac{\sigma_0 c_1}{\lambda+2\mu}H\left(t-\frac{z}{c_1}\right) \tag{1.17}$$

Since the another member, by the crude estimate, in time t by two degrees is a less quantity,

$$\dot{u}_z \approx t^2 O(t) + O(t)$$

for small values of t.

Now we give the inverse functions of images that are the parts of the solutions (1.15):

1) $\dfrac{q^2+v_2^2}{v_2^3}\dfrac{I_1(v_2 a)}{I_0(a v_2)} \overset{L}{\underset{\cdot}{\overset{\cdot}{=}}} c_2 a \sin(q c_2 t) - 2\sum\limits_{k=0}^{\infty}\dfrac{c_2}{a}\dfrac{\left(q^2-\dfrac{\alpha_k^2}{a^2}\right)\sin\left(c_2\sqrt{q^2+\dfrac{\alpha_k^2}{a^2}}\right)t}{\alpha_k^2\sqrt{q^2+\dfrac{\alpha_k^2}{a^2}}}$;

2) $\dfrac{1}{v_i^2}\dfrac{I_0(v_i r)}{I_0(v_i a)} \overset{L}{\underset{\cdot}{\overset{\cdot}{=}}} \dfrac{c_i}{q}\sin(q c_i t) - 2\sum\limits_{k=0}^{\infty}\dfrac{J_0\left(\alpha_k\dfrac{r}{a}\right)}{J_1(\alpha_k)}\dfrac{\sin\left(c_i\sqrt{q^2+\dfrac{\alpha_k^2}{a^2}}\right)t}{\alpha_k\sqrt{q^2+\dfrac{\alpha_k^2}{a^2}}}$

3) $\dfrac{1}{v_1}\dfrac{I_1(v_1 a)}{I_0(v_1 a)} \overset{L}{\underset{\cdot}{\overset{\cdot}{=}}} \dfrac{2c_1}{a}\sum\limits_{k=0}^{\infty}\dfrac{\sin\left(c_1\sqrt{q^2+\dfrac{\alpha_k^2}{a^2}}\right)t}{\sqrt{q^2+\dfrac{\alpha_k^2}{a^2}}}$

4) $\dfrac{1}{v_2} \overset{\cdot}{\underset{\cdot}{=}} c_2 J_0(c_2 q t)$

5) $\dfrac{1}{v_2^3} \overset{\cdot}{\underset{\cdot}{=}} \dfrac{\sqrt{\pi}c_2^2 t}{q\Gamma(3/2)} I_1(c_2 q t)$

here α_k are the zeros of the function $J_0(x)$ and

$$v_i = \sqrt{\frac{p^2}{c_i^2}+q^2};\;\; i=1,2.$$

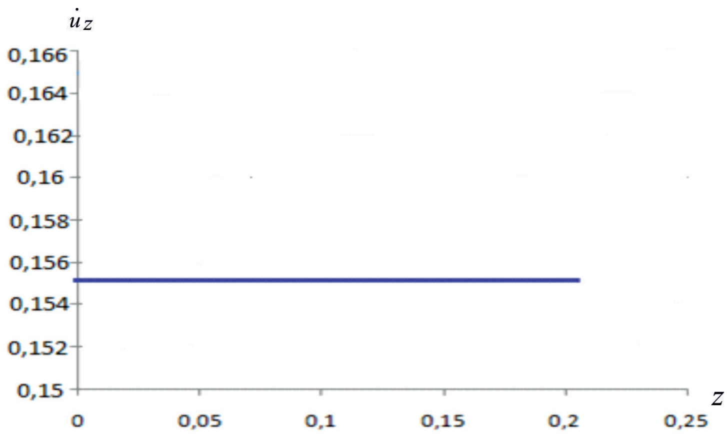

Fig.1. Distribution of longitudinal speed $\dot{u}_z$ along central axis at moment $t_1 = \dfrac{a}{c_1}$.

The solution (1.6) is a plane longitudinal wave that starts from the end area and propagates at a rate of c_1. To the moment $t_0 = \frac{a}{c_1}$ the solution on the axis oz will be presented by this plane wave, since the diffraction waves start from the side surface $r = a$ with speeds c_1 and c_2, reach the central axis with the moments $t_0 = \frac{a}{c_1}$ and $t_1 = \frac{a}{c_2}$ respectively. The calculation of $\underset{z}{\overset{\bullet}{u}}$ the following values of parameter and subsequent time intervals:

$$c_1 = 2c_2 = 6.2 \cdot 10^3 m/c$$
$$\lambda = 2\mu = 1,5 \cdot 10^{10} k\Gamma/m^2$$
$$\sigma_0 = -2 \cdot 10^6 k\Gamma/m^2$$

$t_1 = \frac{a}{c_1}$; $t_2 = \frac{1,5a}{c_1}$; $t_3 = \frac{2a}{c_1}$; $t_4 = \frac{2,5a}{c_1}$; $t_5 = \frac{3a}{c_1}$ and etc. Fig.1 –Fig. 5.

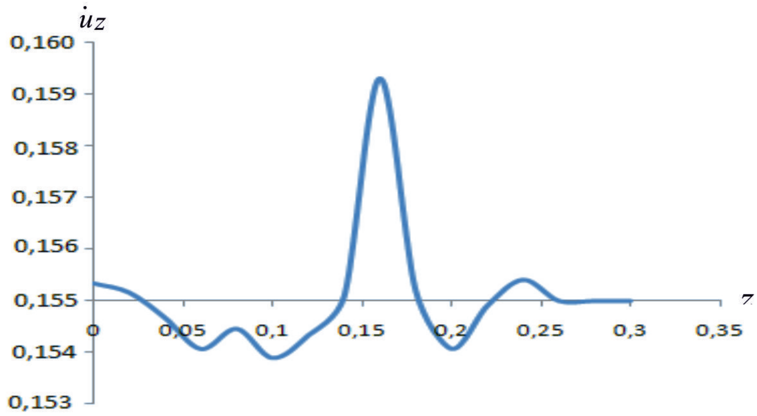

Fig.2. Distribution of longitudinal speed $\dot{u}_z$ along central axis at moment $t_1 = \frac{1.5a}{c_1}$.

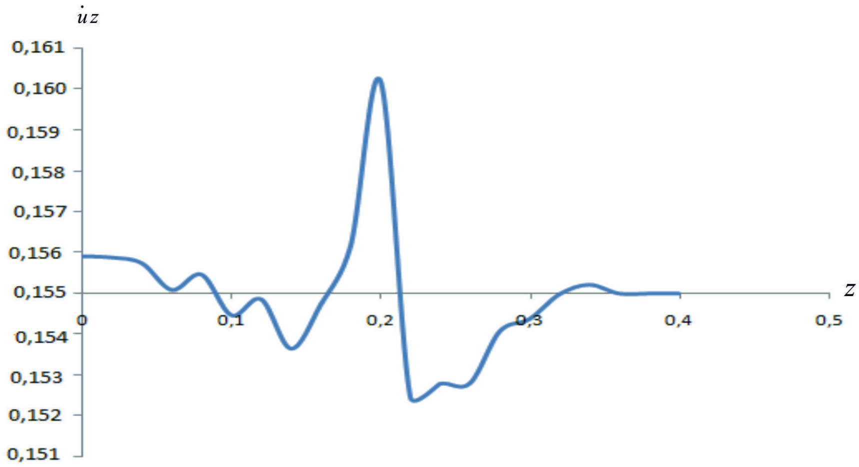

Fig.3. Distribution of longitudinal speed $\dot{u}_z$ along central axis at moment $t_1 = \frac{2a}{c_1}$

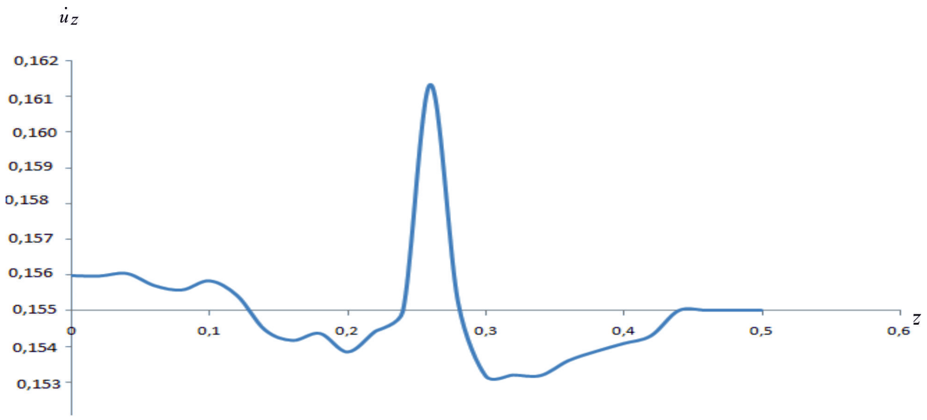

Fig.4. Distribution of longitudinal speed $\dot{u}_z$ along central axis at moment $t_1 = \dfrac{2.5\ a}{c_1}$.

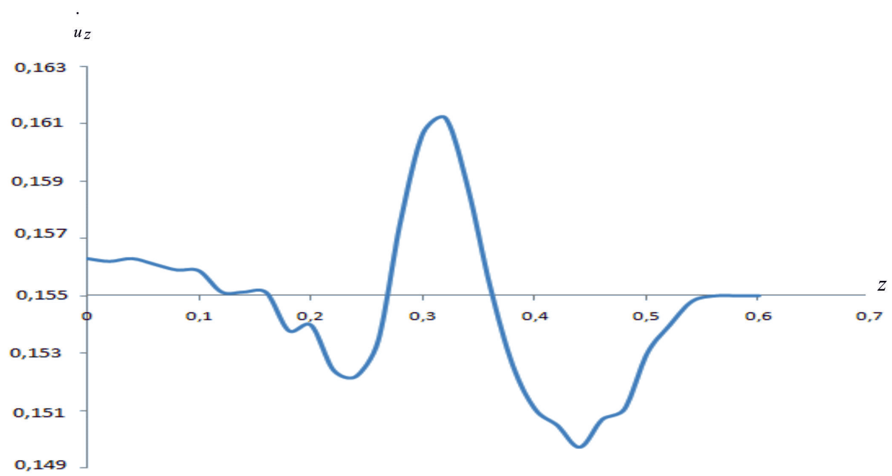

Fig.5. Distribution of longitudinal speed $\dot{u}_z$ along central axis at moment $t_1 = \dfrac{3\ a}{c_1}$.

These results are nearly identical with the results obtained in the [6], and these solutions also satisfy theoretical expectations, and agree well with the experimental results.

2 Conclusions

1. For the integration of an axially symmetric system of Lame equations, a new method has been applied as a result of which, firstly, this system takes the simple form, and secondly, the boundary defined function at the end $z = 0$, appears in the right parts of the equations of motion as a free member. Following fact allows you to quickly build solutions across the area of the cylinder, for a sufficiently broad class of functions, appearing in the boundary conditions at the end $z = 0$.
2. Solutions to these problems more complex configuration cylindrical solids (hollow, layered cylinder, etc.), apparently, it can be easily built in transformations. Then, for the representation of solutions in

real domains (as above the case shown with round cylinder) transformations found necessary to be represented as a product of two members; the first of them gives analytic transformation to the real variables, and the second expands into series of types (1.6), which can be converted easily term wise. It should be noted that with increasing complexity of the area in the process of building a complete solution only increases the difficulty of algebraic nature.

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